CALCULUS QUESTIONS
FOR MATHEMATICS 101

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▲ Legal stuff

How to use this book

▲ Introduction

First of all, welcome back to Calculus!

This book is written as a companion to the CLP notes.

▶▶ How to Work Questions

This book is organized into four sections: Questions, Hints, Answers, and Solutions. As you are working problems, resist the temptation to prematurely peek at the back! It's important to allow yourself to struggle for a time with the material. Even professional mathematicians don't always know right away how to solve a problem. The art is in gathering your thoughts and figuring out a strategy to use what you know to find out what you don't.

If you find yourself at a real impasse, go ahead and look for a hint in the Hints section. Think about it for a while, and don't be afraid to read back in the notes to look for a key idea that will help you proceed. If you still can't solve the problem, well, we included the Solutions section for a reason! As you're reading the solutions, try hard to understand why we took the steps we did, instead of memorizing step-by-step how to solve that one particular problem.

If you struggled with a question quite a lot, it's probably a good idea to return to it in a few days. That might have been enough time for you to internalize the necessary ideas, and you might find it easily conquerable. Pat yourself on the back–sometimes math makes you feel good! If you're still having troubles, read over the solution again, with an emphasis on understanding why each step makes sense.

One of the reasons so many students are required to study calculus is the hope that it will improve their problem-solving skills. In this class, you will learn lots of concepts, and be asked to apply them in a variety of situations. Often, this will involve answering one really big problem by breaking it up into manageable chunks, solving those chunks, then putting the pieces back together. When you see a particularly long question, remain calm and look for a way to break it into pieces you can handle.

▸▸ Working with Friends

Study buddies are fantastic! If you don't already have friends in your class, you can ask your neighbours in lecture to form a group. Often, a question that you might bang your head against for an hour can be easily cleared up by a friend who sees what you've missed. Regular study times make sure you don't procrastinate too much, and friends help you maintain a positive attitude when you might otherwise succumb to frustration. Struggle in mathematics is desirable, but suffering is not.

When working in a group, make sure you try out problems on your own before coming together to discuss with others. Learning is a process, and getting answers to questions that you haven't considered on your own can rob you of the practice you need to master skills and concepts, and the tenacity you need to develop to become a competent problem-solver.

▸▸ Types of Questions

Q[1]: Questions outlined in blue make up the *representative question set*. This set of questions is intended to cover the most essential ideas in each section. These questions are usually highly typical of what you'd see on an exam, although some of them are atypical but carry an important moral. If you find yourself unconfident with the idea behind one of these, it's probably a good idea to practice similar questions.
This representative question set is our suggestion for a minimal selection of questions to work on. You are highly encouraged to work on more.

Q[2](*): In addition to original problems, this book contains problems pulled from quizzes and exams given at UBC for Math 101 and 105 (second–semester calculus) and Math 121 (honours second–semester calculus). These problems are marked with a star. The authors would like to acknowledge the contributions of the many people who collaborated to produce these exams over the years.

The questions are organized into Stage 1, Stage 2, and Stage 3.

▸▸ Stage 1

The first category is meant to test and improve your understanding of basic underlying concepts. These often do not involve much calculation. They range in difficulty from very basic reviews of definitions to questions that require you to be thoughtful about the concepts covered in the section.

▸▸ Stage 2

Questions in this category are for practicing skills. It's not enough to understand the philosophical grounding of an idea: you have to be able to apply it in appropriate situations. This takes practice!

▸▸ Stage 3

The last questions in each section go a little farther than Stage 2. Often they will combine more than one idea, incorporate review material, or ask you to apply your understanding of a concept to a new situation.

In exams, as in life, you will encounter questions of varying difficulty. A good skill to practice is recognizing the level of difficulty a problem poses. Exams will have some easy question, some standard questions, and some harder questions.

Contents

Part I

THE QUESTIONS

Chapter 1

INTEGRATION

1.1 ▲ Definition of the Integral

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](∗): Let f be a function on the whole real line. Express $\int_{-1}^{7} f(x)\,\mathrm{d}x$ as a limit of Riemann sums, using the right endpoints.

Q[2](∗): $\sum_{k=1}^{4} f(1+k)\cdot 1$ is a left Riemann sum for a function $f(x)$ on the interval $[a,b]$ with n subintervals. Find the values of a, b and n.

Q[3](∗): Fill in the blanks with right, left, or midpoint; an interval; and a value of n.

$\sum_{k=0}^{3} f(1.5+k)\cdot 1$ is a ______ Riemann sum for f on the interval [___ , ___] with $n =$ ___.

▸▸ Stage 2

Q[4](∗): The value of the following limit is equal to the area below a graph of $y = f(x)$, integrated over the interval $[0,b]$:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\left[\sin\left(2+\frac{4i}{n}\right)\right]^2$$

Find $f(x)$ and b.

Q[5](*): For a certain function $f(x)$, the following equation holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}\sqrt{1-\frac{k^2}{n^2}}=\int_0^1 f(x)\,\mathrm{d}x$$

Find $f(x)$.

Q[6](*): Use sigma notation to write the midpoint Riemann sum for $f(x)=x^8$ on $[5,15]$ with $n=50$. Do not evaluate the Riemann sum.

Q[7](*): Estimate $\int_{-1}^{5} x^3\,\mathrm{d}x$ using three approximating rectangles and left hand end points.

Q[8](*): Express $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}e^{-i/n}\cos(3i/n)$ as a definite integral.

Q[9](*): Let $\displaystyle R_n=\sum_{i=1}^{n}\frac{ie^{i/n}}{n^2}$. Express $\displaystyle\lim_{n\to\infty}R_n$ as a definite integral. *Do not evaluate this integral.*

Q[10](*): Express $\displaystyle\lim_{n\to\infty}\left(\sum_{i=1}^{n}e^{-1-2i/n}\cdot\frac{2}{n}\right)$ as a integral in three different ways.

Q[11](*): Use elementary geometry to calculate $\displaystyle\int_0^3 f(x)\,\mathrm{d}x$, where

$$f(x)=\begin{cases}x, & \text{if } x\leqslant 1,\\ 1, & \text{if } x>1.\end{cases}$$

Q[12](*): Evaluate $\displaystyle\int_{-1}^{2}|2x|\,\mathrm{d}x$.

Q[13](*): A car's gas pedal is applied at $t=0$ seconds and the car accelerates continuously until $t=2$ seconds. The car's speed at half-second intervals is given in the table below. Find the best possible **upper** estimate for the distance that the car traveled during these two seconds.

t (s)	0	0.5	1.0	1.5	2
v (m/s)	0	14	22	30	40

▸▸ Stage 3

Q[14](*): (a) Express

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\sqrt{4-\left(-2+\frac{2i}{n}\right)^2}$$

as a definite integal.

(b) Evaluate the integral of part (a).

Q[15](∗): Consider the integral:

$$\int_0^3 (7+x^3)\,\mathrm{d}x. \qquad (*)$$

(a) Approximate this integral using the left Riemann sum with $n = 3$ intervals.

(b) Write down the expression for the right Riemann sum with n intervals and calculate the sum. Now take the limit $n \to \infty$ in your expression for the Riemann sum, to evaluate the integral (∗) exactly.

You may use the identity

$$\sum_{i=1}^{n} i^3 = \frac{n^4+2n^3+n^2}{4}$$

Q[16](∗): Using a limit of right–endpoint Riemann sums, evaluate $\int_2^4 x^2\,\mathrm{d}x$. You may use the formulas $\sum\limits_{i=1}^{n} i = n(n+1)/2$ and $\sum\limits_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.

Q[17](∗): Find $\displaystyle\int_0^2 (x^3+x)\,\mathrm{d}x$ using the definition of the definite integral. You may use the summation formulas $\sum\limits_{i=1}^{n} i^3 = \frac{n^4+2n^3+n^2}{4}$ and $\sum\limits_{i=1}^{n} i = \frac{n^2+n}{2}$.

Q[18](∗): Using a limit of right–endpoint Riemann sums, evaluate $\displaystyle\int_1^4 (2x-1)\,\mathrm{d}x$. Do not use anti-differentiation, except to check your answer. You may use the formula $\sum\limits_{i=1}^{n} i = \frac{n(n+1)}{2}$.

1.2▲ Basic properties of the definite integral

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](∗): Decide whether each of the following statements is true or false. If false, provide a counterexample. If true provide a brief justification. (Assume that $f(x)$ and $g(x)$ are continuous functions.)

