$\boxed{\text{Short Answer}}$

$\boxed{1.}$ (a) $I = \int u \cdot (u+4) \cdot (2u+1)\, du$

$\downarrow$

polynomial of degree 3,
just expand and split

$$I = \int (u^2+4u)(2u+1)\,du = \int (2u^3 + \underbrace{u^2 + 8u^2}_{\text{or } 9u^2} + 4u)\,du$$

$$= \cancel{2}\frac{u^4}{\cancel{4}2} + \underbrace{\frac{u^3}{3} + \frac{8u^3}{3}}_{\frac{3\cancel{9}u^3}{\cancel{3}}} + 2\cancel{4}\frac{u^2}{\cancel{2}} + C$$

$$= \frac{u^4}{2} + 3u^3 + 2u^2 + C$$

(b) $I = \int_3^4 \frac{1}{y^2-4y-12}\,dy$ $\longrightarrow$ try to factor

quadratic formula: $y_{1,2} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(-12)}}{2}$

$$= \frac{4 \pm \sqrt{\underbrace{16+48}_{64}}}{2} = \frac{4 \pm 8}{2} = \begin{cases} 6 \\ -2 \end{cases}$$

$$\frac{1}{y^2-4y-12} = \frac{1}{(y-6)(y+2)} \overset{\text{partial}}{\underset{\text{fraction}}{=}} \frac{A}{y-6} + \frac{B}{y+2}$$

$\Rightarrow y(A+B) + (2A - 6B) = 1 \Rightarrow \left.\begin{array}{l} A+B=0 \\ 2A-6B=1 \end{array}\right\} \Rightarrow \begin{array}{l} -A = B \longrightarrow B = -\frac{1}{8} \\ 8A = 1 \Rightarrow A = \frac{1}{8} \end{array}$

$$I = \int_3^4 \frac{1}{y^2-4y-12}\,dy = \int_3^4 \left(\frac{1/8}{y-6} + \frac{(-1/8)}{y+2} \right) dy$$

$$= \frac{1}{8}\int_3^4 \frac{1}{y-6}\,dy - \frac{1}{8}\int_3^4 \frac{1}{y+2}\,dy$$

$$= \frac{1}{8} \ln|y-6| \Big|_3^4 - \frac{1}{8} \ln|y+2| \Big|_3^4$$

$$= \frac{1}{8}\left\{ \left[\ln 2 - \ln 3 \right] - \left[\ln 6 - \ln 5 \right] \right\}$$

(c) $$I = \int_0^{\frac{a\sqrt{2}}{2}} \frac{1}{\sqrt{a^2-x^2}}\,dx$$

$x = a \sin\theta \rightarrow dx = a \cos\theta \, d\theta$

- $0 = a \sin\theta \rightarrow \boxed{\theta = 0}$
- $\frac{\cancel{a}\sqrt{2}}{2} = \cancel{a} \sin\theta \rightarrow \boxed{\theta = \frac{\pi}{4}}$

$$I = \int_0^{\pi/4} \frac{\cancel{a \cos\theta}}{\cancel{a \cos\theta}}\,d\theta = \theta \Big|_0^{\pi/4} = \frac{\pi}{4}$$

$$(d)\quad I = \int \cos^2 x \, \tan^3 x \, dx = \int \cancel{\cos^2 x} \left(\frac{\sin^{3} x}{\cos^3 x} \right) dx \quad \leftarrow \text{odd}$$

$$= \int \frac{\sin^2 x}{\cos x} \boxed{\sin x \, dx} = \int \frac{(1-\cos^2 x)}{\cos x} \sin x \, dx$$

set $u = \cos x \rightarrow du = -\sin x \, dx$

$$I = -\int \frac{1-u^2}{u} du = \int \frac{u^2-1}{u} du = \int \left(\frac{u^2}{u} - \frac{1}{u} \right) du$$

