u-$\boxed{\text{Substitution}}$ (look for $f(g(x))\,\boxed{g'(x)}$)

① $I = \int \cos^4\theta \, \underbrace{\sin\theta \, d\theta}$

if $u = \cos\theta$

$du = -\sin\theta \, d\theta$

$\xrightarrow[u=\cos\theta]{\text{pick}} \quad -\int u^4 \, du = -\frac{u^5}{5} + C \underset{\text{back to } \theta}{\overset{\text{go}}{=}} -\frac{\cos^5\theta}{5} + C$

② $I = \int \frac{1}{1+3t} \, dt$, $u = 1+3t \Rightarrow du = 3dt$

$I = \frac{1}{3} \int \frac{1}{u} du = \frac{1}{3} \ln|u| + C = \frac{1}{3} \ln|1+3t| + C$

③ $I = \int \frac{x}{x^2+1} \, dx = \int \frac{1}{x^2+1} \, x\,dx$

$u = x^2+1 \to du = 2x\,dx \to \frac{1}{2} du = x\,dx$

$I = \frac{1}{2}\int \frac{1}{u} \, du = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln(\underbrace{x^2+1}_{>0}) + C$ (don't need $|\cdot|$)

④ $I = \int \underbrace{x^2 e^{x^3} dx}$, $u = x^3 \to du = 3\,\boxed{x^2\,dx}$

$\to I = \frac{1}{3}\int e^u \, du = \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3} + C$

⑤ $I = \int t\sqrt{1-t^2}\,dt$

$u = 1-t^2 \Rightarrow du = -2\boxed{t\,dt}$

$I = -\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{2}\int u^{1/2}\,du = -\frac{1}{\cancel{2}}\frac{\cancel{2}}{3}u^{3/2} + C$

$= -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(1-t^2)^{3/2} + C$

⑥ $I = \int s^2(2+s^3)^4\,ds$

set $u = 2+s^3$

$du = 3\,\boxed{s^2\,ds}$

Moral of the story:

I am not afraid of high powers :)

$I = \frac{1}{3}\int u^4\,du = \frac{u^5}{15} + C = \frac{(2+s^3)^5}{15} + C$

⑦ $\boxed{\text{Tricky}}$ $I = \int_1^{\sqrt{3}} x^3\sqrt{x^2+1}\,dx$

I will set $u = x^2+1$ although I have $X^3\,dx$ instead of (what I'd have liked better) $X\,dx$

Notice however, $\quad x^3 = x^2 \boxed{x\,dx}$

$x^2 \xrightarrow{u=x^2+1} x^2 = u-1$, $\quad \boxed{x\,dx} \to$ nice

! hm! There is hope!

$x^2 = u-1$

$u = x^2+1$

$du = 2\,x\,dx$

$$x^3\sqrt{x^2+1}\,dx = x^2\,\sqrt{x^2+1}\,\boxed{x\,dx}$$

$$= (u-1)\,\sqrt{u}\,\frac{1}{2}\,du$$

Limits of integration:

$x = 1 \overset{u=x^2+1}{\Longrightarrow} u = 2$

$x = \sqrt{3} \overset{u=x^2+1}{\Longrightarrow} u = 4$

$$I = \frac{1}{2}\int_2^4 \sqrt{u}\,(u-1)\,du = \frac{1}{2}\int_2^4 \Big(\underbrace{\sqrt{u}}_{u^{1/2}}\,u - \underbrace{\sqrt{u}}_{u^{1/2}}\Big)\,du$$

$$= \frac{1}{2}\left[\int_2^4 u^{3/2}\,du - \int_2^4 u^{1/2}\,du\right] =$$

$$= \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{5}\,u^{5/2}\Big|_2^4 - \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{3}\,u^{3/2} = \frac{1}{5}\left(4^{5/2} - 2^{5/2}\right) - \frac{1}{3}\left(4^{3/2} - 2^{3/2}\right)$$

§ Trig integrals

Useful Identities:

- $\sin^2 x + \cos^2(x) = 1 \quad (A)$
- $\sin^2 x = \dfrac{1-\cos(2x)}{2} \quad (B)$
- $\cos^2 x = \dfrac{1+\cos(2x)}{2} \quad (C)$
- $\sin(2x) = 2\sin x \cdot \cos x \quad (D)$

A. Odd powers of $\sin x$ / $\cos x$ $\Longrightarrow$ single one power out, use (A)

(also applies to when I have one in odd, the other in even)