(a) $\displaystyle\int_{-3}^{-2} f(x)\mathrm{d}x = -\int_{3}^{2} f(x)\mathrm{d}x.$

(b) If $f(x)$ is an odd function, then $\displaystyle\int_{-3}^{-2} f(x)\,\mathrm{d}x = \int_{2}^{3} f(x)\,\mathrm{d}x.$

(c) $\displaystyle\int_0^1 f(x)\cdot g(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x \cdot \int_0^1 g(x)\,\mathrm{d}x.$

▸▸ Stage 2

Q[2](*): Suppose $\displaystyle\int_2^3 f(x)\,\mathrm{d}x = -1$ and $\displaystyle\int_2^3 g(x)\,\mathrm{d}x = 5$. Evaluate $\displaystyle\int_2^3 \big(6f(x) - 3g(x)\big)\,\mathrm{d}x.$

Q[3](*): If $\displaystyle\int_0^2 f(x)\,\mathrm{d}x = 3$ and $\displaystyle\int_0^2 g(x)\,\mathrm{d}x = -4$, calculate $\displaystyle\int_0^2 \big(2f(x) + 3g(x)\big)\,\mathrm{d}x.$

Q[4](*): The functions $f(x)$ and $g(x)$ obey

$$\int_0^{-1} f(x)\,\mathrm{d}x = 1 \qquad \int_0^2 f(x)\,\mathrm{d}x = 2 \qquad \int_{-1}^0 g(x)\,\mathrm{d}x = 3 \qquad \int_0^2 g(x)\,\mathrm{d}x = 4$$

Find $\int_{-1}^2 \big[3g(x) - f(x)\big]\,\mathrm{d}x$.

Q[5](*): Evaluate $\displaystyle\int_{-1}^2 |2x|\,\mathrm{d}x.$

▸▸ Stage 3

Q[6](*): Evaluate $\int_{-2}^2 \left(5 + \sqrt{4 - x^2}\right)\,\mathrm{d}x.$

Q[7](*): Evaluate $\displaystyle\int_{-2012}^{+2012} \frac{\sin x}{\log(3 + x^2)}\mathrm{d}x.$

Q[8](*): Evaluate $\displaystyle\int_{-2012}^{+2012} x^{1/3}\cos x\,\mathrm{d}x.$

1.3▴ The Fundamental Theorem of Calculus

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](∗): Suppose that $f(x)$ is a function and $F(x) = e^{(x^2-3)} + 1$ is an antiderivative of $f(x)$. Evaluate the definite integral $\displaystyle\int_1^{\sqrt{5}} f(x)\,\mathrm{d}x$.

Q[2](∗): For the function $f(x) = x^3 - \sin 2x$, find its antiderivative $F(x)$ that satisfies $F(0) = 1$.

Q[3](∗): Decide whether each of the following statements is true or false. Provide a brief justification.

(a) If $f(x)$ is continuous on $[1, \pi]$ then $\int_1^{\pi} f'(x)\,\mathrm{d}x = f(\pi) - f(1)$.

(b) $\int_{-1}^{1} \frac{1}{x^2}\,\mathrm{d}x = 0$.

(c) If f is continuous on $[a, b]$ then $\int_a^b x f(x)\,\mathrm{d}x = x \int_a^b f(x)\,\mathrm{d}x$.

▸▸ Stage 2

Q[4](∗): Evaluate $\displaystyle\int_0^2 \left(x^3 + \sin x\right)\mathrm{d}x$.

Q[5](∗): Evaluate $\displaystyle\int_1^2 \frac{x^2+2}{x^2}\,\mathrm{d}x$.

Q[6](∗): If

$$F(x) = \int_0^x \log(2 + \sin t)\,\mathrm{d}t \quad \text{and} \quad G(y) = \int_y^0 \log(2 + \sin t)\,\mathrm{d}t$$

find $F'\left(\frac{\pi}{2}\right)$ and $G'\left(\frac{\pi}{2}\right)$.

Q[7](∗): Let $f(x) = \int_1^x 100(t^2 - 3t + 2)e^{-t^2}\,\mathrm{d}t$. Find the interval(s) on which f is increasing.

Q[8](∗): If $F(x) = \displaystyle\int_0^{\cos x} \frac{1}{t^3+6}\,\mathrm{d}t$, find $F'(x)$.

Q[9](∗): Compute $f'(x)$ where $f(x) = \int_0^{1+x^4} e^{t^2}\mathrm{d}t$.

Q[10](∗): Evaluate $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{\sin x} (t^6 + 8)\mathrm{d}t\right)$.

Q[11](∗): Let $F(x) = \int_0^{x^3} e^{-t} \sin\left(\frac{\pi t}{2}\right)\mathrm{d}t$. Calculate $F'(1)$.

Q[12](∗): Find $\displaystyle\frac{\mathrm{d}}{\mathrm{d}u}\left(\int_{\cos u}^{0} \frac{\mathrm{d}t}{1+t^3}\right)$.

Q[13](∗): If $x\sin(\pi x) = \int_0^x f(t)\,\mathrm{d}t$ where f is a continuous function, find $f(4)$.

Q[14](*): Consider the function $F(x) = \int_0^{x^2} e^{-t}\,\mathrm{d}t + \int_{-x}^{0} e^{-t^2}\,\mathrm{d}t$.

(a) Find $F'(x)$.

(b) Find the value of x for which $F(x)$ takes its minimum value.

Q[15](*): If $F(x)$ is defined by $F(x) = \int_{x^4-x^3}^{x} e^{\sin t}\,\mathrm{d}t$, find $F'(x)$.

Q[16](*): Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{x^5}^{-x^2} \cos\left(e^t\right)\mathrm{d}t\right]$.

Q[17](*): Differentiate $\int_x^{e^x} \sqrt{\sin t}\,\mathrm{d}t$.

▸▸ Stage 3

Q[18](*): Evaluate $\int_1^5 f(x)\,\mathrm{d}x$, where $f(x) = \begin{cases} 3 & \text{if } x \leqslant 3 \\ x & \text{if } x \geqslant 3 \end{cases}$.

Q[19](*): Find $f(x)$ if $x^2 = 1 + \int_1^x f(t)\,\mathrm{d}t$.

Q[20](*): If $f'(1) = 2$ and $f'(2) = 3$, find $\int_1^2 f'(x)f''(x)\,\mathrm{d}x$.

Q[21](*): A car traveling at 30 m/s applies its brakes at time $t = 0$, its velocity (in m/s) decreasing according to the formula $v(t) = 30 - 10t$. How far does the car go before it stops?

Q[22](*): Compute $f'(x)$ where $f(x) = \int_0^{2x-x^2} \log\left(1+e^t\right)\mathrm{d}t$. Does $f(x)$ have an absolute maximum? Explain.

Q[23](*): Find the minimum value of $\int_0^{x^2-2x} \frac{\mathrm{d}t}{1+t^4}$. Express your answer as an integral.

Q[24](*): Define the function $F(x) = \int_0^{x^2} \sin(\sqrt{t})\,\mathrm{d}t$ on the interval $0 < x < 4$. On this interval, where does $F(x)$ have a maximum?

Q[25](*): Evaluate $\lim\limits_{n\to\infty} \frac{\pi}{n} \sum\limits_{j=1}^{n} \sin\left(j\pi/n\right)$ by interpreting it as a limit of Riemann sums.

Q[26](*): Use Riemann sums to find the limit $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{j=1}^{n} \frac{1}{1+\frac{j}{n}}$.

Q[27](*): Define $f(x) = x^3 \int_0^{x^3+1} e^{t^3}\,\mathrm{d}t$.

(a) Find a formula for the derivative $f'(x)$.

(b) Find the equation of the tangent line to the graph of $y = f(x)$ at $x = -1$.

1.4▲ Substitution

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.

▸▸ Stage 1

Q[1](*): What is the integral which results when the substitution $u = \sin x$ is applied to the integral $\displaystyle\int_0^{\pi/2} f(\sin x)\,\mathrm{d}x$?

▸▸ Stage 2

Q[2](*): Use substitution to evaluate $\displaystyle\int_0^1 xe^{x^2}\cos(e^{x^2})\,\mathrm{d}x$.

Q[3](*): Let $f(t)$ be any function for which $\int_1^8 f(t)\,\mathrm{d}t = 1$. Calculate the integral $\int_1^2 x^2 f(x^3)\,\mathrm{d}x$.

Q[4](*): Evaluate $\displaystyle\int \frac{x^2}{(x^3+1)^{101}}\mathrm{d}x$.

Q[5](*): Evaluate $\displaystyle\int_e^{e^4} \frac{\mathrm{d}x}{x\log x}$.

Q[6](*): Evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\,\mathrm{d}x$.

Q[7](*): Evaluate $\displaystyle\int_0^{\pi/2} \cos x \cdot (1+\sin^2 x)\,\mathrm{d}x$.

Q[8](*): Evaluate $\int_1^3 (2x-1)e^{x^2-x}\,\mathrm{d}x$.

Q[9](*): Evaluate $\displaystyle\int \frac{(x^2-4)x}{\sqrt{4-x^2}}\,\mathrm{d}x$.

▸▸ Stage 3

Q[10](*): Calculate $\displaystyle\int_{-2}^{2} xe^{x^2}\,dx$.

Q[11](*): Calculate $\displaystyle\lim_{n\to\infty}\sum_{j=1}^{n} \tfrac{j}{n^2}\sin\left(1+\tfrac{j^2}{n^2}\right)$.

Q[12](*): Evaluate $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{j}{n^2} \cos\left(\frac{j^2}{n^2}\right)$.

Q[13](*): Calculate $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{j}{n^2} \sqrt{1+\frac{j^2}{n^2}}$.

