$\boxed{\text{The Integral Test}}$

suppose : f — continuous, positive, decreasing

on $[1, \infty)$; let $a_n = f(n)$, Then

$\sum_{n=1}^{\infty} a_n$ is : convergent **IFF** $\int_1^{\infty} f(x)\,dx$ convergent

- if $\int_1^{\infty} f(x)\,dx$ converges $\Rightarrow \sum a_n$ converges
- if $\int_1^{\infty} f(x)\,dx$ diverges $\Rightarrow \sum a_n$ diverges

* Not necessary to start from $n = 1$, e.g

for $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$ $\overset{\text{use}}{\rightsquigarrow}$ $\int_4^{\infty} \frac{1}{(x-3)^2}\,dx$

$\Rightarrow$ eventually decreasing a_n vs fine

Examples:

① $\boxed{\sum_{n=1}^{\infty} \frac{3}{n^2+1} \; ; \; C/D \; ?}$

Sln: $f(x) = \frac{3}{x^2+1}$ — positive ✓, continuous ✓, decreasing ✓

$$\int_1^{\infty} \frac{3}{x^2+1}\, dx = 3 \lim_{t \to \infty} \int_1^t \frac{1}{x^2+1}\, dx =$$

$$= 3 \lim_{t \to \infty} \left[\arctan t - \frac{\pi}{2} \right] = 3\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{3\pi}{4}$$

$\Rightarrow$ Integral Converges $\xRightarrow{\text{I. Test}}$ $\sum \frac{3}{n^2+1} < +\infty$

② (p-series) $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \; ; \; C/D \; ?}$

Sln: • If $p < 0$: $\frac{1}{n^p} \to \infty \xRightarrow[\text{Test}]{\text{Divergence}} \sum$ diverges

• $p = 0$: $\frac{1}{n^p} = 1 \to 1 \xRightarrow[\text{Test}]{\text{Divergence}} \sum$ diverges a $\underline{p > 0}$, $f(x) = \dfrac{1}{x^p}$ — positive on $[1, \infty)$ ✓, continuous ✓, decreasing

$$I = \int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{a\to\infty} \int_1^{a} x^{-p}\,dx =$$

$$= \lim_{a\to\infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^{a} = \frac{1}{1-p} \lim_{a\to\infty} \left(a^{-p+1} - 1 \right)$$

as in HW2, P3 (consider $p=1$ on its own)

$I =$ — converges for $p > 1$; diverges for $p \le 1$

$\xRightarrow[\text{Test}]{\text{Integral}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ — $\boxed{\text{converges}}$ if $\boxed{p > 1}$; $\boxed{\text{diverges}}$ if $\boxed{p \le 1}$

③ $\sum_{n=1}^{\infty} \frac{\ln n}{n}$; C / D

makes you think of $\boxed{\int \ln x \frac{1}{x}\, dx \quad u = \ln x}$

Use integral test

$\underline{\text{Sln}}$: $f(x) = \frac{\ln x}{x}$ — positive ✓, continuous ✓, decreasing ?

$\underline{\text{check}}$: $f'(x) = \frac{\frac{1}{x} x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

$\Rightarrow f'(x) < 0$ when $1 - \ln x < 0 \Rightarrow \ln x > 1 = \ln e$

$\Rightarrow \boxed{x > e} \approx 2.7\ldots$

$\Rightarrow$ eventually decreasing $\to$ can apply Integral Test

$u = \ln x,\ du = \frac{1}{x} dx$

$$\int_1^{\infty} \frac{\ln x}{x}\, dx = \lim_{a \to \infty} \int_1^a \ln x \frac{1}{x}\, dx = \lim_{a \to \infty} \frac{(\ln x)^2}{2} \Big|_1^a = \infty$$

↳ integral diverges

↳ series diverges
I. Test

(2) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$; C/D ?

Sln: • $b_n = \frac{1}{n}$ and $\frac{1}{n+1} < \frac{1}{n} \Rightarrow \boxed{b_{n+1} < b_n}$ decreasing

• $\lim_{n \to \infty} b_n = \frac{1}{n} = 0$

$\underset{\text{Series Test}}{\overset{\text{Alternating}}{\Longrightarrow}}$ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges

(3) $\sum_{n=1}^{\infty} (-1)^{n} \frac{5n}{7n-2}$ → don't care

Sln: (ii) not satisfied: indeed $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{5n}{7n-2} = \frac{5}{7} \neq 0$

Can't apply A.S.T.