① $I = \int \sin^3 x \, dx = \int \sin^2 x \, \sin x \, dx = \int (1-\cos^2 x) \sin x \, dx$

$u = \cos x \Rightarrow du = -\sin x \, dx$

$I = \int (1-u^2)\, du = u - \frac{u^3}{3} + C = \cos x - \frac{\cos^3 x}{3} + C$

<u>Reflection</u> : → isolate one power of <u>sin x</u>

→ put it with the dx (sin x dx)

→ write rest in terms of <u>cos x</u>

→ go for $\boxed{u = \cos x}$

(same idea for <u>$\cos^n x$</u> n odd)

(2) $I = \int \sin^5 x \cos^2 x \, dx$

→ single $\boxed{\sin x}$ out

$$= \int \cos^2 x \, \sin^4 x \, \sin x \, dx$$

$$= \int \cos^2 x \, (1 - \cos^2 x)^2 \, \sin x \, dx$$

→ $\boxed{u = \cos x}$ ⇒ $du = -\sin x \, dx$

$$I = -\int u^2 \underbrace{(1-u^2)^2}_{u^4 - 2u^2 + 1} du = -\int (u^6 - 2u^4 + u^2) \, du$$

$$= -\frac{u^7}{7} + \frac{2u^5}{5} - \frac{u^3}{3} + C$$

$$= -\frac{\cos^7 x}{7} + \frac{2\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

$\boxed{\text{Even}}$ powers : $\rightarrow$ use (B) + (C)

(3) $I = \int_0^{\pi} \sin^2 x \, dx = \int_0^{\pi} \frac{1-\cos(2x)}{2} \, dx$

$= \int_0^{\pi} \left[\frac{1}{2} - \frac{\cos(2x)}{2} \right] dx =$

$= \int_0^{\pi} \frac{1}{2} \, dx - \frac{1}{2} \int_0^{\pi} \cos(2x) \, dx$

$= \frac{x}{2} \Big|_0^{\pi} - \cancel{\frac{\sin(2x)}{4}} \Big|_0^{\pi} = \frac{\pi}{8}$

0 $\leftarrow$ check

(4) $I = \int \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx = \int \left(\frac{1+\cos(2x)}{2} \right)^2 dx$

$= \frac{1}{4} \int \left(1 + \cos^2(2x) + 2\cos(2x) \right) dx =$

again!

$= \frac{1}{4} \int \left(1 + \frac{1+\cos(4x)}{2} + 2\cos(2x) \right) dx$

$= \frac{1}{4} \left[\int 1 \, dx + \int \frac{1}{2} \, dx + \frac{1}{2} \int \cos(4x) \, dx + 2 \int \cos(2x) \, dx \right] =$

$= \frac{1}{4} \left[x + \frac{x}{2} + \frac{\sin(4x)}{4} + \sin(2x) \right] + C$

<u>Recap</u>: $\int \sin^m x \cos^n x \, dx$, m, n positive integers

(a) Both m, n: <u>even</u> $\rightarrow$ use (B) // (C)

(b) $m = $ odd $\rightsquigarrow$ single one $\boxed{\sin x}$ out, use $\sin^2 x = 1 - \cos^2 x$

+ use $\boxed{u = \cos x}$

(c) $n = $ odd $\rightsquigarrow$ –||– $\boxed{\cos x}$ out, –||– $\cos^2 x = 1 - \sin^2 x$

+ use $\boxed{u = \sin x}$

On $\boxed{\text{Trig Substitutions}}$

Integrals of the form

$$\boxed{\int \frac{1}{a^2 \pm x^2}\,dx,\quad \int \frac{1}{\sqrt{a^2 \pm x^2}}\,dx,\quad \int \sqrt{a^2 \pm x^2}\,dx}$$

① $I = \int_0^a \sqrt{a^2-x^2}\,dx$

Let $x = a\sin\theta$, $\boxed{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$ $\to (*)$

$dx = a\cos\theta\,d\theta$

$\to \quad a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta)$

$= a^2\cos^2\theta$

Endpoints of integration : $x=0 \Rightarrow a\sin\theta = 0 \Rightarrow \sin\theta = 0 \overset{(*)}{\Rightarrow} \theta = 0$

$x = a \Rightarrow a\sin\theta = a \Rightarrow \sin\theta = 1 \overset{(*)}{\Rightarrow} \theta = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} a \sqrt{a^2(1-\sin^2\theta)} \cos\theta \, d\theta$$