1.5▲ Area between curves

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](*): Write down a definite integral that represents the finite area bounded by the curves $y = x^3 - x$ and $y = x$ for $x \geqslant 0$. *Do not evaluate the integral explicitly.*

Q[2](*): Write down a definite integral that represents the area of the region bounded by the line $y = -\frac{x}{2}$ and the parabola $y^2 = 6 - \frac{5x}{4}$. *Do not evaluate the integral explicitly.*

Q[3](*): Write down a definite integral that represents the area of the finite plane region bounded by $y^2 = 4ax$ and $x^2 = 4ay$, where $a > 0$ is a constant. *Do not evaluate the integral explicitly.*

Q[4](*): Write down a definite integral that represents the area of the region bounded between the line $x + 12y + 5 = 0$ and the curve $x = 4y^2$. *Do not evaluate the integral explicitly.*

▸▸ Stage 2

Q[5](*): Find the area of the region bounded by the graph of $f(x) = \frac{1}{(2x-4)^2}$ and the x–axis between $x = 0$ and $x = 1$.

Q[6](*): Find the area between the curves: $y = x$ and $y = 3x - x^2$, by first identifying the points of intersection and then integrating.

Q[7](*): Calculate the area of the region enclosed by $y = 2^x$ and $y = \sqrt{x} + 1$.

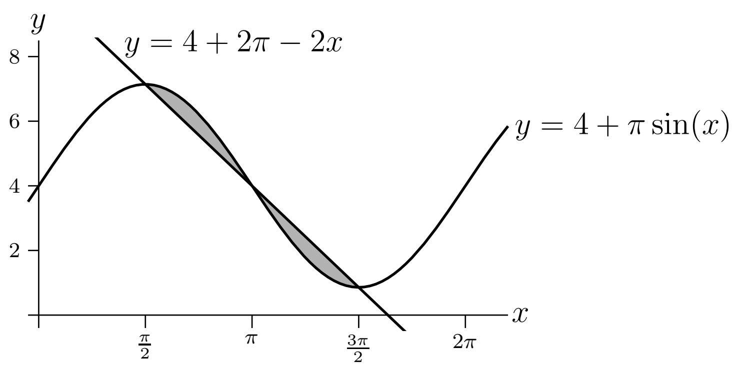
Q[8](*): Find the area of the finite region bounded between the two curves $y = \sqrt{2}\cos(\pi x/4)$ and $y = |x|$.

Q[9](*): Find the area of the finite region that is bounded by the graphs of $f(x) = x^2\sqrt{x^3+1}$ and $g(x) = 3x^2$.

Q[10](*): Find the area to the left of the y–axis and to the right of the curve $x = y^2 + y$.

▸▸ Stage 3

Q[11](∗): The graph below shows the region between $y = 4 + \pi \sin x$ and $y = 4 + 2\pi - 2x$.



Find the area of this region.

Q[12](∗): Compute the area of the finite region bounded by the curves $x = 0$, $x = 3$, $y = x + 2$ and $y = x^2$.

Q[13](∗): Find the total area between the curves $y = x\sqrt{25 - x^2}$ and $y = 3x$, on the interval $0 \leqslant x \leqslant 4$.

1.6▴ Volumes

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](∗): Write down definite integrals that represent the following quantities. *Do not evaluate the integrals explicitly.*

(a) The volume of the solid obtained by rotating around the x–axis the region between the x–axis and $y = \sqrt{x}\, e^{x^2}$ for $0 \leqslant x \leqslant 3$.

(b) The volume of the solid obtained by revolving the region bounded by the curves $y = x^2$ and $y = x + 2$ about the line $x = 3$.

Q[2](∗): Write down definite integrals that represent the following quantities. *Do not evaluate the integrals explicitly.*

(a) The volume of the solid obtained by rotating the finite plane region bounded by the curves $y = 1 - x^2$ and $y = 4 - 4x^2$ about the line $y = -1$.

(b) The volume of the solid obtained by rotating the finite plane region bounded by the curve $y = x^2 - 1$ and the line $y = 0$ about the line $x = 5$.

Q[3](*): Write down a definite integral that represents the volume of the solid obtained by rotating around the line $y = -1$ the region between the curves $y = x^2$ and $y = 8 - x^2$. *Do not evaluate the integrals explicitly.*

Q[4](*): Write definite integrals that represent the following quantities. *Do not evaluate the integrals.*

(a) The area of the finite plane region bounded by $y^2 = 4ax$ and $x^2 = 4ay$, where $a > 0$ is a constant.

(b) The volume of the solid obtained by rotating the finite plane region bounded by the curves $y = 1 - x^2$ and $y = 4 - 4x^2$ about the line $y = -1$.

(c) The volume of the solid obtained by rotating the finite plane region bounded by the curve $y = x^2 - 1$ and the line $y = 0$ about the line $x = 5$.

▸▸ Stage 2

Q[5](*): Let $a > 0$ be a constant. Let R be the finite region bounded by the graph of $y = 1 + \sqrt{x}e^{x^2}$, the line $y = 1$, and the line $x = a$. Using vertical slices, find the volume generated when R is rotated about the line $y = 1$.

Q[6](*): Let R be the region between the curves $T(x) = \sqrt{x}e^{3x}$ and $B(x) = \sqrt{x}(1 + 2x)$ on the interval $0 \leqslant x \leqslant 3$. (It is true that $T(x) \geqslant B(x)$ for all $0 \leqslant x \leqslant 3$.) Compute the volume of the solid formed by rotating R about the x-axis.

Q[7](*): Find the volume of the solid generated by rotating the finite region bounded by $y = 1/x$ and $3x + 3y = 10$ about the x–axis.

Q[8](*): Let R be the region inside the circle $x^2 + (y - 2)^2 = 1$. Let S be the solid obtained by rotating R about the x-axis.

(a) Write down an integral representing the volume of S.

(b) Evaluate the integral you wrote down in part (a).

Q[9](*): The region R is the portion of the first quadrant which is below the parabola $y^2 = 8x$ and above the hyperbola $y^2 - x^2 = 15$.

(a) Sketch the region R.

(b) Find the volume of the solid obtained by revolving R about the x axis.

Q[10](*): The region R is bounded by $y = \log x$, $y = 0$, $x = 1$ and $x = 2$. (Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\log x$.)

(a) Sketch the region R.

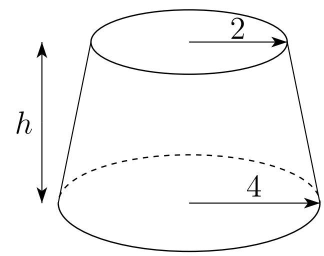
(b) Find the volume of the solid obtained by revolving this region about the y axis.

Q[11](*): The finite region between the curves $y = \cos(\frac{x}{2})$ and $y = x^2 - \pi^2$ is rotated about the line $y = -\pi^2$. Using vertical slices (disks and/or washers), find the volume of the resulting solid.

Q[12](∗): The solid V is 2 meters high and has square horizontal cross sections. The length of the side of the square cross section at height x meters above the base is $\frac{2}{1+x}$ m. Find the volume of this solid.

Q[13](∗): Consider a solid whose base is the finite portion of the xy–plane bounded by the curves $y = x^2$ and $y = 8 - x^2$. The cross–sections perpendicular to the x–axis are squares with one side in the xy–plane. Compute the volume of this solid.

Q[14](∗): A frustrum of a right circular cone (as shown below) has height h. Its base is a circular disc with radius 4 and its top is a circular disc with radius 2. Calculate the volume of the frustrum.



▸▸ Stage 3

Q[15](∗): Let R be the bounded region that lies between the curve $y = 4 - (x-1)^2$ and the line $y = x + 1$.

(a) Sketch R and find its area.

(b) Write down a definite integral giving the volume of the region obtained by rotating R about the line $y = 5$. *Do not evaluate this integral.*

Q[16](∗): Let $\mathcal{R} = \big\{(x,y) \ : \ (x-1)^2 + y^2 \leqslant 1 \text{ and } x^2 + (y-1)^2 \leqslant 1 \big\}$.

(a) Sketch $\mathcal{R}$ and find its area.

(b) If $\mathcal{R}$ rotates around the y–axis, what volume is generated?

Q[17](∗): Let $\mathcal{R}$ be the plane region bounded by $x = 0,\ x = 1,\ y = 0$ and $y = c\sqrt{1+x^2}$, where c is a positive constant.

(a) Find the volume V_1 of the solid obtained by revolving $\mathcal{R}$ about the x–axis.

(b) Find the volume V_2 of the solid obtained by revolving $\mathcal{R}$ about the y–axis.

(c) If $V_1 = V_2$, what is the value of c?

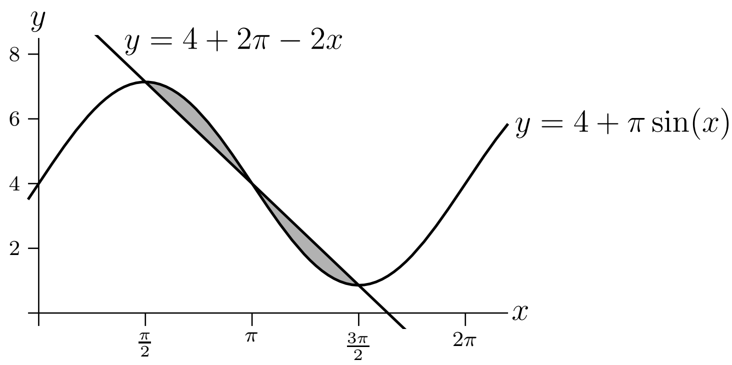
Q[18](∗): The region R is the portion of the first quadrant where $3 \leqslant x \leqslant 4$ and $0 \leqslant y \leqslant \frac{10}{\sqrt{25-x^2}}$.