I On the other hand: $a_n = (-1)^n b_n$ $\begin{cases} \xrightarrow{\text{odd}} -\frac{5}{7} \\ \xrightarrow{\text{even}} \frac{5}{7} \end{cases}$ ⇝ a_n doesn't have a limit

↓ EXTRAPOLATE ~~A.S.T.~~ - Div. Test

$\boxed{\sum \text{ diverges}}$ ← Divergence Test

④ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$; C/D

sln: • $\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0 \rightarrow$ (ii) satisfied

• Check (i) : is $b_n = \frac{n^2}{n^3+1}$ decreasing ?

Define, for $\boxed{x \geq 1}$ (since $n \geq 1$) $f(x) = \frac{x^2}{x^3+1}$

$$f'(x) = \frac{2x(x^3+1) - 3x^2 \cdot x^2}{(x^3+1)^2} = \frac{-x^4+2x}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

$\rightarrow f'(x) < 0$ when $x > \sqrt[3]{2}$

$\Downarrow$

$f \downarrow$ on $(\sqrt[3]{2}, \infty) \Rightarrow \boxed{b_{n+1} < b_n \text{ when } n \geq 2}$

$\Rightarrow$ (i) ✓, (ii) ✓ Series converges

(possibly 2nd hour)

not just on alternating

Dfn (Absolutely Convergent) :

$\sum a_n$:= absolutely convergent

if the $\sum |a_n|$ is convergent

Examples :

① $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$:

$\rightarrow \sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{1}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} \xrightarrow{\text{p-test}}$ converges

$\rightsquigarrow$ original series is abs. convergent!

$\sum (-1)^{n-1} \frac{1}{n}$

- Convergent $\not\Longrightarrow$ abs. convergent

HOWEVER

- absolutely convergent $\Longrightarrow$ C

Example:

$\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^{3/2}}$; C/D ?

both positive + negative terms

but not alternating

Will show : $\sum$ = A.C $\Longrightarrow$ C

p-test

$\sum \left| \frac{\sin(2n)}{n^{3/2}} \right|$: $\left| \frac{\sin(2n)}{n^{3/2}} \right| \le \frac{1}{n^{3/2}}$, $\sum \frac{1}{n^{3/2}} < +\infty$

$\xrightarrow[\text{Test}]{\text{Comparison}}$ A.C $\Rightarrow$ C

• $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (notice this - by A. S. T -

(converges)

but

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \xrightarrow{p\text{-test}}$ diverges

Not abs. convergent

$\rightsquigarrow$ Dfn 2: $\sum a_n$ :: Conditionally convergent

if it is
- convergent
- not absolutely convergent

(e.g. $\sum \frac{(-1)^{n-1}}{n}$ cond. convergent)

Next criterion:

both

positive

sign-changing

(shows$\overset{\text{if true}}{\vee}$ A.C. $\Rightarrow$ C)

Σa_n

Ratio Test

(i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \rightsquigarrow$ A.C.

$\Downarrow$

C.

(ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ (can be ∞)

$\rightsquigarrow$ diverges

(iii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ inconclusive

① $\sum_{n=1}^{\infty} (-1)^n \frac{n^5}{2^n}$; C/D ?

<u>sln</u>: Could use A.S.T. but (i), (ii) need work to verify

⇝ instead Ratio Test :

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} \frac{(n+1)^5}{2^{n+1}}}{(-1)^n \frac{n^5}{2^n}}\right|$$

$$= \left(\frac{n+1}{n}\right)^5 \frac{2^n}{2^{n+1}} = \left(\frac{n+1}{n}\right)^5 \frac{1}{2} \longrightarrow 1 \cdot \frac{1}{2} = \frac{1}{2} < 1$$

Ratio Test

$$\boxed{A.C. \Rightarrow C}$$

(2) $\sum_{n=1}^{\infty} \frac{1}{n!}$; C/D ?

Sln: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| =$

$= \frac{n!}{(n+1)!} = \frac{\cancel{n!}}{\cancel{n!}(n+1)} = \frac{1}{n+1} \to 0$

RATIO TEST

Converges

(3) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$; C/D ?

Sln: $\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{\frac{(n+1)^{n+1}}{(n+1)\cancel{n!}}}{\frac{n^n}{\cancel{n!}}} = \frac{\frac{(n+1)^n \cancel{(n+1)}}{\cancel{(n+1)}}}{n^n}$

$= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \xrightarrow{n \to \infty} e > 1$

$\overset{R.T.}{\Longrightarrow}$ diverges