Examples – July 10th

FTC part 1

① Differentiate the following:

(a) $\int_1^x e^{t^2}\,dt$.

Let $F(x) = \int_1^x e^{t^2}\,dt$. (then $f(t) = e^{t^2}$).

FTC $\Rightarrow F'(x) = f(x)$ $(= e^{x^2})$

So, $\frac{d}{dx}\left(\int_1^x e^{t^2}\,dt\right) = F'(x) = e^{x^2}$ //

(b) $\int_x^1 e^{t^2}\,dt$

Why define F(x) like this? Because this is the type of integral FTC tells me how to differentiate!

Again, let $F(x) = \int_1^x e^{t^2}\,dt$ $\left(\begin{array}{l} f(t) = e^{t^2} \text{ and} \\ \text{FTC} \Rightarrow F'(x) = f(x) \end{array}\right)$

Using a property of integrals $\left(\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt\right)$ we have

$$\int_x^1 e^{t^2}\,dt = -F(x)$$

$$\Rightarrow \frac{d}{dx}\left(\int_x^1 e^{t^2}\,dt\right) = \underbrace{-F'(x) = -f(x)}_{F'(x) = f(x) \text{ by FTC}} = -e^{x^2} //$$

(c) $\displaystyle\int_x^{\cos x} e^{t^2}\,dt$

Again, define $F(x)$ like this because this is what FTC tells me how to differentiate. Now write given integral in terms of $F(x)$.

Again, let $F(x) = \int_1^x e^{t^2}\,dt$. Using another property of integrals $\left(\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt\right)$ we have

could choose any number

$$\int_x^{\cos x} e^{t^2}\,dt = \int_x^{1} e^{t^2}\,dt + \int_1^{\cos x} e^{t^2}\,dt$$

$$= -\int_1^x e^{t^2}\,dt + \int_1^{\cos x} e^{t^2}\,dt$$

$$= -F(x) + F(\cos(x))$$

So

$$\frac{d}{dx}\left(\int_x^{\cos x} e^{t^2}\,dt\right) = -F'(x) + \underbrace{F'(\cos(x))\cdot(-\sin(x))}_{\text{CHAIN RULE!!}}$$

$$\underset{\text{FTC}}{=} -f(x) + f(\cos(x))\cdot(-\sin(x))$$

$$= -e^{x^2} + e^{(\cos(x))^2}\cdot(-\sin(x))$$

ANTIDERIVATIVES

① Find an antiderivative of $\underbrace{f(x) = 4x + \cos(x)}$

Which function's derivative is equal to $4x + \cos x$???

Differentiating decreases the power by 1, so when we antidifferentiate, the power should increase by 1.

GUESS: $F(x) = 2x^{2} + \sin(x)$.

CHECK: $F'(x) = 2 \cdot 2x + \cos(x) = 4x + \cos(x)$ ✓

ANSWER: $F(x) = 2x^2 + \sin(x)$ is one antiderivative of $f(x) = 4x + \cos x$. The general form of an antiderivative of $f(x)$ is

$$2x^2 + \sin(x) + \underbrace{C}_{} \quad (\text{where } C \text{ is a constant})$$

since the derivative of constants is zero, adding C still gives us an antiderivative.

FTC part 2

① Calculate the following integrals.

(a) $\int_0^1 x^7 dx$

$\frac{1}{8}x^8$ is an antiderivative of x^7 (check: $\left(\frac{1}{8}x^8\right)' = x^7$)

SO FTC ⇒

$$\int_0^1 x^7 dx = \frac{1}{8}x^8 \Big|_0^1 = \frac{1}{8}(1)^8 - \frac{1}{8}(0)^8 = \frac{1}{8}$$

(b) $\int_0^3 e^{2t}\,dt$ $\left(\text{Remember: } (e^{2t})' = 2e^{2t} \Rightarrow \left(\frac{1}{2}e^{2t}\right)' = \frac{1}{2}\cdot 2e^{2t} = e^{2t}\right)$

$\frac{1}{2}e^{2t}$ is an antiderivative of e^{2t} so

FTC $\Rightarrow$ $\int_0^3 e^{2t}\,dt = \frac{1}{2}e^{2t}\Big|_0^3 = \frac{1}{2}e^6 - \frac{1}{2}e^0$

$= \frac{1}{2}e^6 - \frac{1}{2}$ //

METHOD OF SUBSTITUTION

① Calculate the following ~~integrals~~ antiderivatives.

(a) $\int \sqrt{\underbrace{1+2x}_{u}}\,dx$

Let $u = 1+2x$. Then $du = (1+2x)'\cdot dx$

$\Rightarrow du = 2\cdot dx$ $\left(\Rightarrow \frac{1}{2}du = dx\right)$

and $\sqrt{1+2x} = \sqrt{u}$. So

Replace all of the xs and dx with us and du. ~~according to~~

$$\int \sqrt{1+2x}\cdot dx = \int \sqrt{u}\cdot \frac{1}{2}du$$

$$= \int \frac{1}{2}u^{1/2}\,du \quad \left(\sqrt{u} = u^{1/2}\right).$$

Now, $\frac{1}{2}\cdot\frac{1}{3/2}u^{3/2}$ is an antiderivative of $\frac{1}{2}u^{1/2}$ $\left(\text{check: } \left(\frac{1}{2}\cdot\frac{1}{3/2}u^{3/2}\right)' = \frac{1}{2}\cdot\frac{3/2}{3/2}u^{1/2} = \frac{1}{2}u^{1/2}\right)$

So

$$\int \sqrt{1+2x}\,dx = \int \frac{1}{2}u^{1/2}\,du = \frac{1}{3}u^{3/2} + C$$

change back into original variable 'x'

$$= \frac{1}{3}(1+2x)^{3/2} + C$$

* An antiderivative of a function of $\underline{x}$ should also be a function of $\underline{\underline{x}}$ (not u). *

(b) $\int_0^1 \sqrt{1+2x}\,dx$

METHOD 1: From (a), we know that $\frac{1}{3}(1+2x)^{3/2}+C$ is an antiderivative of $\sqrt{1+2x}$ for any constant C. So,

FTC says we can use any antiderivative, so use the one with $c=0$.

$$\int_0^1 \sqrt{1+2x}\,dx = \frac{1}{3}(1+2x)^{3/2}\Big|_0^1 = \frac{3^{3/2}}{3} - \frac{1^{3/2}}{3}$$

METHOD 2: As in (a), if we make the substitution

$u = 1+2x$

Then $\int_0^1 \sqrt{1+2x}\,dx = \int_{?}^{?} \frac{1}{2}\sqrt{u}\,du$

We can evaluate the original integral by evaluating the second integral (the 'u' one)

BUT our change of variables changes the limits of integration.

use equation $u = 1+2x$

$$\left\{\begin{array}{l} x = 0 \\ x = 1 \end{array}\right. \Rightarrow \left.\begin{array}{l} u = 1 + 2(0) = 1 \\ u = 1 + 2(1) = 3 \end{array}\right\}$$

old limits of integration for variable x.

new limits of integration for variable u.

$$\Rightarrow \int_0^1 \sqrt{1+2x}\, dx = \int_1^3 \tfrac{1}{2}\sqrt{u}\, du$$

from (a)

$$= \frac{1}{3} u^{3/2} \Big|_1^3 = \underbrace{\frac{3^{3/2}}{3} - \frac{1^{3/2}}{3}}_{\text{same as in METHOD 1!}}$$