(a) Sketch the region R.

(b) Determine the volume of the solid obtained by revolving R around the x–axis.

(c) Determine the volume of the solid obtained by revolving R around the y–axis.

Q[19](*): The graph below shows the region between $y = 4 + \pi \sin x$ and $y = 4 + 2\pi - 2x$.



The region is rotated about the line $y = -1$. Express in terms of definite integrals the volume of the resulting solid. Do not evaluate the integrals.

1.7▲ Integration by parts

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

▸▸ Stage 2

Q[1](*): Evaluate $\displaystyle\int x \log x \, \mathrm{d}x$.

Q[2](*): Evaluate $\displaystyle\int \frac{\log x}{x^7} \, \mathrm{d}x$.

Q[3](*): Evaluate $\displaystyle\int_0^{\pi} x \sin x \, \mathrm{d}x$.

Q[4](*): Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x \cos x \, \mathrm{d}x$.

Q[5](*): Evaluate $\displaystyle\int \cos^{-1} y \, \mathrm{d}y$.

▸▸ Stage 3

Q[6](*): Evaluate $\displaystyle\int 4y \arctan(2y) \, \mathrm{d}y$.

Q[7](*): A reduction formula.

(a) Derive the reduction formula $\int \sin^n(x)\,\mathrm{d}x = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\,\mathrm{d}x$.

(b) Calculate $\int_0^{\pi/2} \sin^8(x)\,\mathrm{d}x$.

Q[8](∗): Let R be the part of the first quadrant that lies below the curve $y = \arctan x$ and between the lines $x = 0$ and $x = 1$.

(a) Sketch the region R and determine its area.

(b) Find the volume of the solid obtained by rotating R about the y–axis.

Q[9](∗): Let $f(0) = 1$, $f(2) = 3$ and $f'(2) = 4$. Calculate $\displaystyle\int_0^4 f''\big(\sqrt{x}\big)\,\mathrm{d}x$.

1.8▴ Trigonometric Integrals

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](∗): Evaluate $\displaystyle\int \cos^3 x\,\mathrm{d}x$.

Q[2](∗): Evaluate $\displaystyle\int \sin^{36} t\,\cos^3 t\,\mathrm{d}t$.

▸▸ Stage 2

Q[3](∗): Evaluate $\displaystyle\int \tan^3 x \sec^5 x\,\mathrm{d}x$.

Q[4](∗): Evaluate $\displaystyle\int \sec^4 x\,\tan^{46} x\,\mathrm{d}x$.

Q[5](∗): Evaluate $\displaystyle\int_0^{\pi} \cos^2 x\,\mathrm{d}x$.

▸▸ Stage 3

Q[6](∗): A reduction formula.

(a) Let n be a positive integer with $n \geqslant 2$. Derive the reduction formula
$\int \tan^n(x)\,\mathrm{d}x = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,\mathrm{d}x$.

(b) Calculate $\int_0^{\pi/4} \tan^6(x)\,\mathrm{d}x$.

1.9▲ Trigonometric Substitution

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.

▸▸ Stage 1

Q[1](∗): For each of the following integrals, choose the substitution that is most beneficial for evaluating the integral.

(a) $\displaystyle\int \frac{2x^2}{\sqrt{9x^2-16}}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{x^4-3}{\sqrt{1-4x^2}}\,\mathrm{d}x$

(c) $\displaystyle\int (25+x^2)^{-5/2}\,\mathrm{d}x$

▸▸ Stage 2

Q[2](∗): Evaluate $\displaystyle\int \frac{1}{(x^2+4)^{3/2}}\,\mathrm{d}x$.

Q[3](∗): Evaluate $\displaystyle\int_0^4 \frac{1}{\left(4+x^2\right)^{3/2}}\,\mathrm{d}x$. Your answer may not contain inverse trigonometric functions.

Q[4](∗): Evaluate $\int \frac{\mathrm{d}x}{\sqrt{x^2+25}}$. You may use that $\int \sec \mathrm{d}x = \log\left|\sec x + \tan x\right| + C$.

Q[5](∗): Evaluate $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+16}}$.

Q[6](∗): Evaluate $\int_0^{5/2} \frac{\mathrm{d}x}{\sqrt{25-x^2}}$.

Q[7](∗): Evaluate $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-9}}$. Do not include any inverse trigonometric functions in your answer.

▸▸ Stage 3

Q[8](∗): Evaluate $\int \sqrt{4-x^2}\,\mathrm{d}x$.

Q[9](∗): Evaluate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}}$.

Q[10](∗): (a) Show that $\int_0^{\pi/4} \cos^4\theta\,\mathrm{d}\theta = (8+3\pi)/32$.

(b) Evaluate $\int_{-1}^{1} \frac{\mathrm{d}x}{(x^2+1)^3}$.

Q[11](∗): Evaluate $\displaystyle\int \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x$.

1.10▴ Partial Fractions

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.

▸▸ Stage 1

Q[1](∗): Find the coefficient of $\dfrac{1}{x-1}$ in the partial fraction expansion of $\dfrac{3x^3-2x^2+11}{x^2(x-1)(x^2+3)}$.

Q[2](∗): Write out the **general form** of the partial-fractions decomposition of $\dfrac{x^3+3}{(x^2-1)^2(x^2+1)}$. You need not determine the values of any of the coefficients.

▸▸ Stage 2

Q[3](∗): Evaluate $\int_1^2 \frac{\mathrm{d}x}{x+x^2}$.

Q[4](∗): Calculate $\displaystyle\int \frac{1}{x^4+x^2}\,\mathrm{d}x$.

Q[5](∗): Calculate $\displaystyle\int \frac{12x+4}{(x-3)(x^2+1)}\,dx$.

Q[6](∗): Evaluate the following indefinite integral using partial fractions:

$$F(x) = \int \frac{3x^2-4}{(x-2)(x^2+4)}\,\mathrm{d}x$$

Q[7](*): Evaluate $\displaystyle\int \frac{x-13}{x^2-x-6}\,\mathrm{d}x$.

Q[8](*): Evaluate $\displaystyle\int \frac{5x+1}{x^2+5x+6}\,\mathrm{d}x$.

▸▸ Stage 3

1.11▲ Numerical Integration

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.

▸▸ Stage 1

Q[1](*): Decide whether the following statement is true or false. If false, provide a counterexample. If true provide a brief justification.

> When $f(x)$ is positive and concave up, any Trapezoid Rule approximation for $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ will be an upper estimate for $\displaystyle\int_a^b f(x)\,\mathrm{d}x$.

▸▸ Stage 2

Q[2](*): Find the midpoint rule approximation to $\int_0^\pi \sin x\,\mathrm{d}x$ with $n=3$.

Q[3](*): A 6 metre long cedar log has cross sections which are approximately circular. The diameters of the log, measured at one metre intervals, are given below:

metres from left end of log	0	1	2	3	4	5	6
diameter in metres	1.2	1	0.8	0.8	1	1	1.2

Use Simpson's Rule to estimate the volume of the log.

Q[4](*): The solid V is 40 cm high and the horizontal cross sections are circular disks. The table below gives the diameters of the cross sections in centimeters at 10 cm intervals. Use the trapezoidal rule to estimate the volume of V.

height	0	10	20	30	40
diameter	24	16	10	6	4

Q[5](*): The circumference of an 8 metre high tree at different heights above the ground is given in the table below. Assume that all horizontal cross–sections of the tree are circular disks.

height (metres)	0	2	4	6	8
circumference (metres)	1.2	1.1	1.3	0.9	0.2

Use Simpson's rule to approximate the volume of the tree.

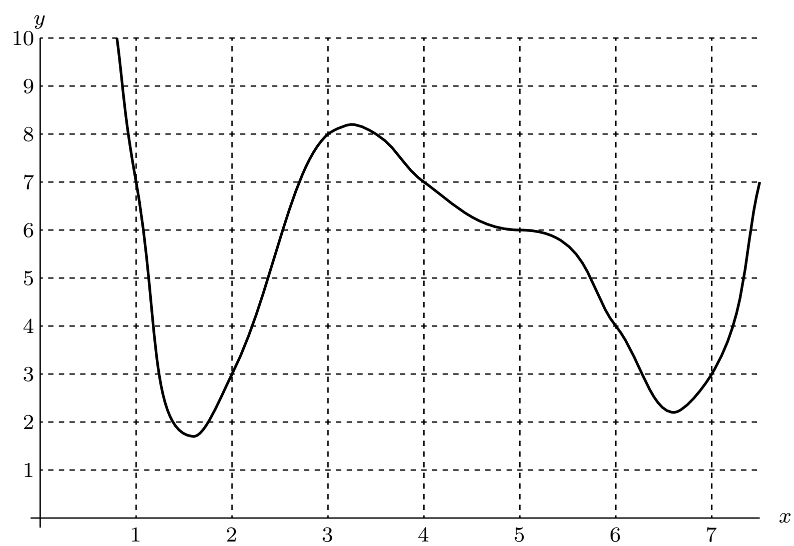
Q[6](∗): By measuring the areas enclosed by contours on a topographic map, a geologist determines the cross sectional areas A in m^2 of a 60 m high hill. The table below gives the cross sectional area $A(h)$ at various heights h. The volume of the hill is $V = \int_0^{60} A(h)\,\mathrm{d}h$.

h	0	10	20	30	40	50	60
A	10,200	9,200	8,000	7,100	4,500	2,400	100

(a) If the geologist uses the Trapezoidal Rule to estimate the volume of the hill, what will be his estimate, to the nearest $1{,}000\mathrm{m}^3$?

