NUMERICAL INTEGRATION-

ERROR IN THE APPROXIMATION OF THE AREA (with the MIDPOINT RULE)

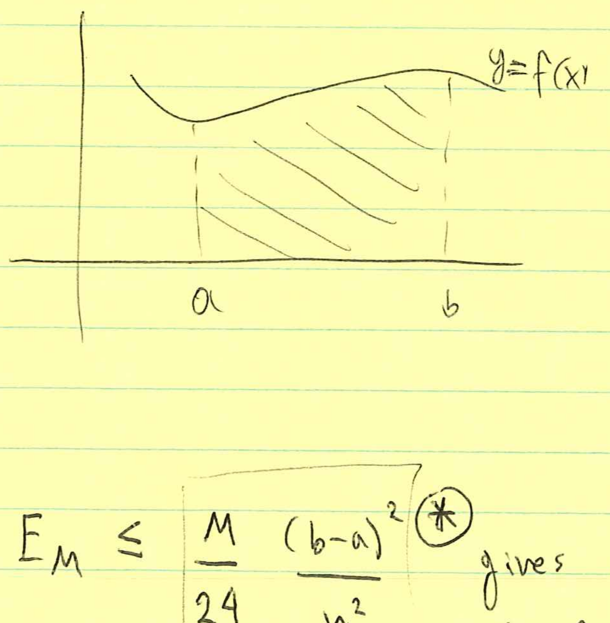
<u>Context</u>: approximating $I = \int_a^b f(x)\,dx$

which (to simplify, for positive $f(x)$) represents the <u>area</u> of the region bounded by
- the graph of f
- the x-axis
- $x = a$
- $x = b$

this procedure, using R_n, M_n, L_n, ... to approximate I is called NUMERICAL INTEGRATION

§ 7.7 in the book

e.g.



by using M_n

approximation with n rectangles based on <u>midpoints</u>

<u>Fact</u>: $E_M \leq \boxed{\dfrac{M}{24} \dfrac{(b-a)^2}{n^2}}$ (*) gives an upper bound on the absolute error $E_M = |I - M_n|$

where M: $|f''(x)| \leq M$ on $[a, b]$

Two categories of problems:

1. I give you $(f(x), a, b, n)$

and I want an estimate on the error $|I - M_n|$
or upper bound

$\rightarrow$ For that, I need to find an M s.t.

$$|f''(x)| \leq M \text{ on } [a,b]$$

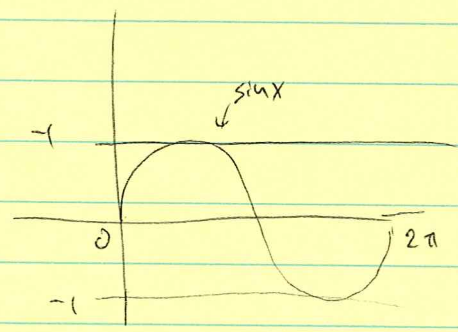
and then plug back in $(*)$ to finish it.

Remark: $|g(x)| \leq M$ on $[a,b]$

means

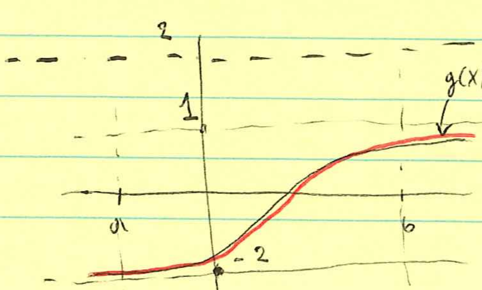
$$-M \leq g(x) \leq M, \ x \text{ in } [a,b]$$

Examples:



$|\sin(x)| \leq 1$

since $-1 \leq \sin(x) \leq 1$



$\rightarrow$ $g(x) \leq 1$ and $g(x) \geq -2$

pick $\Rightarrow$ larger in absolute value

$|g(x)| \leq 2$

To find M (things you can try)

① Inspection!

② Check whether the function is increasing or decreasing (because then, the max/min occur at the endpoints a, b)

③ if ①, ② are not useful, look for critical points and then compare with $\underbrace{g(a), g(b)}_{\text{endpoints}}$ to find max/min on [a, b]

** For such problems, it helps to call $f''(x) := g(x)$, forget about f completely and focus on $g(x)$.

(end of Remark)

② → category of problems.

I give you ($f(x)$, a, b, my tolerance for error) and I want to find an $\boxed{n}$ = # rectangles which produces an acceptable approximation.

$\boxed{\text{Example}}$ (type ② problem):

How large should $\boxed{n}$ be so that the M_n approximation for $I = \int_1^2 \frac{1}{x}\,dx$ is accurate to within 10^{-4}?

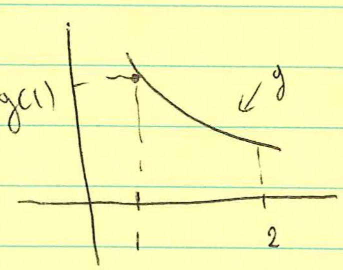
Solution: given $\left(f(x), a, b, \begin{matrix}\text{tolerance} \\ \text{for error}\end{matrix}\right) = \left(\frac{1}{x}, 1, 2, 10^{-4}\right)$ respectively

Recall ⊛: $E_M = |I - M_n| \le \frac{M}{24} \cdot \frac{(b-a)^3}{n^2} \underset{b=2}{\overset{a=1}{=}} \frac{M}{24} \frac{1}{n^2}$

M: $|f''(x)| \le M$ on $[1,2]$

↳ need to find such an M

- $f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2} = -x^{-2} \Rightarrow f''(x) = 2x^{-3} = \frac{2}{x^3}$



decreasing $\Rightarrow$ if I define $g(x) := f''(x)$

$g(x) \le g(1)$ for all x in $[1,2]$

(Remark: You don't need to draw it to notice:

$g'(x) = f^{(3)}(x) = -\frac{6}{x^4} < 0 \Rightarrow g$ decreasing by 1st derivative test) this $\Rightarrow$ means $\quad f''(x) \leq f''(1) = 2 \quad$ on $[1,2]$

and also $\overset{\text{implies}}{\checkmark}$ $|f''(x)| \leq 2$

$\rightsquigarrow$ can take $M = 2$

Go back to ⊛ to solve for n:

Want $E_M < 10^{-4} \Rightarrow \dfrac{2}{24\,n^2} < 10^{-4}$

$\Rightarrow n^2 > \dfrac{1}{12 \cdot 10^{-4}} = \dfrac{10^4}{12}$

$\overset{\text{sqrt}}{\Longrightarrow} n > 28.87$ (**) $\Rightarrow$ I can pick $\boxed{n = 29}$ rectangles to satisfy the tolerance requirement

needs to be an integer

$\boxed{\text{Rmk}}$ (on (**)): I used a calculator for that.

$\rightarrow$ do that for WebWork! (also HW!)

but no calculators will be available on the test, what do I do?

$\rightarrow$ need to get something which has an easy to compute square root.

Could try: $n^2 > \dfrac{10^4}{12} \overset{\text{check}}{>} \dfrac{10^4}{16} = \dfrac{(10^2)^2}{4^2} = \left(\dfrac{10^2}{4}\right)^2 \Rightarrow n > \dfrac{10^2}{4} = 25$

$\Rightarrow$ pick 26 $\rightarrow$ full marks