DIFFERENTIAL EQUATIONS

(*) If $Q'(t) = K\,Q(t) \Rightarrow Q(t) = Ce^{Kt}$, for any C ← remember

$\downarrow t=0$

→ If $Q(0) = Q_0$: $Q_0 = Ce^0 = C \Rightarrow C = Q_0$

$$\Rightarrow \boxed{Q(t) = Q_0 e^{Kt}}$$

→ If $\underline{K>0}$: $\lim_{t\to\infty} Q(t) = +\infty$

→ If $K<0$: $\lim_{t\to\infty} Q(t) = 0$

Separable DEs : $y' = g(x)\,f(y)$ (DE)

If $y(0) = y_0$ given (an initial condition)

$$\Rightarrow \begin{cases} y' = g(x)\,f(y) \\ y(0) = y_0 \end{cases} =: \text{initial value problem (I.V.P.)}$$

→ without $y(0)=y_0$ → solution has arbitrary constant

→ with $y(0)=y_0$ → Constant becomes specific

$$y' = g(x)\, f(y)$$

$$\Downarrow$$

$$\frac{dy}{dx} = g(x)\, f(y)$$

"Cross"-multiply: $\frac{1}{f(y)}\, dy = g(x)\, dx$

Integrate

with respect to x

with respect to y

$$\boxed{\int \frac{1}{f(y)}\, dy} = \int g(x)\, dx$$

Examples: (1) $\frac{dy}{dx} = -\frac{x}{y}$

$\rightarrow \quad y\, dy = -x\, dx$

$\rightarrow \quad \int y\, dy = -\int x\, dx$

$\rightarrow \quad \frac{1}{2} y^2 = -\frac{x^2}{2} + C_1 \Rightarrow \boxed{y^2 = -x^2 + C_1}$

Implicit solution

② $$\frac{dy}{dx} = x\,y$$

- If $y=0$: $\frac{dy}{dx} = 0 \Rightarrow y = \text{Constant}$

(independent of x)

$\rightarrow y = 0$ for all x (A)

- If $y \neq 0$:

$$\frac{1}{y}\,dy = x\,dx \rightarrow \int \frac{1}{y}\,dy = \int x\,dx$$

$$\Rightarrow \ln|y| = \frac{x^2}{2} + C$$

just a constant

$$\Rightarrow |y| = e^{\frac{x^2}{2}+C} = e^{\frac{x^2}{2}} \cdot \boxed{e^C} = C\,e^{\frac{x^2}{2}}$$

$$\Rightarrow y = \pm C\,e^{\frac{x^2}{2}} = \hat{C}\,e^{\frac{x^2}{2}} \qquad (B)$$

incorporated "$\pm$" in it

<u>Rmk</u>: (A), (B) can be merged into $y = A\,e^{\frac{x^2}{2}}$, allowing A to be 0 (to get (A))

(3) $\dfrac{dy}{dX} = \dfrac{X^2}{2y + \cos y}$, $y(0) = \pi$ $\Big\}$ I.V.P.

__sln__ : $(2y + \cos y)\, dy = X^2 dX$

$\Rightarrow \int (2y + \cos y)\, dy = \int x^2 dx$

$\Rightarrow y^2_{(x)} + \sin y(x) = \dfrac{X^3}{3} + C$ (general implicit solution)

To find $\underline{C}$: plug in $X=0$

$y^2(0) + \sin y(0) = \dfrac{0^3}{3} + C \overset{y(0)=\pi}{\Longrightarrow}$

$\pi^2 + \underbrace{\sin \pi}_{0} = 0 + C \Rightarrow \boxed{C = \pi^2}$

$\Rightarrow \boxed{y^2(x) + \sin(y(x)) = \dfrac{X^3}{3} + \pi^2}$

$\hookrightarrow$ sln to the I.V.P.