§ Improper Integrals

(area under the graph)

So far: we have looked at the integral of a continuous function on a closed interval $[a,b]$

good condition

Now: → either not continuous functions on $[a,b]$

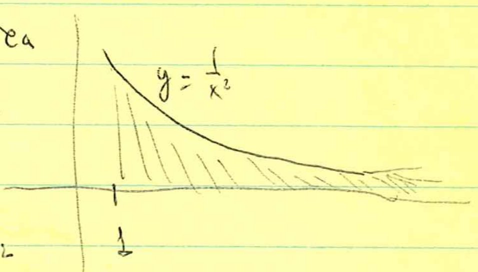
→ on infinite intervals

→ in either case, we call the corresponding integral

$\boxed{\text{IMPROPER}}$

I. Infinite Intervals

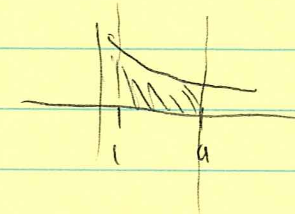
e.g. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ ⟷ net area under $y = \frac{1}{x^2}$



As before $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{\infty}$ → but $\boxed{\infty}$ → is not a number!

<u>What to do:</u>

- replace ∞ by a number (e.g. a)
- calculate for fixed a
- send $a \to \infty$



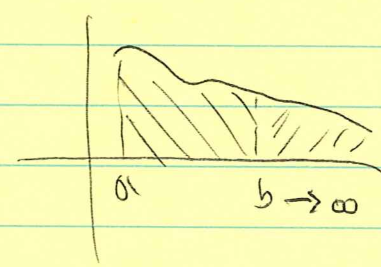
$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x^2}\,dx = \lim_{a\to\infty} \left.\frac{-1}{x}\right|_1^a =$$

$$= \lim_{a\to\infty} \left[-\frac{1}{a} - \frac{(-1)}{1}\right] = \lim_{a\to\infty} \left(\cancelto{0}{-\frac{1}{a}} + 1\right) = \boxed{1}$$

<u>Definition</u>: <u>Improper</u> Integrals over <u>Infinite</u> Intervals

① If $f(x)$ is cts on $[a, \infty)$, then

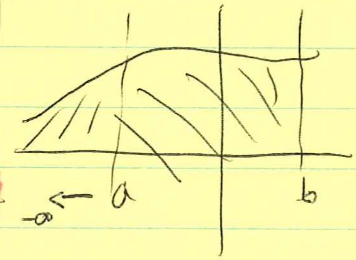
$$\boxed{\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx}$$



e.g. $\displaystyle \int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x^2}\,dx$

② If $f(x) = \text{cts}$ on $(-\infty, b]$, then

$$\boxed{\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\,dx}$$

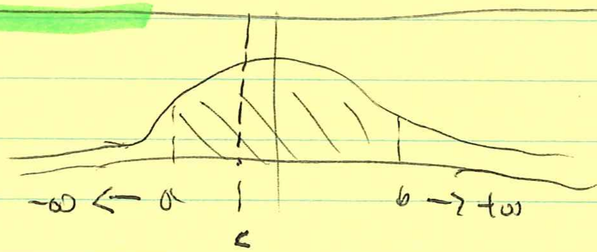


e.g., $\int_{-\infty}^{1} f(x)\,dx = \lim_{a \to -\infty} \int_{a}^{1} f(x)\,dx$

③ If $f(x) = \text{cts}$ on $(-\infty, \infty)$

$$\boxed{\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a \to -\infty} \int_{a}^{c} f(x)\,dx + \lim_{b \to \infty} \int_{c}^{b} f(x)\,dx}$$

Where $\boxed{c = \text{ANY real number}}$



If all the limits in ① – ③ exist ⇒ we say Integral ~~[illegible]~~ **CONVERGES**

→ if not, **DIVERGES**

Examples

(1) $$\int_1^{\infty} 5e^{-2x}\,dx = 5\int_1^{\infty} e^{-2x}\,dx = 5\lim_{a\to\infty}\int_1^{a} e^{-2x}\,dx$$

$$= 5\lim_{a\to\infty} \frac{e^{-2x}}{-2}\Big|_1^{a} = 5\lim_{a\to\infty}\left[\cancel{\frac{e^{-2a}}{-2}}^{\to 0} - \frac{e^{-2}}{-2}\right] = \boxed{\frac{5}{2e^2}}$$