$$= \int \left(u - \frac{1}{u} \right) du = \frac{u^2}{2} - \ln|u| + C$$

$$= \frac{\cos^2 x}{2} - \ln|\cos x| + C$$

$$(e)\quad I = \int \cos^4 \theta \, d\theta = \int (\cos^2\theta)^2 \, d\theta = \int \left(\frac{1+\cos(2\theta)}{2} \right)^2 d\theta$$

$$= \frac{1}{4} \int \left(1 + 2\cos(2\theta) + \cos^2(2\theta) \right) d\theta$$

$$= \frac{1}{4} \int \left(1 + 2\cos(2\theta) + \frac{1+\cos(4\theta)}{2} \right) d\theta$$

$$= \frac{1}{4} \left[\theta + \cancel{2} \frac{\sin(2\theta)}{\cancel{2}} + \frac{\theta}{2} + \frac{1}{2} \frac{\sin(4\theta)}{4} \right] + C$$

(f) $I = \int \frac{x^2+2}{x+2}\,dx$

long division gives $(x^2+2) = (x+2)(x-2)+6$

$$I = \int \frac{(x+2)(x-2)+6}{x+2}\,dx =$$

$$= \int (x-2) + \frac{6}{x+2}\,dx =$$

$$= x^2 - 2x + 6\ln|x+2| + C$$

(g) $I = \int \sin x \left[\cos(\cos x)\right] dx$, $u = \cos x$

$du = -\sin x\,dx$

$$I = -\int \cos u\,du = -\sin u + C = -\sin(\cos x) + C$$

(h) $I = \int \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx$; set $u = \frac{\pi}{x}$

$\rightsquigarrow du = -\pi \boxed{\frac{1}{x^2}\,dx}$

$$\rightarrow I = \int \frac{\cos u}{-\pi}\,du = -\frac{1}{\pi}\sin u + C =$$

$$= -\frac{1}{\pi}\sin\left(\frac{\pi}{x}\right) + C$$

(i) $I = \int t \cos(t^2)\,dt = \int \cos(t^2)\, t\,dt$

$u = t^2 \rightarrow du = 2\,\boxed{t\,dt}$

$I = \frac{1}{2}\int \cos u\, du = \frac{1}{2} \sin u + C = \frac{1}{2} \sin(t^2) + C$

(j) $I = \int x^{3/2} \ln x\, dx = \int \underbrace{\ln x}_{u}\ \underbrace{x^{3/2}\,dx}_{dv}$

(because the derivative will make it easier)

$u = \ln x \qquad dv = x^{3/2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{2}{5}\, x^{\frac{5}{2}}$

$I = \overbrace{\frac{2}{5} \ln x \cdot x^{5/2}}^{u\ v} - \overbrace{\int \frac{2}{5}\, x^{\frac{5}{2}}\, \frac{1}{x}\, dx}^{\int v\,du}$ $\left(\frac{x^{\frac{5}{2}}}{x} = x^{\frac{5}{2}} \cdot x^{-1} = x^{\frac{3}{2}} \right)$

$= \frac{2}{5}\, x^{5/2} \ln x - \frac{2}{5} \int x^{\frac{3}{2}}\, dx =$

$= \frac{2}{5}\, x^{5/2} \ln x - \frac{4}{25}\, x^{\frac{5}{2}} + C$

(k) $I = \int \underbrace{\ln x}_{u} \; \underbrace{\overset{1\,dx}{\overset{\shortparallel}{dx}}}_{dv}$; $\quad u = \ln x \quad du = \frac{1}{x}\,dx \quad \Big| \quad dv = dx \quad v = x$

$$I = \overset{u\cdot v}{x \cdot \ln x} - \overset{\int v\,du}{\int \cancel{x}\,\frac{1}{\cancel{x}}\,dx} = x\ln x - x + C$$

