Solutions to HW2

(1) Solve the I.V.P. $\left.\begin{aligned} (x-2)\frac{dy}{dx} &= \frac{1}{2y(x^2+1)} \\ y(0) &= 1 \end{aligned}\right\}$

Sln: DE = separable. Indeed,

$$2y\,dy = \frac{1}{(x-2)(x^2+1)}\,dx$$

$$\Rightarrow \underbrace{\int 2y\,dy}_{I_1} = \underbrace{\int \frac{1}{(x-2)(x^2+1)}\,dx}_{I_2}$$

$I_1 = y^2$; for I_2 we need Partial Fractions

$$\frac{1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow \overset{0x^2+0x+}{1} = x^2(A+B) + x(-2B+C) + (A-2C)$$

$$\Rightarrow \left.\begin{aligned} &A+B=0 \Rightarrow A=-B \\ &C=2B \\ &A=2C+1 = 4B+1 \end{aligned}\right\} \quad \boxed{\begin{aligned} &B=-\tfrac{1}{5}, \quad A=\tfrac{1}{5} \\ &C=-\tfrac{2}{5} \end{aligned}}$$

$$\Rightarrow I_2 = \int \frac{1}{(x-2)(x^2+1)}\,dx = \int \frac{1/5}{x-2}\,dx + \int \frac{-\frac{1}{5}x - \frac{2}{5}}{x^2+1}\,dx$$

$$= \frac{1}{5}\ln|x-2| - \frac{1}{5}\int \underbrace{\frac{x}{x^2+1}\,dx}_{u=x^2+1} - \frac{2}{5}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{5}\ln|x-2| - \frac{1}{10}\ln(x^2+1) - \frac{2}{5}\arctan x + C$$

$$\rightsquigarrow \quad y^2 = \frac{1}{5}\ln|x-2| - \frac{1}{10}\ln(x^2+1) - \frac{2}{5}\arctan x + c$$

$y(0)=1 \rightsquigarrow$ find c

plug in $x=0$:

$$y^2(0) = \frac{1}{5}\ln|0-2| - \frac{1}{10}\underbrace{\cancel{\ln(0+1)}}_{\ln 1 = 0} - \frac{2}{5}\underbrace{\cancel{\arctan(0)}}_{0} + c$$

$$\Rightarrow 1 = \frac{1}{5}\ln 2 + c \Rightarrow \boxed{c = 1 - \frac{1}{5}\ln 2}$$

$$\boxed{y^2(x) = \frac{1}{5}\ln|x-2| - \frac{1}{10}\ln(x^2+1) - \frac{2}{5}\arctan x + \overbrace{\left(1 - \frac{1}{5}\ln 2\right)}^{c}}$$

implicit solution is fine here

(2) Evaluate $\int_{-1}^{5/2} g(s)\,ds$, where

$$g(s) = \begin{cases} -1, & -1 \le s < 0 \\ e^{\sin(s)}\cos(s), & 0 \le s < 2 \\ \dfrac{1}{\sqrt{27-3s^2}}, & 2 \le s \le \frac{5}{2} \end{cases}$$

Sln: Since the formula of $g(s)$ changes at $s=0$ and $s=2$:

$$I = \int_{-1}^{5/2} g(s)\,ds = \int_{-1}^{0} g(s)\,ds + \int_{0}^{2} g(s)\,ds + \int_{2}^{5/2} g(s)\,ds$$

$$= \underbrace{\int_{-1}^{0} -1\,ds}_{I_1} + \underbrace{\int_{0}^{2} e^{\sin(s)}\cos(s)\,ds}_{I_2} + \underbrace{\int_{2}^{5/2} \frac{1}{\sqrt{27-3s^2}}\,ds}_{I_3}$$

I_1: $I_1 = \int_{-1}^{0} -1\,ds = -s\Big|_{-1}^{0} = 0 \overset{F(b)-F(a)}{-} (-(-1)) = \boxed{-1}$

(from $-s$) limit

I_2: $\int_0^2 e^{\sin(s)} \underbrace{\cos(s)\,ds}$; $u = \sin(s)$
$du = \cos(s)\,ds$

$s = 0 \Rightarrow \quad u = \sin(0) = 0$

$s = 2 \Rightarrow \quad u = \sin(2)$

$$\Rightarrow \boxed{I_2} = \int_0^{\sin(2)} e^u \, du = e^u \Big|_0^{\sin(2)} =$$

$$= e^{\sin(2)} - e^0 = \boxed{e^{\sin(2)} - 1}$$

$$I_3: \int_2^{5/2} \frac{1}{\sqrt{27 - 3s^2}} \, ds = \int_2^{5/2} \frac{1}{\sqrt{3(9 - s^2)}} \, ds$$

$$= \frac{1}{\sqrt{3}} \int_2^{5/2} \frac{1}{\sqrt{\underset{3^2}{9} - s^2}} \, ds$$