(2) $$\int_{-\infty}^{-1}\frac{1}{x^3}\,dx = \lim_{b\to-\infty}\int_b^{-1}\frac{1}{x^3}\,dx = \lim_{b\to-\infty}\int_b^{-1}x^{-3}\,dx$$

$$= \lim_{b\to-\infty} -\frac{1}{2}x^{-2}\Big|_b^{-1} = \lim_{b\to-\infty}\left[\frac{-1}{2(1)^2} - \cancel{\frac{1}{2(b)^2}}^{\to 0}\right] = \boxed{-\frac{1}{2}}$$

(3) ← random

$$\int_{-\infty}^{+\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{\boxed{0}}\frac{1}{1+x^2}\,dx + \int_{\boxed{0}}^{\infty}\frac{1}{1+x^2}\,dx$$

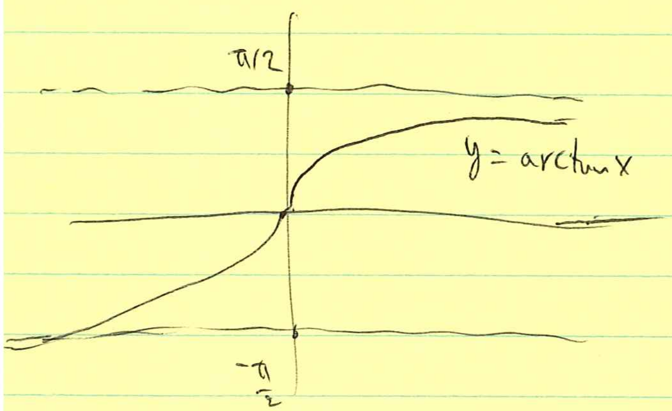
$$= \lim_{b\to-\infty}\int_b^{0}\frac{1}{1+x^2}\,dx + \lim_{a\to\infty}\int_0^{a}\frac{1}{1+x^2}\,dx =$$

$$= \lim_{b\to-\infty}\arctan x\Big|_b^{0} + \lim_{a\to\infty}\arctan x\Big|_0^{a}$$

$$= \lim_{b \to -\infty} \left[\cancel{\arctan(0)}^{\,0} - \arctan(b) \right] +$$

$$+ \lim_{a \to \infty} \left[\arctan a - \cancel{\arctan(0)}^{\,0} \right] =$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$



II. Unbounded integrands

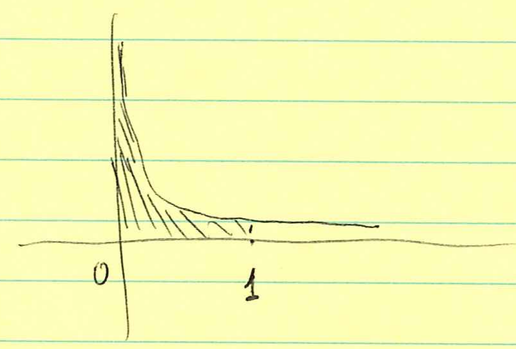
e.g.. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$, $x = 0$???

→ $\boxed{\text{can't}}$ find antiderivative and plug in $x=0$

→ use limits as before

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{b\to 0^+}\int_b^1 \frac{1}{\sqrt{x}}\,dx = \lim_{b\to 0^+}\int_b^1 x^{-\frac{1}{2}}\,dx$$

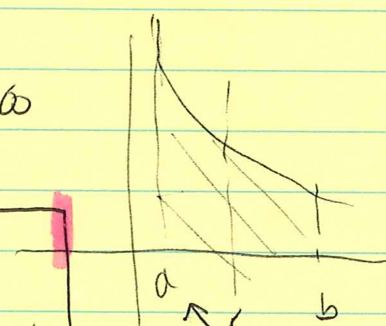
$$= \lim_{b\to 0^+} 2\sqrt{x}\,\Big|_b^1 = \lim_{b\to 0^+}\left(2\sqrt{1} - 2\underbrace{\sqrt{b}}_{0}\right) = \boxed{2}$$