(2) By the F.T.C (Part 2) :

$$\int_1^4 f'(x)\,dx = f(4) - f(1)$$

$$\Rightarrow \quad f(4) = \int_1^4 f'(x)\,dx + f(1)$$

$$= 17 + 12 = 29$$

(3) $I = \int_1^2 7\, f(3x-1)\,dx$

$$u = 3x - 1 \Rightarrow du = 3\,dx \Rightarrow dx = \frac{1}{3}\,du$$

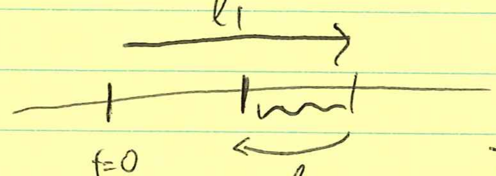
- $x = 1 \rightarrow u = 2$
- $x = 2 \rightarrow u = 5$

$$I = \frac{7}{3}\int_2^5 f(u)\,du = \frac{7}{3}\int_2^5 f(x)\,dx = \frac{7}{\cancel{3}}\left(\overset{-10}{\cancel{-30}}\right) = -70$$

dummy, name doesn't matter

$\rightarrow$ this is NOT going back to x; just renaming

④ Total distance! Not displacement



$T.d. = l_1 + l_2$

$(\text{not } l_1 - l_2)$

$$T.D. = \int_0^4 |t \cdot (t-2)| \, dt$$

need to remove!

But

t		0		2		4
t		0	+		+	
$t-2$			−	0	+	
$t(t-2)$		0	−	0	+	

$$\rightarrow |t(t-2)| = \begin{cases} -t(t-2), & 0 \le t < 2 \\ t(t-2), & 2 \le t \le 4 \end{cases}$$

$$T.D. = \int_0^2 -\underbrace{t(t-2)}_{t^2-2t} \, dt + \int_2^4 t(t-2) \, dt$$

$$= -\frac{t^3}{3} + 2t \Big|_0^2 + \frac{t^3}{3} - 2t \Big|_2^4 =$$

$$= \left(-\frac{8}{3} + 4\right) - 0 + \left(\frac{4^3}{3} - 8\right) - \left(\frac{8}{3} - 4\right) =$$

$$= \frac{4^3}{3} - 8 - 2 \cdot \left(\frac{8}{3} - 4\right) \text{ m}$$

5 Separable D.E.:

$$(x-2)\frac{dy}{dx} = \frac{1}{2y(x^2+1)}, \quad y(0)=1$$

$$\Rightarrow \quad 2y\,dy = \frac{1}{(x-2)(x^2+1)}\,dx$$

$$\Rightarrow \quad 2\int y\,dy = \int \frac{1}{(x-2)(x^2+1)}\,dx \quad \rightarrow y^2$$

$$\frac{1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow \quad x^2(A+B) + x(-2B+C) + (A-2C) = 1$$

$$\Rightarrow \quad \begin{array}{lll} A+B=0 & \Rightarrow & -A = B \\ C-2B=0 & \Rightarrow & C=2B \\ A-2C=1 & \Rightarrow & A=2C+1 \end{array}$$

$A = 4B+1$

$A = -4A+1$

$5A = 1$

$$\boxed{C = -\frac{2}{5}} \leftarrow \boxed{B = -\frac{1}{5}} \leftarrow \boxed{A = \frac{1}{5}}$$

$$\Rightarrow y^2(x) = \int \frac{1}{(x-2)(x^2+1)}\,dx = \frac{1}{5}\int \frac{1}{x-2}\,dx - \frac{1}{5}\int \frac{x+2}{x^2+1}\,dx$$

$$= \frac{1}{5}\left[\ln|x-2| - \int \frac{x}{x^2+1}\,dx - 2\int \frac{1}{x^2+1}\,dx \right]$$

