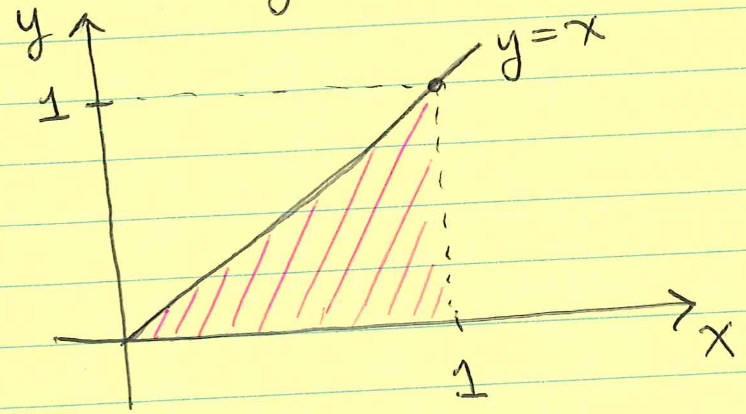
Examples from Friday, July 7th part II.

Definition of Area

Ex: Use the definition of area to find the area under $y=x$ from $x=0$ to $x=1$.



Solution: Since $f(x)=x$ is integrable, we can choose our "starred points" x_k^* however is most convenient.

Let's use right endpoints $(x_k^* = x_k)$.

Then,

$$\text{Riemann Sum} = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$\left(\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} \right.$$

$$\left. x_k^* = x_k = a + k\Delta x = 0 + k\cdot\frac{1}{n} = \frac{k}{n} \right)$$

$$= \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n}$$

$$\left(f(x)=x \Rightarrow f\left(\frac{k}{n}\right) = \frac{k}{n} \right)$$

$$= \sum_{k=1}^{n} \frac{k}{n^2}$$

Now, using the properties of sigma notation, we can pull the constant $\frac{1}{n^2}$ out of the sum:

$$\text{Riemann Sum} = \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2} \sum_{k=1}^{n} k \qquad \left(\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \right)$$

$$= \frac{1}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2n^2}$$

Therefore,

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x$$

$$= \lim_{n \to \infty} \frac{n(n+1)}{2n^2}$$

$$= \lim_{n \to \infty} \frac{n^2+n}{2n^2}$$

Factor out the highest power of n in the numerator and denominator

$$= \lim_{n \to \infty} \frac{\cancel{n^2}(1+1/n)}{\cancel{n^2}(2)}$$

$$= \lim_{n \to \infty} \frac{1+\overset{\to 0}{1/n}}{2}$$

$$\frac{1+0}{h} = \frac{1}{h}$$

$$= \frac{1+0}{\quad} = \frac{1}{\quad}$$

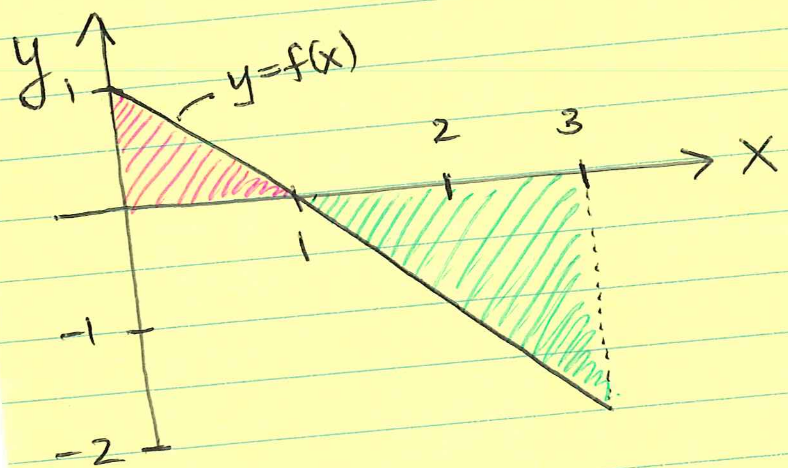
Signed Area

<u>Ex:</u> Given the graph of $f(x)$, calculate the following (signed) area's.

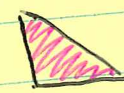
(a) $\int_0^1 f(x)\,dx$

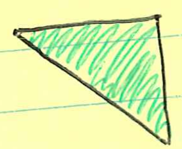
(b) $\int_1^3 f(x)\,dx$

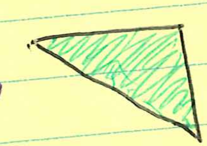
(c) $\int_0^3 f(x)\,dx$



<u>Solution:</u>

(a) $\int_0^1 f(x)\,dx =$ area of  $= \frac{(1)\cdot(1)}{2} = \frac{1}{2}$

(b) $\int_1^3 f(x)\,dx =$ '$-$' area of  $= -\frac{(2)(2)}{2} = -2$

(c) $\int_0^3 f(x)\,dx =$ (area of ) $-$ (area of )

$= -1.5$