Definition: Improper Integrals with an Unbounded Integrand

(1) $f(x) = $ Cts on $(a,b]$, $\lim_{x\to a^+} f(x) = \pm\infty$

LEFT



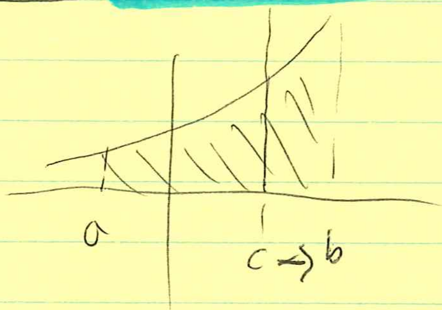
Then $$\boxed{\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx}$$

e.g. $\int_1^3 \frac{dx}{x-1} = \lim_{c\to 1^+}\int_c^3 \frac{dx}{x-1}$

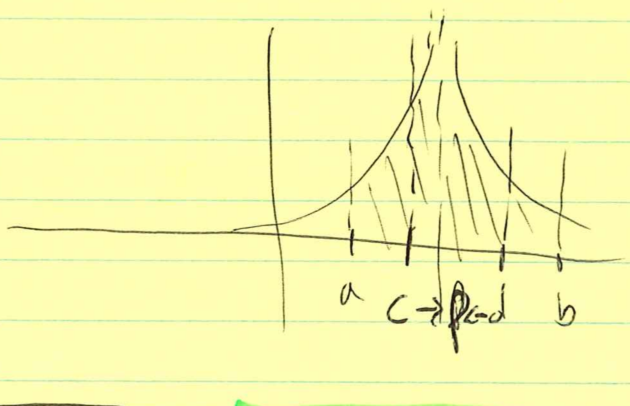
② RIGHT $f(x) = $ cts on $[a,b)$ with $\lim_{x \to b} f(x) = \pm\infty$, then

$$\boxed{\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx}$$

$$\boxed{\underline{\text{e.g.}} \quad \int_{-5}^{0} \frac{1}{x}\,dx = \lim_{c \to 0^-} \int_{-5}^{c} \frac{1}{x}\,dx}$$



③ $f(x) = $ cts on $[a,b]$ $\boxed{\text{except at } p}$, $a < p < b$



e.g. $\displaystyle\int_0^2 \frac{1}{\sqrt{x-1}}\,dx$

$$= \lim_{c \to 1^-} \int_0^c \frac{1}{\sqrt{x-1}}\,dx + \lim_{d \to 1^+} \int_d^2 \frac{1}{\sqrt{x-1}}\,dx$$

then

$$\boxed{\int_a^b f(x)\,dx = \lim_{c \to p^-} \int_a^c f(x)\,dx + \lim_{d \to p^+} \int_d^b f(x)\,dx}$$

If the limits in ①-③, integral CONVERGES, else DIVERGES

Examples:

① $\int_0^3 \frac{1}{\sqrt[3]{x}}\,dx$, $x=0$???

$$= \lim_{a\to 0^+} \int_a^3 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{a\to 0^+} \int_a^3 x^{-\frac{1}{3}}\,dx = \lim_{a\to 0^+} \frac{3}{2} x^{\frac{2}{3}} \Big|_a^3$$

$$= \lim_{a\to 0^+} \Big[\frac{3}{2}(3)^{2/3} - \frac{3}{2} \underbrace{a^{2/3}}_{0}\Big] = \boxed{\frac{3}{2}\sqrt[3]{9}}$$

② $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$, $x=1$???

$$= \lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-} \arcsin x \Big|_0^b =$$

$$= \lim_{b\to 1^-} \arcsin b = \boxed{\frac{\pi}{2}}$$

③ $\int_{-1}^3 \frac{1}{\sqrt[3]{x}}$, $x=0$!??