$\downarrow$ $u = x^2+1$

$$= \frac{1}{5}\left[\ln|x-2| - \frac{1}{2}\ln|x^2+1| - 2\arctan x \right] + C$$

$\ln|x^2+1| = \ln(x^2+1)$ $\quad\downarrow y(0)=1$

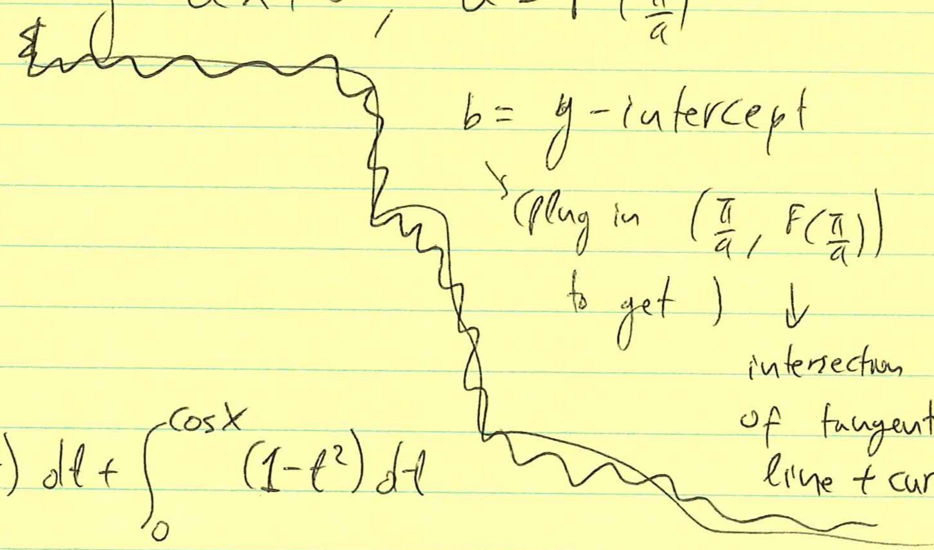
$$\Rightarrow y^2(x) = \frac{1}{5}\ln|x-2| - \frac{1}{2}\ln|x^2+1| - 2\arctan x + \left(1 - \frac{1}{2}\ln 2\right)$$

$\left(1 = \frac{1}{5}\left(0 - \frac{1}{2}\ln 2 + 0\right) + C,\ C = 1 - \frac{1}{2}\ln 2\right)$

$\Rightarrow y = \cancel{\pm}$ ....

$\boxed{\text{Long Answer}}$

① tangent line to $F(x)$ at $x = \frac{\pi}{a}$: $y = ax + b$, $a = F'\left(\frac{\pi}{a}\right)$



b = y-intercept (plug in $\left(\frac{\pi}{a}, F\left(\frac{\pi}{a}\right)\right)$ to get) $\downarrow$ intersection of tangent line + curve

- $a = F'\left(\frac{\pi}{a}\right)$, first

$F'(x) = ?$

$$F(x) = \int_{\sin x}^{0} (1-t^2)\,dt + \int_0^{\cos x} (1-t^2)\,dt$$

$$= -\int_0^{\sin x} (1-t^2)\,dt + \int_0^{\cos x} (1-t^2)\,dt \quad \overset{\text{F.T.C. 1}}{\Longrightarrow}$$

$$\Rightarrow \quad F'(x) = -\left(1-\sin^2 x\right)\overset{(\sin x)'}{\overset{\downarrow}{\cos x}} + \left(1-\cos^2 x\right)\overset{(\cos x)'}{\overset{\downarrow}{(-\sin x)}}$$

$$= -\cos^3 x - \sin^3 x$$

at $x = \frac{\pi}{4}$: $\quad \cos\frac{\pi}{4} = \sin\frac{\pi}{2} = \frac{\sqrt{2}}{2}$

$$\Rightarrow \quad \underset{\underset{a}{\parallel}}{F'\left(\frac{\pi}{4}\right)} = -2\left(\frac{\sqrt{2}}{2}\right)^3 = -\frac{\left(\sqrt{2}\right)^3}{4}$$

To find b:
$\downarrow$
plug in $\left(\frac{\pi}{4}, F\left(\frac{\pi}{4}\right)\right)$

$$F\left(\frac{\pi}{4}\right) = \int_{\sin\frac{\pi}{4}}^{\cos\frac{\pi}{4}} (1-t^2)\,dt = \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} (1-t^2)\,dt = 0$$

$$\Rightarrow \quad y = ax + b$$

$$\xrightarrow{x=\frac{\pi}{4}} \quad 0 = -\frac{\left(\sqrt{2}\right)^3}{4}\,\frac{\pi}{4} + b \Rightarrow b = \frac{\pi}{16}\left(\sqrt{2}\right)^3$$