$\circledast$ $a\cos\theta$

IDEA behind method

$$\overset{\circledast}{=} \int_0^{\pi/2} a^2 \cos^2\theta \, d\theta$$

to hold $| \cdot |$

(remember $\sqrt{b^2} = |b|$)

$$= a^2 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= a^2 \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2} \, d\theta$$

$$= a^2 \left[\frac{\theta + \sin(2\theta)}{2} \right]_0^{\pi/2}$$

check

$$= \frac{a^2}{2} \frac{\pi}{2} = \frac{\pi a^2}{4}$$

it's ok if you omit

Integrand	Identity	Substitution	Range
$\sqrt{a^2-x^2}$	$\sin^2x+\cos^2x=1$	$\boxed{x=a\sin\theta}$ or $\boxed{x=a\cos\theta}$	$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ $0\le\theta\le\pi$
$\sqrt{a^2+x^2}$ or $\frac{1}{a^2+x^2}$	$\boxed{1+\tan^2\theta=\sec^2\theta}$ * $\sec\theta=\frac{1}{\cos\theta}$	$\boxed{x=a\tan\theta}$	$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$$\boxed{(\tan\theta)'=\sec^2\theta}$$

(2) $I=\displaystyle\int\frac{1}{(9-x^2)^{3/2}}\,dx$, $9=3^2$

$x=3\sin\theta \Rightarrow dx=3\cos\theta\,d\theta$

$$I=\int\frac{\overbrace{3\cos\theta\,\cancel{d\theta}}^{dx}}{\underbrace{(9-9\cos^2\theta)^{3/2}}_{=\,27\cdot\cos^3\theta}}\,d\theta=\frac{1}{9}\int\frac{1}{\cos^2\theta}\,d\theta=\frac{1}{9}\int\sec^2\theta\,d\theta$$

$$=\frac{1}{9}\tan\theta+C$$

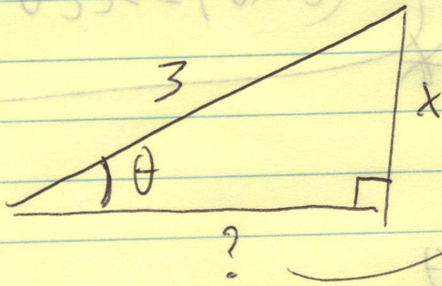
Need to go back to x ...

$\tan\theta \to x$???

- $x = 3 \sin\theta \Rightarrow \boxed{\sin\theta = \frac{x}{3}}$

Draw a reference triangle with an angle θ and with the sides labeled such that

$$\sin\theta = \frac{x}{3} = \frac{\text{opposite}}{\text{hypotenuse}}$$



Pythagoras: $? = \sqrt{9-x^2}$

Now I know all 3 sides (with respect to x)

$$\Rightarrow \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{9-x^2}}$$

$$\Rightarrow \boxed{I = \frac{1}{9}\frac{x}{\sqrt{9-x^2}} + C}$$

(3) $I = \int \frac{1}{(1+x^2)^2}\,dx$. From table: $\boxed{\text{try } x = \tan\theta}$

$dx = \sec^2\theta\, d\theta$

$$\rightarrow I = \int \frac{\overbrace{\sec^2\theta\, d\theta}^{dx}}{\sec^4\theta} = \int \frac{1}{\sec^2\theta}\,d\theta \overset{*}{=} \int \cos^2\theta\, d\theta$$

$(1+\tan^2\theta)^2$
$\downarrow$
$(\sec^2\theta)^2$

$$= \int \frac{1+\cos(2\theta)}{2}\,d\theta$$

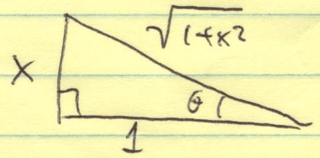
$$= \frac{\theta}{2} + \frac{\sin(2\theta)}{\cancel{4}} + C$$

<u>Go back to x:</u>

- $x = \tan\theta \Rightarrow \tan\theta = \frac{x}{1}$

$\Rightarrow \boxed{\theta = \arctan x}$

- $\sin 2\theta = 2\sin\theta\cos\theta$



$\rightarrow \sin\theta = \frac{x}{\sqrt{1+x^2}}$

$\rightarrow \cos\theta\ \frac{1}{\sqrt{1+x^2}}$

$\left(\frac{2}{4}\right)$

$$\Rightarrow \boxed{I = \frac{\arctan x}{2} + \frac{1}{2}\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}} + C = \frac{1}{2}\arctan x + \frac{x}{2(1+x^2)} + C}$$