(b) What will be the geologist's estimate of the volume of the hill if he uses Simpson's Rule instead of the Trapezoidal Rule?

Q[7](∗): The graph below applies to both parts (a) and (b).



(a) Use the Trapezoidal Rule, with $n = 4$, to estimate the area under the graph between $x = 2$ and $x = 6$. Simplify your answer completely.

(b) Use Simpson's Rule, with $n = 4$, to estimate the area under the graph between $x = 2$ and $x = 6$.

Q[8](∗): The integral $\int_{-1}^{1} \sin(x^2)\,\mathrm{d}x$ is estimated using the Midpoint Rule with 1000 points. Show that the error in this approximation is at most $2 \cdot 10^{-6}$ in absolute value.

You may use the fact that when approximating $\int_a^b f(x)\,\mathrm{d}x$ with the Midpoint Rule using n points, the absolute value of the error is at most $K(b-a)^3/24n^2$ when $|f''(x)| \leqslant K$ for all $x \in [a, b]$.

Q[9](∗): The total error using the midpoint rule with n subintervals to approximate the integral of $f(x)$ over $[a, b]$ is bounded by $M(b-a)^3/(24n^2)$, if $|f''(x)| \leqslant M$ for all $a \leqslant x \leqslant b$.

If the integral $\int_{-2}^{1} 2x^4\,\mathrm{d}x$ is approximated using the midpoint rule with 60 subintervals, what is the largest possible error between the approximation M_{60} and the true value of the integral?

Q[10](*): Both parts of this question concern the integral $I = \int_0^2 (x-3)^5\,\mathrm{d}x$.

(a) Write down the Simpson's Rule approximation to I with $n = 6$. Leave your answer in calculator-ready form.

(b) Which method of approximating I results in a smaller error bound: the Midpoint Rule with $n = 100$ intervals, or Simpson's Rule with $n = 10$ intervals? You may use the formulas

$$|E_M| \leqslant \frac{K(b-a)^3}{24n^2} \qquad \text{and} \qquad |E_S| \leqslant \frac{L(b-a)^5}{180n^4},$$

where K is an upper bound for $|f''(x)|$ and L is an upper bound for $|f^{(4)}(x)|$.

Q[11](*): Consider the Trapezoid Rule for making numerical approximations to $\int_a^b f(x)\,\mathrm{d}x$. The error for the Trapezoid Rule satisfies $|E_T| \leqslant \frac{K(b-a)^3}{12n^2}$, where $|f''(x)| \leqslant K$ for $a \leqslant x \leqslant b$. If $-2 < f''(x) < 0$ for $1 \leqslant x \leqslant 4$, find a value of n to guarantee the Trapezoid Rule will give an approximation for $\int_1^4 f(x)\,\mathrm{d}x$ with absolute error, $|E_T|$, less than 0.001.

Q[12](*): Find a bound for the error in approximating $\int_1^5 \frac{1}{x}\,\mathrm{d}x$ using Simpson's rule with $n = 4$. Do not write down the Simpson's rule approximation S_4.

In general the error in approximating $\int_a^b f(x)\,\mathrm{d}x$ using Simpson's rule with n steps is bounded by $\frac{K(b-a)}{180}(\Delta x)^4$ where $\Delta x = \frac{b-a}{n}$ and $K \geqslant |f^{(4)}(x)|$ for all $a \leqslant x \leqslant b$.

Q[13](*): Find a bound for the error in approximating

$$\int_0^1 \left[e^{-2x} + 3x^3\right]\mathrm{d}x$$

using Simpson's rule with $n = 6$. Do not write down the Simpson's rule approximation S_n.

In general the error in approximating $\int_a^b f(x)\,\mathrm{d}x$ using Simpson's rule with n steps is bounded by $\frac{K(b-a)}{180}(\Delta x)^4$ where $\Delta x = \frac{b-a}{n}$ and $K \geqslant |f^{(4)}(x)|$ for all $a \leqslant x \leqslant b$.

Q[14](*): Let $I = \int_1^2 (1/x)\,\mathrm{d}x$.

(a) Write down the trapezoidal approximation T_4 for I. *You do not need to simplify your answer.*

(b) Write down the Simpson's approximation S_4 for I. *You do not need to simplify your answer.*

(c) Without computing I, find an upper bound for $|I - S_4|$. You may use the fact that if $\left|f^{(4)}(x)\right| \leqslant K$ on the interval $[a, b]$, then the error in using S_n to approximate $\int_a^b f(x)\,\mathrm{d}x$ has absolute value less than or equal to $K(b-a)^5/180n^4$.

Q[15](*): A function $s(x)$ satisfies $s(0) = 1.00664$, $s(2) = 1.00543$, $s(4) = 1.00435$, $s(6) = 1.00331$, $s(8) = 1.00233$. Also, it is known to satisfy $\left|s^{(k)}(x)\right| \leqslant \frac{k}{1000}$ for $0 \leqslant x \leqslant 8$ and all positive integers k.

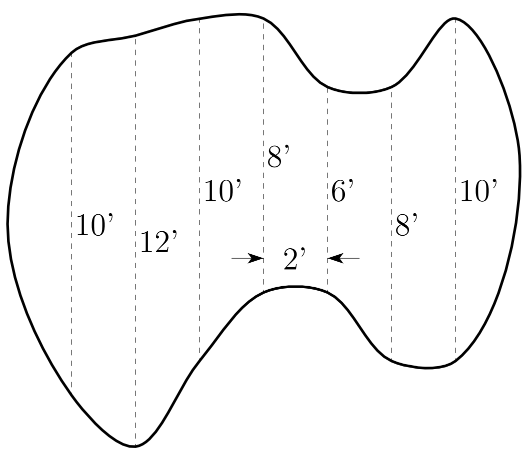
(a) Find the best Trapezoidal Rule and Simpson's Rule approximations that you can for $I = \int_0^8 s(x)\,dx$.

(b) Determine the maximum possible sizes of errors in the approximations you gave in part (a). Recall that if a function $f(x)$ satisfies $\left|f^{(k)}(x)\right| \leqslant K_k$ on $[a, b]$, then

$$\left|\int_a^b f(x)\,dx - T_n\right| \leqslant \frac{K_2(b-a)^3}{12n^2} \quad \text{and} \quad \left|\int_a^b f(x)\,dx - S_n\right| \leqslant \frac{K_4(b-a)^5}{180n^4}$$

▸▸ Stage 3

Q[16](*): A swimming pool has the shape shown in the figure below. The vertical cross–sections of the pool are semi–circular disks. The distances in feet across the pool are given in the figure at 2 foot intervals along the sixteen foot length of the pool. Use Simpson's Rule to estimate the volume of the pool.



Q[17](*): A piece of wire 1m long with radius 1mm is made in such a way that the density varies in its cross–section, but is radially symmetric (that is, the local density $g(r)$ in kg/m^3 depends only on the distance r in mm from the centre of the wire). Take as given that the total mass M of the wire in kg is given by

$$M = 2\pi 10^{-6} \int_0^1 r g(r)\,dr$$

Data from the manufacturer is given below:

r	0	1/4	1/2	3/4	1
$g(r)$	8051	8100	8144	8170	8190

(a) Find the best Trapezoidal Rule approximation that you can for M based on the data in the table.

(b) Suppose that it is known that $|g'(r)| < 200$ and $|g''(r)| < 150$ for all values of r. Determine the maximum possible size of the error in the approximation you gave in part (a). Recall that if a function $f(x)$ satisfies $|f''(x)| \leqslant K$ on $[a, b]$, then

$$|I - T_n| \leqslant \frac{K(b-a)^3}{12n^2}$$

where $I = \int_a^b f(x)\,\mathrm{d}x$ and T_n is the Trapezoidal Rule approximation to I using n subintervals.

Q[18](∗): Simpson's rule can be used to approximate $\log 2$, since $\log 2 = \int_1^2 \frac{1}{x}\,\mathrm{d}x$.

(a) Use Simpson's rule with 6 subintervals to approximate $\log 2$.

(b) How many subintervals are required in order to guarantee that the absolute error is less than 0.00001?