$\Rightarrow$ tangent line: $\quad y = -\frac{\left(\sqrt{2}\right)^3}{4}x + \frac{\pi}{16}\left(\sqrt{2}\right)^3$

② Will use F.T.C. $\sqrt{1}$, but first

$$F(x) = \int_0^x x e^t dt = \boxed{x} \int_0^x e^t dt$$

because x is indepentent of t

$\Leftrightarrow$ is a constant for the integral

now use product rule: $(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$

with $f(x) = x \Rightarrow f'(x) = 1$

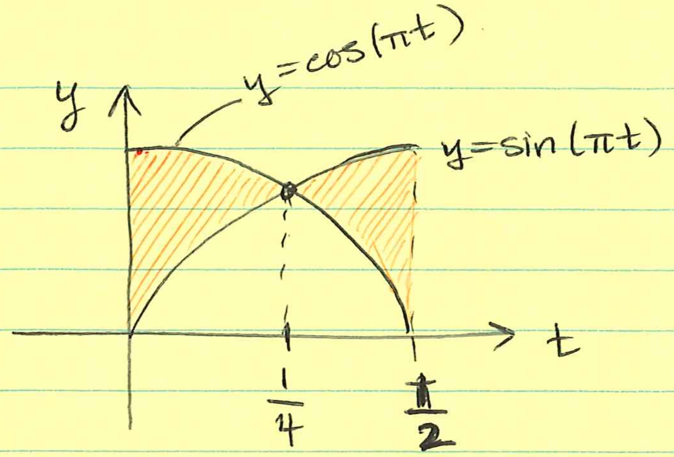
$g(x) = \int_0^x e^t dt \Rightarrow g'(x) = e^x$

$$\Rightarrow F'(x) = \underset{\underset{f'}{\uparrow}}{1} \cdot \boxed{\int_0^x e^t dt} + x \cdot e^x$$

Now $\int_0^x e^t dt = e^t \Big|_0^x = e^x - 1$

$$\Rightarrow F'(x) = 1 \cdot (e^x - 1) + x e^x = e^x - 1 + x e^x$$ ▣

3.



Area between curves $= \int_0^{1/4} [\cos(\pi t) - \sin(\pi t)]\,dt + \int_{1/4}^{1/2} [\sin(\pi t) - \cos(\pi t)]\,dt$

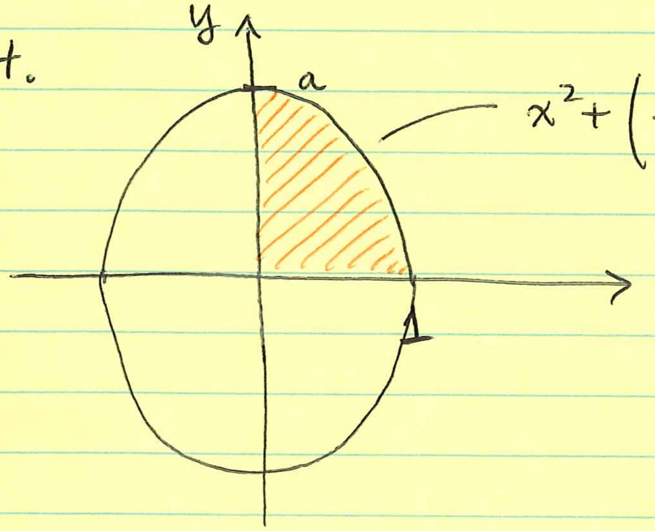
$$= \left(\frac{1}{\pi}\sin(\pi t) + \frac{1}{\pi}\cos(\pi t) \Big|_0^{1/4} \right) + \left(\frac{-1}{\pi}\cos(\pi t) - \frac{1}{\pi}\sin(\pi t) \Big|_{1/4}^{1/2} \right)$$

