# Lecture Examples — Frrday, July 27th

## Power Series - Interval and Radius of Convergence

1. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \sum_{n=1}^{\infty} x^{n-1}$ Geometric Series with $a=1$, $r=x$.

From geometric series, we know that this converges when $|x|<1$ ($|r|<1$) and diverges when $|x| \geq 1$ ($|r| \geq 1$).

So, $R=1$, and the interval of convergence is

$(-1,1)$ ← all x-values for which $|x|<1$.

2. For which values of $x$ does $\sum_{n=1}^{\infty} \frac{(x-7)^n}{n}$ converge?

Use Ratio Test (to test for absolute convergence):

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(\frac{(x-7)^{n+1}}{n+1}\right)}{\left(\frac{(x-7)^n}{n}\right)}\right| = \left|\frac{(x-7)^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{(x-7)^n}}\right| = |x-7| \cdot \frac{n}{n+1}$$

Therefore,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \overbrace{|x-7|}^{\text{doesn't depend on } n} \cdot \frac{n}{n+1}$$

$$= |x-7| \cdot \lim_{n\to\infty} \frac{n}{n+1}$$

$$= |x-7| \cdot \lim_{n\to\infty} \frac{\cancel{n} \cdot (1)}{\cancel{n}(1+1/n)} = |x-7|$$

So, from the Ratio Test, we know that the series converges (absolutely) if $|x-7| < 1$

and diverges if $|x-7| > 1$.

Note: The Ratio Test is inconclusive when $|x-7| = 1$.

So, the radius of convergence is $R = 1$.

To find the interval of convergence, we first expand the inequality $|x-7| < 1$:

$$-1 < x-7 < 1.$$
$$\quad +7 \qquad +7 \quad +7$$
$$\Leftrightarrow \quad 6 < x < 8$$

} Ratio Test $\Rightarrow$ converges for all $x$ such that $6 < x < 8$.

$\Rightarrow$ The endpoints of the interval of convergence are $x=6$ and $x=8$. These are the points where the Ratio Test is inconclusive (i.e. $|x-7|=1$), so we need to test them separately using other tests for convergence.

Plug the endpoints into the series:

When $x=6$: $\displaystyle\sum_{n=1}^{\infty} \frac{(6-7)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ ] Alternating Series with $b_n = \frac{1}{n}$.

Now, $1/n$ is always positive and is a decreasing function. Furthermore

$$\lim_{n\to\infty} 1/n = 0,$$

so by the Alternating series Test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Since the series converges at $x=6$, 6 should be <u>included</u> in the interval of convergence.

When $x=8$: $\sum_{n=1}^{\infty} \frac{(8-7)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ ] p-series with $p=1 \Rightarrow$ <u>divergent</u>.

Since the series diverges at $x=8$, 8 should be <u>excluded</u> from the interval of convergence.

Therefore, the interval of convergence is $[6,8)$.

③ Find the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{n(x+4)^n}{2^{n+1}}$

<u>Ratio Test:</u>

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(\frac{(n+1)(x+4)^{n+1}}{2^{n+2}}\right)}{\left(\frac{n(x+4)^n}{2^{n+1}}\right)}\right| = \left|\frac{(n+1)(x+4)^{\cancel{n+1}}}{2^{\cancel{n+2}}} \cdot \frac{\cancel{2^{n+1}}}{n\,\cancel{(x+4)^n}}\right|$$

$$= |x+4| \cdot \frac{(n+1)}{2n}$$

So, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x+4| \cdot \left(\frac{n+1}{2n}\right)$

$$= |x+4| \cdot \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}(2)} = \frac{|x+4|}{2}$$

So, by Ratio Test, the series converges (absolutely) when

$$\frac{|x+4|}{2} < 1 \iff |x+4| < 2.$$

(and diverges when $\frac{|x+4|}{2} > 1 \iff |x+4| > 2$).

$\therefore$ the radius of convergence is $R=2$.

Expand the inequality:

Now $|x+4| < 2$ is the same as

$$\begin{array}{ccccc} -2 & < & x+4 & < & 2. \\ -4 & & -4 & & -4 \\ -6 & < & x & < & -2. \end{array}$$

(so endpoints are $x=-6,-2$).