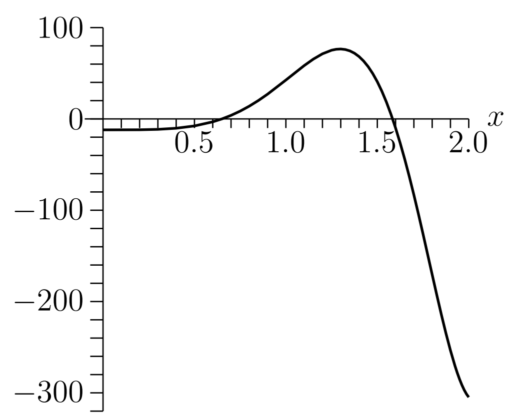
Note that if E_n is the error using n subintervals, then $|E_n| \leqslant \frac{K(b-a)^5}{180n^4}$ where K is the maximum absolute value of the fourth derivative of the function being integrated and a and b are the end points of the interval.

Q[19](∗): Let $I = \displaystyle\int_0^2 \cos(x^2)\,\mathrm{d}x$ and let S_n be the Simpson's rule approximation to I using n subintervals.

(a) Estimate the maximum absolute error in using S_8 to approximate I.

(b) How large should n be in order to insure that $|I - S_n| \leqslant 0.0001$?

Note: The graph of $f''''(x)$, where $f(x) = \cos(x^2)$ is shown below. The absolute error in the Simpson's rule approximation is bounded by $\frac{K(b-a)^5}{180n^4}$ when $|f''''(x)| \leqslant K$ on the interval $[a, b]$.



Q[20](∗): Define a function $f(x)$ and an integral I by

$$f(x) = \int_0^{x^2} \sin(\sqrt{t})\,\mathrm{d}t, \qquad I = \int_0^1 f(t)\,\mathrm{d}t$$

Estimate how many subdivisions are needed to calculate I to five decimal places of accuracy using the trapezoidal rule.

Note that if E_n is the error using n subintervals, then $|E_n| \leqslant \frac{M(b-a)^3}{12n^2}$ where M is the maximum absolute value of the second derivative of the function being integrated and a and b are the end points of the interval of integration.

Q[21](*):

A piece of wire 1m long with radius 1mm is made in such a way that the density varies in its cross–section, but is radially symmetric (that is, the local density $g(r)$ in kg/m^3 depends only on the distance r in mm from the centre of the wire). Take as given that the total mass M of the wire in kg is given by

$$M = 2\pi 10^{-6} \int_0^1 r g(r)\,\mathrm{d}r$$

Data from the manufacturer is given below:

r	0	1/4	1/2	3/4	1
$g(r)$	8051	8100	8144	8170	8190

(a) Find the best Trapezoidal Rule approximation that you can for M based on the data in the table.

(b) Suppose that it is known that $|g'(r)| < 200$ and $|g''(r)| < 150$ for all values of r. Determine the maximum possible size of the error in the approximation you gave in part (a). Recall that if a function $f(x)$ satisfies $|f''(x)| \leqslant K$ on $[a, b]$, then

$$|I - T_n| \leqslant \frac{K(b-a)^3}{12n^2}$$

where $I = \int_a^b f(x)\,\mathrm{d}x$ and T_n is the Trapezoidal Rule approximation to I using n subintervals.

1.12▲ Improper Integrals

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](*): Decide whether the following statement is true or false. If false, provide a counterexample. If true provide a brief justification. (Assume that $f(x)$ and $g(x)$ are continuous functions.)

If $\displaystyle\int_1^\infty f(x)\,\mathrm{d}x$ converges and $g(x) \geqslant f(x) \geqslant 0$ for all x, then $\displaystyle\int_1^\infty g(x)\,\mathrm{d}x$ converges.

Q[2](*): What is the largest value of q for which the integral $\displaystyle\int_1^\infty \frac{1}{x^{5q}}\,\mathrm{d}x$ diverges?

▸▸ Stage 2

Q[3](∗): Evaluate the integral $\displaystyle\int_0^1 \frac{x^4}{x^5-1}\,\mathrm{d}x$ or state that it diverges.

Q[4](∗): Determine whether the integral $\displaystyle\int_{-2}^{2} \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x$ is convergent or divergent. If it is convergent, find its value.

Q[5](∗): Does the improper integral $\int_1^\infty \frac{1}{\sqrt{4x^2-x}}\,\mathrm{d}x$ converge? Justify your answer.

Q[6](∗): Does the integral $\int_0^\infty \frac{\mathrm{d}x}{x^2+\sqrt{x}}$ converge or diverge? Justify your claim.

Q[7](∗): Determine (with justication!) whether the integral $\int_{-\infty}^{+\infty} \frac{x}{x^2+1}\mathrm{d}x$ converges absolutely, converges but not absolutely, or diverges.

Q[8](∗): Decide whether $I = \int_0^\infty \frac{|\sin x|}{x^{3/2}+x^{1/2}}\,\mathrm{d}x$ converges or diverges. Justify.

Q[9](∗): Does the integral $\displaystyle\int_0^\infty \frac{x+1}{x^{1/3}(x^2+x+1)}\,\mathrm{d}x$ converge or diverge?

▸▸ Stage 3

Q[10](∗): Is the integral $\int_0^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x$ convergent or divergent? Explain why.

Q[11](∗): Let $M_{n,t}$ be the Midpoint Rule approximation for $\int_0^t \frac{e^{-x}}{1+x}\,\mathrm{d}x$ with n equal subintervals. Find a value of t and a value of n such that $M_{n,t}$ differs from $\int_0^\infty \frac{e^{-x}}{1+x}\,\mathrm{d}x$ by at most 10^{-4}. Recall that the error E_n introduced when the Midpoint Rule is used with n subintervals obeys

$$|E_n| \leqslant \frac{M(b-a)^3}{24n^2}$$

where M is the maximum absolute value of the second derivative of the integrand and a and b are the end points of the interval of integration.

1.13▲ More Integration Examples

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 2

Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\log x$.

Q[1](*): Evaluate $\int \frac{x}{x^2-3} \mathrm{d}x$

Q[2](*): Evaluate the following integrals.

(a) $\int_0^4 \frac{x}{\sqrt{9+x^2}} \, \mathrm{d}x$

(b) $\int_0^{\pi/2} \cos^3 x \ \sin^2 x \, \mathrm{d}x$

(c) $\int_1^e x^3 \log x \, \mathrm{d}x$

Q[3](*): Evaluate the following integrals.

(a) $\int_0^{\pi/2} x \sin x \, \mathrm{d}x$

(b) $\int_0^{\pi/2} \cos^5 x \, \mathrm{d}x$

Q[4](*): Evaluate the following integrals.

(a) $\int_0^2 xe^x \, \mathrm{d}x$

(b) $\int_0^1 \frac{1}{\sqrt{1+x^2}} \, \mathrm{d}x$

(c) $\int_3^5 \frac{4x}{(x^2-1)(x^2+1)} \, \mathrm{d}x$

Q[5](*): Calculate the following integrals.

(a) $\int_0^{\pi/2} \cos^5(x) \, \mathrm{d}x$

(b) $\int_0^3 \sqrt{9-x^2} \, \mathrm{d}x$

(c) $\int_0^1 \log(1+x^2) \, \mathrm{d}x$

(d) $\int_3^\infty \frac{x}{(x-1)^2(x-2)} \, \mathrm{d}x$

Q[6](*): Evaluate the following integrals. *Show your work.*

(a) $\int_0^{\frac{\pi}{4}} \sin^2(2x) \cos^3(2x) \, \mathrm{d}x$

(b) $\int \left(9+x^2\right)^{-\frac{3}{2}} \, \mathrm{d}x$

(c) $\int \frac{\mathrm{d}x}{(x-1)(x^2+1)}$

(d) $\int x \tan^{-1} x \, \mathrm{d}x$

Q[7](*): Evaluate the following integrals.

(a) $$\int_0^{\pi/4} \sin^5(2x) \ \cos(2x) \, \mathrm{d}x$$

(b) $$\int \sqrt{4-x^2} \, \mathrm{d}x$$

(c) $$\int \log(1+x^2) \, \mathrm{d}x$$

(d) $\displaystyle\int \frac{x+1}{x^2(x-1)}\,\mathrm{d}x$

Q[8](*): Calculate the following integrals.

(a) $\int_0^\infty e^{-x}\sin(2x)\,\mathrm{d}x$

(b) $\int_0^{\sqrt{2}} \frac{1}{(2+x^2)^{3/2}}\,\mathrm{d}x$

(c) $\int_0^1 x\log(1+x^2)\,\mathrm{d}x$

(d) $\int_3^\infty \frac{1}{(x-1)^2(x-2)}\,\mathrm{d}x$

Q[9](*): Evaluate the following integrals.

(a) $\int x\ \log x\ \mathrm{d}x$

(b) $\int \frac{(x-1)\,\mathrm{d}x}{x^2+4x+5}$

(c) $\int \frac{\mathrm{d}x}{x^2-4x+3}$

(d) $\int \frac{x^2\,\mathrm{d}x}{1+x^6}$

Q[10](*): Evaluate the following integrals.

(a) $\int_0^1 \tan^{-1} x\ \mathrm{d}x$.