$$= \left[\frac{1}{\pi}\overbrace{\sin\left(\frac{\pi}{4}\right)}^{\sqrt{2}/2} + \frac{1}{\pi}\overbrace{\cos\left(\frac{\pi}{4}\right)}^{\sqrt{2}/2} - \frac{1}{\pi}\overbrace{\sin(0)}^{0} - \frac{1}{\pi}\overbrace{\cos(0)}^{1} \right] + \left[\frac{-1}{\pi}\overbrace{\cos\left(\frac{\pi}{2}\right)}^{0} - \frac{1}{\pi}\overbrace{\sin\left(\frac{\pi}{2}\right)}^{1} + \frac{1}{\pi}\overbrace{\cos\left(\frac{\pi}{4}\right)}^{\sqrt{2}/2} + \frac{1}{\pi}\overbrace{\sin\left(\frac{\pi}{4}\right)}^{\sqrt{2}/2} \right]$$

$$= \left(\frac{1}{\pi}\cdot\frac{\sqrt{2}}{2} + \frac{1}{\pi}\cdot\frac{\sqrt{2}}{2} - \frac{1}{\pi} \right) + \left(\frac{-1}{\pi} + \frac{1}{\pi}\frac{\sqrt{2}}{2} + \frac{1}{\pi}\frac{\sqrt{2}}{2} \right)$$

$$= \frac{2\sqrt{2} - 2}{\pi}.$$

4.

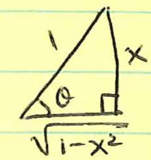


$x^2 + \left(\frac{y}{a}\right)^2 = 1$

upper half ellipse: $\frac{y}{a} = +\sqrt{1-x^2}$

$y = a\sqrt{1-x^2}$

Area of quarter ellipse $= \int_0^1 a\sqrt{1-x^2}\,dx$



$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$\sqrt{1-x^2} = \cos\theta$

when $x = 0$, $0 = \sin\theta \Rightarrow \theta = 0$

when $x = 1$, $1 = \sin\theta \Rightarrow \theta = \pi/2$.

$$= \int_0^{\pi/2} a\cos\theta \cdot \cos\theta\,d\theta = \int_0^{\pi/2} a\cos^2\theta\,d\theta$$

$$= a\int_0^{\pi/2}\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta$$ (Double Angle Formula)

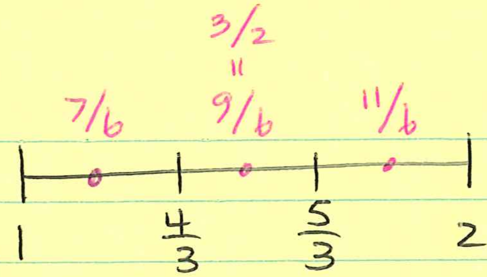
$$= a\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta \Big|_0^{\pi/2}\right)$$

$$= a\left(\frac{\pi}{4} + \cancelto{0}{\frac{1}{4}\sin\left(\frac{2\pi}{2}\right)} - 0 - \cancelto{0}{\frac{1}{4}\sin(0)}\right).$$

$$= a \cdot \pi/4$$

$\therefore$ area of full ellipse is $4 \cdot (a\pi/4) = \pi a$ //

5(a)



$a=1,\ b=2,\ n=3,\ \Delta x = \frac{2-1}{3} = \frac{1}{3}.$

$$\therefore \int_1^2 e^{1/t}\,dt \approx \left(e^{6/7}\cdot\frac{1}{3}\right) + \left(e^{2/3}\cdot\frac{1}{3}\right) + \left(e^{6/11}\cdot\frac{1}{3}\right) \quad //$$

(b) $n = 10 \Rightarrow \Delta x = \frac{2-1}{10} = \frac{1}{10}$

$$x_k^* = a + (k-0.5)\Delta x = 1 + \frac{(k-0.5)}{10} = \frac{9.5+k}{10}$$

$$\therefore \int_1^2 e^{1/t}\,dt \approx \sum_{k=1}^{10} e^{\left(\frac{10}{9.5+k}\right)} \cdot \frac{1}{10} \quad //$$

