Examples from lecture — Thursday, July 13th

Partial Fractions:

① $\int_2^4 \frac{1}{(x-1)(x+2)}\,dx$

Step 1: "UNDO" the common denominator:

$$\frac{1}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

GOAL: Find the numbers A and B.

Step 2: Redo! CD on right side and equate numerators:

$$\frac{1}{(x-1)(x+2)} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$\Rightarrow \quad 1 = A(x+2) + B(x-1).$$

Step 3:

Method #1: Equate coefficients of powers of x:

$$\underline{0}\cdot x + 1 = Ax + 2A + Bx - B = (\underline{A+B})x + (2A-B)$$

$$\Rightarrow \quad A+B=0 \quad \text{and} \quad 2A-B=1$$

Solve system of equations for A and B:

$A+B=0 \Rightarrow A=-B$

$2A-B=1 \Rightarrow 2(-B)-B=1$

$\Rightarrow -3B = 1$

$\Rightarrow \boxed{B=-1/3}$ and $\boxed{A=-B=1/3}$

Method #2: Plug in 2 different x-values to generate 2 equations in A and B:

$$1 = A(x+2) + B(x-1).$$

when

$x=1 \rightarrow \quad 1 = A(1+2) + B(1-1)$

$\Rightarrow \quad 1 = A(3) \quad \Rightarrow \boxed{A = 1/3}$

when

$x=-2 \rightarrow \quad 1 = A(-2+2) + B(-2-1)$

$\Rightarrow \quad 1 = -3B$

$\Rightarrow \quad \boxed{B = -1/3}$

4 Step 4: Plug A and B back into decomposition:

$$\frac{1}{(x-1)(x+2)} = \frac{1/3}{(x-1)} + \frac{-1/3}{(x+2)}.$$

Step 5: Integrate!

$$\int_2^4 \frac{1}{(x-1)(x+2)}\,dx = \int_2^4 \frac{1/3}{(x-1)} + \frac{-1/3}{(x+2)}\,dx$$

subs $u=x-1$ subs $w=x+2$.

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| \Bigg|_2^4$$

$$= \frac{1}{3}\ln(3) - \frac{1}{3}\ln(6) - \frac{1}{3}\ln(1) + \frac{1}{3}\ln(4)$$

2 $$\int \frac{1}{\underbrace{(x^2+1)}_{\text{irreducible}}(x-1)}\,dx$$

1 $$\frac{1}{(x^2+1)(x-1)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)}$$

decomposition has different form due to irreducible quadratic!

2 $$\frac{1}{(x^2+1)(x-1)} = \frac{(Ax+B)(x-1)+C(x^2+1)}{(x^2+1)(x-1)}$$

$$\Rightarrow \quad 1 = (Ax+B)(x-1) + C(x^2+1).$$

3 Method #1:

$$1 = Ax^2 + Bx - Ax - B + Cx^2 + C$$

$$\Rightarrow \quad 1 = (A+C)x^2 + (B-A)x + (C-B)$$

$$\Rightarrow \quad A+C=0 \quad \text{and} \quad B-A=0 \quad \text{and} \quad C-B=1.$$

$$A=-C \qquad\qquad B=A \; (=-C)$$

$$C-B=1 \Rightarrow \quad C-(-C)=1$$

$$2C=1$$

$$\boxed{C=1/2}. \Rightarrow \boxed{A=-C=-1/2}$$

$$\Rightarrow \boxed{B=A=-1/2}$$

Method #2:

$$1 = (Ax+B)(x-1) + C(x^2+1).$$

when $x=1 \rightarrow 1 = (A+B)(1-1) + C(1+1)$

$$1 = 2C \Rightarrow \boxed{C = 1/2}$$

when $x=0 \rightarrow 1 = (0+B)(0-1) + C(0+1).$

Death of purple pen. :(

$$= -B + C$$

$$1 = -B + 1/2 \Rightarrow \boxed{B = 1/2 - 1 = -1/2}$$

when $x=-1 \rightarrow 1 = (-A+B)(-1-1) + C((-1)^2+1)$

$$1 = -2(B-A) + 2C$$

$$1 = -2\left(-\tfrac{1}{2} - A\right) + 2\left(\tfrac{1}{2}\right)$$

$$1 = 1 + 2A + 1 \Rightarrow 2A = -1$$

$$\boxed{A = -1/2}$$

* Note: You don't have to choose $x = 1, 0, -1$. Any 3 different x-values will work!!!

4 $$\frac{1}{(x^2+1)(x-1)} = \frac{-\frac{1}{2}x - \frac{1}{2}}{(x^2+1)} + \frac{1/2}{x-1}$$

5 $$\int \frac{1}{(x^2+1)(x-1)}\,dx = \int \frac{-\frac{1}{2}x - \frac{1}{2}}{(x^2+1)} + \frac{1/2}{(x-1)}\,dx$$