(b) $\int \frac{2x-1}{x^2-2x+5}\,\mathrm{d}x$.

Q[11](*):

(a) Evaluate $\int x\ \log x\ \mathrm{d}x$.

(b) Evaluate $\displaystyle\int \frac{x^2}{(x^3+1)^{101}}\mathrm{d}x$.

(c) Evaluate $\int \cos^3 x\ \sin^4 x\ \mathrm{d}x$.

(d) Evaluate $\int \sqrt{4-x^2}\ \mathrm{d}x$.

Q[12](*): Evaluate the following integrals.

(a) $\int \frac{e^x}{(e^x+1)(e^x-3)}\mathrm{d}x$.

(b) $\int_2^4 \frac{x^2-4x+4}{\sqrt{12+4x-x^2}}\mathrm{d}x$.

Q[13](*): Evaluate these integrals.

(a) $\displaystyle\int \frac{\sin^3 x}{\cos^3 x}\,\mathrm{d}x$

(b) $\displaystyle\int_{-2}^{2} \frac{x^4}{x^{10}+16}\,\mathrm{d}x$

(c) $\displaystyle\int_0^1 \log(1+x^2)\,\mathrm{d}x$

Q[14](∗): Evaluate (with justification)

(a) $\displaystyle\int_0^3 (x+1)\sqrt{9-x^2}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{4x+8}{(x-2)(x^2+4)}\,\mathrm{d}x$

(c) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{e^x+e^{-x}}\,\mathrm{d}x$

Q[15](∗): Evaluate these integrals.

(a) $\displaystyle\int \sin(\log x)\,\mathrm{d}x$

(b) $\displaystyle\int_0^1 \frac{1}{x^2-5x+6}\,\mathrm{d}x$

Chapter 2

APPLICATIONS OF INTEGRATION

2.1▲ Work

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](∗): Find the work (in joules) required to stretch a string 10 cm beyond equilibrium, if its spring constant is $k = 50$ N/m.

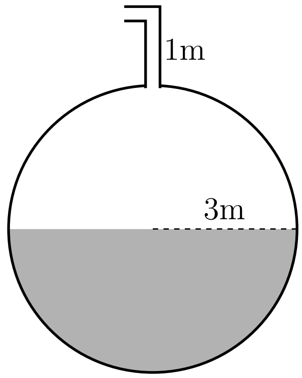
▸▸ Stage 2

Q[2](∗): A variable force $F(x) = \frac{a}{\sqrt{x}}$ Newtons moves an object along a straight line when it is a distance of x meters from the origin. If the work done in moving the object from $x = 1$ meters to $x = 16$ meters is 18 Joules, what is the value of a? Don't worry about the units of a.

Q[3](∗): A force of 10N (newtons) is required to hold a spring stretched 5cm beyond its natural length. How much work, in joules (J), is done in stretching the spring from its natural length to 50cm beyond its natural length?

Q[4](∗): A 5-meter-long cable of mass 8 kg is used to lift a bucket off the ground. How much work is needed to raise the entire cable to height 5 m? Ignore the weight of the bucket and its contents. Use $g = 9.8$ m/s^2 for the acceleration due to gravity.

Q[5](∗): A spherical tank of radius 3 metres is half–full of water. It has a spout of length 1 metre sticking up from the top of the tank. Find the work required to pump all of the water in the tank out the spout. The density of water is 1000 kilograms per cubic metre. The acceleration due to gravity is 9.8 metres per second squared.



Q[6](*): A sculpture, shaped like a pyramid 3m high sitting on the ground, has been made by stacking smaller and smaller (very thin) iron plates on top of one another. The iron plate at height zm above ground level is a square whose side length is $(3 - z)$m. All of the iron plates started on the floor of a basement 2 m below ground level.

Write down an integral that represents the work, in joules, it took to move all of the iron from its starting position to its present position. Do **not** evaluate the integral. (You can use 9.8 $\mathrm{m/s^2}$ for the force of gravity and 8000 $\mathrm{kg/m^3}$ for the density of iron.)

▸▸ Stage 3

2.2▴ Averages

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.

▸▸ Stage 1

▸▸ Stage 2

Q[1](*): Find the average value of $f(x) = \sin(5x) + 1$ over the interval $-\pi/2 \leqslant x \leqslant \pi/2$.

Q[2](*): Find the average value of the function $y = x^2 \log x$ on the interval $1 \leqslant x \leqslant e$.

Q[3](*): Find the average value of the function $f(x) = 3\cos^3 x + 2\cos^2 x$ on the interval $0 \leqslant x \leqslant \frac{\pi}{2}$.

Q[4](*): Let k be a positive constant. Find the average value of the function $f(x) = \sin(kx)$ on the interval $0 \leqslant x \leqslant \pi/k$.

Q[5](*): The temperature in Celsius in a 3 m long rod at a point x metres from the left end of the rod is given by the function $T(x) = \frac{80}{16-x^2}$. Determine the average temperature in the rod.

Q[6](*): What is the average value of the function $f(x) = \frac{\log x}{x}$ on the interval $[1, e]$?

Q[7](*): Find the average value of $f(x) = \cos^2(x)$ over $0 \leqslant x \leqslant 2\pi$.

▸▸ Stage 3

Q[8](*):

A car travels two hours without stopping. The driver records the car's speed every 20 minutes, as indicated in the table below:

time in hours	0	1/3	2/3	1	4/3	5/3	2
speed in km/hr	50	70	80	55	60	80	40

(a) Use the trapezoidal rule to estimate the total distance traveled in the two hours.

(b) Use the answer to part (a) to estimate the average speed of the car during this period.

2.3▲ Centre of Mass and Torque

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](*): Express the x–coordinate of the centroid of the triangle with vertices $(-1,-3)$, $(-1,3)$, and $(0,0)$ in terms of a definite integral. *Do not evaluate the integral.*

▸▸ Stage 2

Q[2](*): Find the y-coordinate of the centroid of the region bounded by the curves $y = 1$, $y = -e^x$, $x = 0$ and $x = 1$. You may use the fact that the area of this region equals e.

Q[3](*): Find the *y-coordinate* of the centre of mass of the (infinite) region lying to the right of the line $x = 1$, above the x–axis, and below the graph of $y = 8/x^3$.

Q[4](*): Consider the region bounded by $y = \frac{1}{\sqrt{16-x^2}}$, $y = 0$, $x = 0$ and $x = 2$.

(a) Sketch this region.

(b) Find the y–coordinate of the centroid of this region.

Q[5](*): Find the centroid of the finite region bounded by $y = \sin(x)$, $y = \cos(x)$, $x = 0$, and $x = \pi/4$.

Q[6](*): Let A denote the area of the plane region bounded by $x = 0$, $x = 1$, $y = 0$ and $y = \frac{k}{\sqrt{1+x^2}}$, where k is a positive constant.

(a) Find the coordinates of the centroid of this region in terms of k and A.

(b) For what value of k is the centroid on the line $y = x$?

Q[7](*): The region R is the portion of the plane which is above the curve $y = x^2 - 3x$ and below the curve $y = x - x^2$.

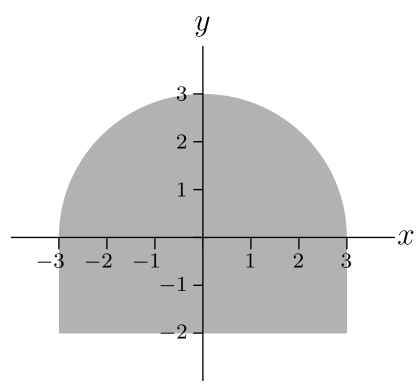
(a) Sketch the region R

(b) Find the area of R.

(c) Find the x coordinate of the centroid of R.

Q[8](*): Let R be the region where $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant \frac{1}{1+x^2}$. Find the x–coordinate of the centroid of R.

Q[9](*): Find the centroid of the region below, which consists of a semicircle of radius 3 on top of a rectangle of width 6 and height 2.



Q[10](*): Let D be the region below the graph of the curve $y = \sqrt{9 - 4x^2}$ and above the x-axis.

(a) Using an appropriate integral, find the *area* of the region D; simplify your answer completely.

(b) Find the *centre of mass* of the region D; simplify your answer completely. (Assume it has constant density ρ.)

▸▸ Stage 3

Q[11](*): Let A be the region to the right of the y-axis that is bounded by the graphs of $y = x^2$ and $y = 6 - x$.

(a) Find the centroid of A, assuming it has constant density $\rho = 1$. The area of A is $\frac{22}{3}$ (you don't have to show this).

(b) Write down an expression, using horizontal slices (disks), for the volume obtained when the region A is rotated around the y-axis. Do not evaluate any integrals; simply write down an expression for the volume.

Q[12](*): (a) Find the y–coordinate of the centroid of the region bounded by $y = e^x$, $x = 0$, $x = 1$, and $y = -1$.

(b) Calculate the volume of the solid generated by rotating the region from part (a) about the line $y = -1$.

2.4▲ Separable Differential Equations

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

▸▸ Stage 2

Q[1](*): Find the solution to the separable initial value problem:

$$\frac{dy}{dx} = \frac{2x}{e^y}, \quad y(0) = \log 2$$

Express your solution explicitly as $y = y(x)$.