$= 1/10$

(c) $f(t) = e^{1/t}$

$f'(t) = e^{1/t} \cdot \frac{-1}{t^2}$

$$f''(t) = \left(e^{1/t}\cdot\frac{\cancel{-}1}{t^2}\right)\cdot\left(\frac{\cancel{-}1}{t^2}\right) + e^{1/t}\cdot\frac{2}{t^3}$$

$$= \frac{e^{1/t}(1+2t)}{t^4}$$

$t \in [1,2] \Rightarrow$ all terms are already positive.
$\Rightarrow |f''(t)| = f''(t)$

Now, $e^{1/t}$ is biggest when $\frac{1}{t}$ is biggest $\Leftrightarrow$ t is smallest $\Leftrightarrow$ when $t=1$. $\therefore e^{1/t} \le e^1$

Also, $(1+2t)$ is biggest when t is biggest $\Leftrightarrow$ when $t=2$

$\Rightarrow 1+2t \le 1+2\cdot 2 = 5.$

And, $\frac{1}{t^4}$ is biggest when t is smallest $\Leftrightarrow$ when $t=1 \Rightarrow \frac{1}{t^4} \leq 1$.

$$\therefore \ |f''(t)| = \frac{e^{1/t}(1+2t)}{t^4}$$

$$= \underbrace{e^{1/t}}_{\leq e} \cdot \underbrace{(1+2t)}_{\leq 5} \cdot \underbrace{\frac{1}{t^4}}_{\leq 1} \leq 5e$$

Therefore choose $M = 5e$. Then

$$E_M \leq \frac{M(b-a)^3}{24n^2} = \frac{5e(2-1)^3}{24 \cdot 10^2} = \frac{5e}{2400}.$$

6(a) $\quad \dfrac{dP}{dt} = -27\ln\left(\dfrac{P}{120}\right)P \qquad P(0)=60.$

$$\Rightarrow \int \frac{1}{P\cdot\ln\left(\frac{P}{120}\right)}\,dP = \int -27\,dt = -27t + C$$

Let $u=\ln\left(\frac{P}{120}\right)$

$$du = \frac{1}{(P/120)}\cdot\frac{1}{120}\,dP$$

$$= \frac{1}{P}\,dP$$

$$\Rightarrow \int \frac{1}{P\ln\left(\frac{P}{120}\right)}\,dP = \int \frac{1}{u}\,du = \ln|u| = \ln|\ln(P/120)|$$

$\therefore$ We have.

$$\ln|\ln(P/120)| = -27t + C$$

Plug in initial condition:

$$\ln|\ln(60/120)| = -\cancel{27\cdot 0}^{\,0} + C$$

$$\ln|\ln(\tfrac{1}{2})| = C$$

$$\ln|-\ln(2)| = C \qquad \left(\ln(a^{-1}) = -\ln(a)\right)$$

$$\ln(\ln(2)) = C$$

So

$$\ln|\ln(P/120)| = -27t + \ln(\ln(2)).$$

Since initially $\overset{60}{\overset{\shortparallel}{P}} < 120 \Rightarrow \ln(P/120) < 0$, for t close to zero, we'd expect $P(t) \approx 60 \Rightarrow P(t) < 120$

$$\Rightarrow \ln\left(P(t)/120\right) < 0$$

$$\Rightarrow \ln\left|\ln(P/120)\right| = \ln\left(-\ln(P/120)\right).$$

So

$$\ln\left(-\ln(P/120)\right) = -27t + \ln(\ln(2)).$$

$$-\ln(P/120) = e^{-27t + \ln(\ln(2))} = \ln(2)\, e^{-27t}$$

$$\Rightarrow \ln(P/120) = -\ln(2)\, e^{-27t}$$

$$\Rightarrow \frac{P}{120} = e^{-\ln(2) e^{-27t}}$$

$$\Rightarrow P = 120\, e^{-\ln(2) e^{-27t}}$$

(b) In the long run,

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} 120\, e^{-\ln(2) e^{-27t}} = 120\, e^{-\ln(2)\cdot 0}$$

$$= 120\, e^{0}$$

$$= 120$$

the population will tend toward 120 phytoplankton.