Plug endpoints into series:

When $x=-6$: $\displaystyle\sum_{n=0}^{\infty} \frac{n(-6+4)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{n(-2)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{n(-1)^n \cdot \cancel{2^n}}{\cancel{2^{n+1}}}$

$$= \sum_{n=0}^{\infty} \frac{n(-1)^n}{2}.$$

So, since $\displaystyle\lim_{n\to\infty} \frac{n(-1)^n}{2}$ DNE (and $\therefore \neq 0$),

from Divergence Test we must have that the

$\displaystyle\sum_{n=0}^{\infty} \frac{n(-1)^n}{2}$ diverges.

When $n=-2$: $\sum_{n=0}^{\infty} \frac{n(-2+4)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{n \cdot \cancel{2^n}}{\cancel{2^{n+1}}} = \sum_{n=0}^{\infty} \frac{n}{2}$.

So, since $\lim_{n\to\infty} \frac{n}{2} = \infty \neq 0$, again, it follows from the Divergence Test that $\sum_{n=0}^{\infty} \frac{n}{2}$ diverges.

Thus, the interval of convergence is $(-6,-2)$.

## Finding Power Series representations of functions.

① $\frac{1}{1-x}$ ] = sum of geometric series with $a=1$ and $r=x$.

So, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\left(\text{or } \sum_{n=1}^{\infty} x^{n-1} \text{ - either is fine}\right)$

② $\frac{1}{2+x^2}$. ] Need to make this look like the sum of a geometric series $\left(\frac{a}{1-r}\right)$ for some $a$ and $r$.

Algebraic Trickery:

$$\frac{1}{2+x^2} = \frac{1}{2(1+x^2/2)} = \frac{1}{2}\cdot\frac{1}{1+x^2/2} = \frac{1}{2}\cdot\boxed{\underbrace{\frac{1}{1-(-x^2/2)}}_{\frac{a}{1-r} \text{ with } a=1,\ r=\frac{-x^2}{2}.}}$$

replace with geometric series.

$$\text{So } \frac{1}{2+x^2} = \frac{1}{2}\cdot\boxed{\sum_{n=0}^{\infty} \overset{a}{(1)}\cdot\overset{r}{\left(\frac{-x^2}{2}\right)}^n} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n\cdot(x^2)^n}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{n+1}}$$

3. $\dfrac{x^3}{2+x^2}$.

Since $x^3$ is a power of $x$, we can write this as.

$$\frac{x^3}{2+x^2} = x^3 \cdot \frac{1}{2+x^2}.$$

then replace $\frac{1}{2+x^2}$ with it's power series representation (from 2), and then multiply the $x^3$ through the series:

$$\frac{x^3}{2+x^2} = x^3 \cdot \boxed{\frac{1}{2+x^2}} = x^3 \cdot \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{n+1}}} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2^{n+1}}}_{\text{multiply the } x^3 \text{ through the series.}}$$

~~replace~~ with series from 2

4. $\dfrac{1}{(1-x)^2}$. Trick: Notice that $\frac{1}{(1-x)^2}$ is the derivative of $\frac{1}{1-x}$ (check: $\left(\frac{1}{1-x}\right)' = \frac{\cancel{-}1}{(1-x)^2}\cdot(\cancel{-}1)$ ✓)

i.e. $\dfrac{d}{dx} \underbrace{\dfrac{1}{1-x}}_{} = \dfrac{1}{(1-x)^2}$.

this is the sum of a geometric series with $a=1, r=x$.

So, we have that

replace with geometric series

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

$$= \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + \cdots\right).$$

(Differentiate term-by-term)

$$= \underline{0} + 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

$$= \sum_{n=\underline{1}}^{\infty} n x^{n-1}$$

sum starts at 1, since the derivative of the $n=0$ term was 0.

⑤ $\ln(1+x)$ ] Trick: Notice that $\ln(1+x)$ is equal to the integral $\int_0^x \frac{1}{1+t}\,dt$. Alternatively, it is <u>the</u> antiderivative of $\frac{1}{1+x}$ with the choice of constant $C=0$.

Futhermore, $\frac{1}{1+t} = \frac{1}{1-(-t)}$ is the sum of a geometric series with $a=1$, $r=-t$.

So, $$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \int_0^x \frac{1}{1-(-t)}\,dt$$

$$= \int_0^x \sum_{n=0}^{\infty} (-t)^n \cdot dt$$

$$= \int_0^x \left( \sum_{n=0}^{\infty} (-1)^n t^n \right) dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n+1} t^{n+1}$$

Integrate term-by-term. Sum still starts at $n=0$ because integrating didn't make $n=0$ term ~~go~~ vanish like differentiating did.