$$= \int_{-1}^0 \frac{1}{\sqrt[3]{x}}\,dx + \int_0^3 \frac{1}{\sqrt[3]{x}} = \lim_{a\to 0^-} \int_{-1}^a \frac{1}{\sqrt[3]{x}}\,dx + \frac{3}{2}\sqrt[3]{9}$$

$$= \lim_{a\to 0^-} \Big(\underbrace{\frac{3}{2}a^{2/3}}_{0} - \frac{3}{2}(-1)^{2/3} + \frac{3}{2}\sqrt[3]{9}\Big)$$

$$= \boxed{\frac{3}{2}\left(\sqrt[3]{9}-1\right)}$$

§ Improper Integrals (MORE ADVANCED EXAMPLES)

$\Updownarrow$

Combine definition + integration techniques

Examples: (for all of them, evaluate if it converges say it diverges if not)

① $I = \int_0^{\boxed{\infty}} \frac{1}{(2x+1)^3}\, dx = \lim_{a\to\infty} \int_0^{a} \frac{1}{(2x+1)^3}\, dx.$

Substitution $u = 2x+1 \Rightarrow \frac{1}{2}du = dx$

$x = 0 \Rightarrow u = 1$

$x = a = u = 2a+1$

$I = \int_0^{\infty} \frac{1}{(2x+1)^3}\, dx = \lim_{a\to\infty} \int_1^{2a+1} \frac{1}{2} \frac{1}{u^3}\, du = \frac{1}{2}\left(-\frac{1}{2} \lim_{a\to\infty} u^{-2}\Big|_1^{2a+1}\right)$ (u^{-3})

$= \frac{1}{2} \cdot \left[-\frac{1}{2} \lim_{a\to\infty} \frac{1}{(2a+1)^2} - \frac{1}{(1)^2}\right] = -\frac{1}{4}(-1) = \boxed{\frac{1}{4}}$

Practice: $\boxed{\int_{-\infty}^{1} \frac{1}{\sqrt[3]{2-x}}\,dx}$, $\boxed{\int_{-3}^{1} \frac{1}{(2x+6)^{2/3}}\,dx}$, $\boxed{\int_{0}^{\ln 3} \frac{e^x}{(e^x-1)^{2/3}}\,dx}$

$\hookrightarrow$ all are improper + a substitution is needed (don't forget the limits!)

Also: $\boxed{\int_{e}^{\infty} \frac{1}{x \ln^{99} x}\,dx}$ Hint: after you express it using limits, use $u = \ln x$

② $\int_0^\infty \cos x\,dx = \lim_{a\to\infty} \int_0^a \cos x\,dx = \lim_{a\to\infty} \sin x \Big|_0^a =$

$= \boxed{\lim_{a\to\infty} \sin a} - \underbrace{\sin(0)}_{0}$ → doesn't exist!!!

integral diverges

(3) $\int_1^\infty \frac{1}{x(x+1)}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x(x+1)}\,dx$

Partial fractions! $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$$\ldots \Rightarrow \boxed{A = 1,\ B = -1}$$

$$\lim_{a\to\infty} \int_1^a \frac{1}{x(x+1)}\,dx = \lim_{a\to\infty}\left[\int_1^a \left(\frac{1}{x} - \frac{1}{x+1}\right)dx\right]$$

$$= \lim_{a\to\infty} \left(\ln|x| - \ln|x+1|\right)\Big|_1^a = \lim_{a\to\infty}\left(\left[\underset{\downarrow\ \infty\,(!)}{\ln|a|} - \underset{\downarrow\ \infty\,(!)}{\ln|a+1|}\right] - [\underset{\downarrow\ 0}{\ln|1|} - \ln|2|]\right)$$

$\ddot{\frown}$ $\boxed{\infty - \infty}$ $\to$ undefined

$$= \left(\lim_{a\to\infty} \ln\left|\frac{a}{a+1}\right|\right) + \ln 2 = \underbrace{\ln 1}_{0} + \ln 2 = \boxed{\ln 2}$$

$\frac{a}{a+1} \to 1$

Practice:

- $\displaystyle\int_1^5 \frac{1}{(x-5)^2}\,dx$ (show it diverges) hint: substitution

- $\displaystyle\int_0^9 \frac{1}{(x-1)^{1/3}}\,dx$ (hint: problem at $x=1$) then substitution

- $\displaystyle\int_0^{\boxed{\infty}} x^2 e^{-x}\,dx$ $\rightarrow$ hint: after expressing through limits, use integration by parts.