$\rightarrow$ trig sub $\boxed{\begin{array}{l} s = 3\sin\theta \\ ds = 3\cos\theta \, d\theta \end{array}}$

$$I_3 = \frac{1}{\sqrt{3}} \int_{\arcsin\left(\frac{2}{3}\right)}^{\arcsin\left(\frac{5}{6}\right)} \frac{3\cos\theta}{\sqrt{3^2(1 - \sin^2\theta)}} \, d\theta =$$

$s = 2 \Rightarrow \quad 2 = 3\sin\theta$

$\Rightarrow \theta = \arcsin\left(\frac{2}{3}\right)$

$s = \frac{5}{2} \Rightarrow \quad \theta = \arcsin\left(\frac{5}{6}\right)$

$$= \frac{1}{\sqrt{3}} \int_{\arcsin\left(\frac{2}{3}\right)}^{\arcsin\left(\frac{5}{6}\right)} \frac{\cancel{3\cos\theta}^{\,1}}{\cancel{3\cos\theta}} \, d\theta = \frac{1}{\sqrt{3}} \left(\theta \Big|_{\arcsin\left(\frac{2}{3}\right)}^{\arcsin\left(\frac{5}{6}\right)} \right)$$

$$= \frac{1}{\sqrt{3}} \left(\arcsin\left(\frac{5}{6}\right) - \arcsin\left(\frac{2}{3}\right) \right) \Rightarrow$$

$$\boxed{\begin{array}{l} I = I_1 + I_2 + I_3 = (-1) + (e^{\sin(2)} - 1) \\ + \frac{1}{\sqrt{3}} \left(\arcsin\left(\frac{5}{6}\right) - \arcsin\left(\frac{2}{3}\right) \right) \end{array}}$$

③ Find all (non-negative) values of p

for which the improper integral

$$I = \int_2^{+\infty} \frac{1}{x(\ln x)^p}\, dx \quad \text{converges.}$$

Sol : I = improper of Type I, thus

$$I = \lim_{a \to \infty} \int_2^{a} \frac{1}{x(\ln x)^p}\, dx$$

Consider for a moment $\int \frac{1}{x(\ln x)^p}\, dx$

• $\boxed{\text{If } p=1}$: $\int \frac{1}{x \ln x} dx \rightarrow u = \ln x$, $du = \frac{1}{x} dx$

$= \int \frac{1}{u} du = \ln|u| + C = \ln|\ln x| + C$

$(x \text{ in } [2,\infty) \rightarrow \ln x > 0)$

$\rightarrow \boxed{\ln(\ln x) + C}$ would be the antiderivative in this case $\Rightarrow$ $$I = \lim_{a\to\infty} \ln(\ln x)\Big|_2^a =$$

$$= \lim_{a\to\infty}\Big[\underbrace{\ln(\ln a)}_{\xrightarrow{a\to\infty}\ \infty} - \underbrace{\ln(\ln 2)}_{\text{just a constant}}\Big] = \infty$$

$\rightarrow$ $\boxed{\beta=1,\ I \text{ diverges}}$

$\boxed{0\le\beta\neq 1}$:

$$I = \lim_{a\to\infty}\int \frac{1}{x(\ln x)^\beta}\,dx$$

Consider $\displaystyle\int \frac{1}{(\ln x)^\beta}\boxed{\frac{1}{x}\,dx}$, $u \equiv \ln x$ as before

$du = \frac{1}{x}\,dx$

$$= \int \frac{1}{u^\beta}\,du = \int u^{-\beta}\,du =$$

$$\overset{\beta\neq 1}{=} \frac{u^{-\beta+1}}{-\beta+1} = \frac{(\ln x)^{-\beta+1}}{-\beta+1} + C$$

that's why one thinks about treating $\beta=1$ separately (only case where $\int \frac{1}{u^\beta}\,du$ has a different form)

Thus, if $0 \leq p \neq 1$

$$I = \lim_{a \to \infty} \frac{1}{-p+1} \left[(\ln a)^{-p+1} - \underbrace{(\ln 2)^{-p+1}}_{\text{constant}} \right]$$

- If $p < 1$, $-p+1 > 0 \Rightarrow (\ln a)^{\overset{>0}{-p+1}} \xrightarrow[a \to \infty]{} \infty$

$\Rightarrow I \xrightarrow{a \to \infty} \infty$, diverges

- If $p > 1$, $-p+1 < 0 \Rightarrow (\ln a)^{-p+1} = \frac{1}{\underbrace{(\ln a)^{p-1}}_{>0}} \xrightarrow{a \to \infty} 0$

$\Rightarrow I \xrightarrow{a \to \infty} \frac{(\ln 2)^{-p+1}}{-1+p}$ which is a finite number

$\rightarrow I =$ converges

To sum it up: $\boxed{0 \leq p \leq 1}$, I diverges

$\boxed{p > 1}$, I converges