Q[2](*): Find the solution $y(x)$ of $\frac{dy}{dx} = \frac{xy}{x^2+1}$, $y(0) = 3$.

Q[3](*): Solve the differential equation $y'(t) = e^{\frac{y}{3}} \cos t$. You should express the solution $y(t)$ in terms of t explicitly.

Q[4](*): Solve the differential equation

$$\frac{dy}{dx} = xe^{x^2 - \log(y^2)}$$

Q[5](*): Let $y = y(x)$. Find the general solution of the differential equation $y' = xe^y$.

Q[6](*): Find the solution to the differential equation $\dfrac{yy'}{e^x - 2x} = \dfrac{1}{y}$ that satisfies $y(0) = 3$. Solve completely for y as a function of x.

Q[7](*): Find the function $y = f(x)$ that satisfies

$$\frac{dy}{dx} = -xy^3 \qquad \text{and} \qquad f(0) = -\frac{1}{4}$$

Q[8]($*$): Find the function $y = y(x)$ that satisfies $y(1) = 4$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{15x^2 + 4x + 3}{y}$$

Q[9]($*$): Find the solution $y(x)$ of $y' = x^3 y$ with $y(0) = 1$.

Q[10]($*$): Find the solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^2$$

that satisfies $y(1) = -1$.

Q[11]($*$): A function $f(x)$ is always positive, has $f(0) = e$ and satisfies $f'(x) = x\,f(x)$ for all x. Find this function.

Q[12]($*$): Solve the following initial value problem:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(x^2 + x)y} \qquad y(1) = 2$$

Q[13]($*$): Find the solution of the differential equation $\dfrac{1+\sqrt{y^2-4}}{\tan x}y' = \dfrac{\sec x}{y}$ that satisfies $y(0) = 2$. You don't have to solve for y in terms of x.

Q[14]($*$): The fish population in a lake is attacked by a disease at time $t = 0$, with the result that the size $P(t)$ of the population at time $t \geqslant 0$ satisfies

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -k\sqrt{P}$$

where k is a positive constant. If there were initially 90,000 fish in the lake and 40,000 were left after 6 weeks, when will the fish population be reduced to 10,000?

Q[15]($*$): An object of mass m is projected straight upward at time $t = 0$ with initial speed v_0. While it is going up, the only forces acting on it are gravity (assumed constant) and a drag force proportional to the square of the object's speed $v(t)$. It follows that the differential equation of motion is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -(mg + kv^2)$$

where g and k are positive constants. At what time does the object reach its highest point?

Q[16]($*$): A motor boat is traveling with a velocity of 40 ft/sec when its motor shuts off at time $t = 0$. Thereafter, its deceleration due to water resistance is given by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -k\,v^2$$

where k is a positive constant. After 10 seconds, the boat's velocity is 20 ft/sec.

(a) What is the value of k?

(b) When will the boat's velocity be 5 ft/sec?

Q[17](∗): Consider the initial value problem $\frac{\mathrm{d}x}{\mathrm{d}t} = k(3-x)(2-x)$, $x(0) = 1$, where k is a positive constant. (This kind of problem occurs in the analysis of certain chemical reactions.)

(a) Solve the initial value problem. That is, find x as a function of t.

(b) What value will $x(t)$ approach as t approaches $+\infty$.

Q[18](∗): The quantity $P = P(t)$, which is a function of time t, satisfies the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 4P - P^2$$

and the initial condition $P(0) = 2$.

(a) Solve this equation for $P(t)$.

(b) What is P when $t = 0.5$? What is the limiting value of P as t becomes large?

Q[19](∗): An object moving in a fluid has an initial velocity v of 400 m/min. The velocity is decreasing at a rate proportional to the square of the velocity. After 1 minute the velocity is 200 m/min.

(a) Give a differential equation for the velocity $v = v(t)$ where t is time.

(b) Solve this differential equation.

(c) When will the object be moving at 50 m/min?

▸▸ Stage 3

Q[20](∗): An investor places some money in a mutual fund where the interest is compounded continuously and where the interest rate fluctuates between 4% and 8%, Assume that the amount of money $B = B(t)$ in the account in dollars after t years satisfies the differential equation

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \big(0.06 + 0.02\sin t\big)B$$

(a) Solve this differential equation for B as a function of t.

(b) If the initial investment is \$1000, what will the balance be at the end of two years?

Q[21](∗): An endowment is an investment account in which the balance ideally remains constant and withdrawals are made on the interest earned by the account. Such an account may be modeled by the initial value problem $B'(t) = aB - m$ for $t \geqslant 0$, with $B(0) = B_0$. The constant a reflects the annual interest rate, m is the annual rate of withdrawal, and B_0 is the initial balance in the account.

(a) Solve the initial value problem with $a = 0.02$ and $B(0) = B_0 = \$30,000$. Note that your answer depends on the constant m.

(b) If $a = 0.02$ and $B(0) = B_0 = \$30,000$, what is the annual withdrawal rate m that ensures a constant balance in the account?

Q[22](∗): A certain continuous function $y = y(x)$ satisfies the integral equation

$$y(x) = 3 + \int_0^x \big(y(t)^2 - 3y(t) + 2\big) \sin t \, dt \tag{$*$}$$

for all x in some open interval containing 0. Find $y(x)$ and the largest interval for which $(*)$ holds.

Q[23](∗): A cylindrical water tank, of radius 3 meters and height 6 meters, is full of water when its bottom is punctured. Water drains out through a hole of radius 1 centimeter. If

- $h(t)$ is the height of the water in the tank at time t (in meters) and
- $v(t)$ is the velocity of the escaping water at time t (in meters per second) then
- Torricelli's law states that $v(t) = \sqrt{2gh(t)}$ where $g = 9.8\ \mathrm{m/sec}^2$. Determine how long it takes for the tank to empty.

Q[24](∗): A spherical tank of radius 6 feet is full of mercury when a circular hole of radius 1 inch is opened in the bottom. How long will it take for all of the mercury to drain from the tank?

Use the value $g = 32\,\mathrm{feet/sec}^2$. Also use Torricelli's law, which states when the height of mercury in the tank is h, the speed of the mercury escaping from the tank is $v = \sqrt{2gh}$.

Q[25](∗): Consider the equation

$$f(x) = 3 + \int_0^x \big(f(t) - 1\big)\big(f(t) - 2\big)\, \mathrm{d}t$$

(a) What is $f(0)$?

(b) Find the differential equation satisfied by $f(x)$.

(c) Solve the initial value problem determined in (a) and (b).

Q[26](∗):

A tank 2m tall is to be made with circular cross–sections with radius $r = y^p$. Here y measures the vertical distance from the bottom of the tank and p is a positive constant to be determined. You may assume that when the tank drains, it obeys Torricelli's law, that is

$$A(y)\frac{\mathrm{d}y}{\mathrm{d}t} = -c\sqrt{y}$$

for some constant c where $A(y)$ is the cross–sectional area of the tank at height y. It is desired that the tank be constructed so that the top half ($y = 2$ to $y = 1$) takes exactly the same amount of time to drain as the bottom half ($y = 1$ to $y = 0$). Determine the value of p so that the tank has this property. *Note:* it is not possible or necessary to find c for this question.

Chapter 3

SEQUENCES AND SERIES

3.1▲ Sequences

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

▸▸ Stage 2

Q[1](∗): Find the limit, if it exists, of the sequence $\{a_k\}$, where

$$a_k = \frac{k! \sin^3 k}{(k+1)!}$$

Q[2](∗): Consider the sequence $\left\{(-1)^n \sin\left(\frac{1}{n}\right)\right\}$. State whether this sequence converges or diverges, and if it converges give its limit.

Q[3](∗): Evaluate $\lim\limits_{n\to\infty} \left[\dfrac{6n^2+5n}{n^2+1} + 3\cos(1/n^2)\right]$.

▸▸ Stage 3

Q[4](∗): Find the limit of the sequence $\left\{\log\left(\sin\dfrac{1}{n}\right) + \log(2n)\right\}$.

Q[5](∗): A sequence $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ satisfies the recursion relation $a_{n+1} = \sqrt{3+\sin a_n}$ for $n \geqslant 0$.

(a) Show that the equation $x = \sqrt{3 + \sin x}$ has a solution.

(b) Show that $\lim_{n\to\infty} a_n = L$, where L is a solution to equation above.

(c) Show that the equation $x = \sqrt{3 + \sin x}$ has a unique solution.

3.2▲ Series

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](∗): Evaluate $\sum_{k=7}^{\infty} \frac{1}{8^k}$

Q[2](∗): To what value does the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots$ converge?

Q[3](∗): Show that the series $\sum_{k=1}^{\infty} \left(\frac{6}{k^2} - \frac{6}{(k+1)^2} \right)$ converges and find its limit.

Q[4](∗): Find the sum of the convergent series $\sum_{n=3}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+1}\right) \right)$.

Q[5](∗): The n^{th} partial sum of a series $\sum_{n=1}^{\infty} a_n$ is known to have the formula $s_n = \frac{1+3n}{5+4n}$.

(a) Find an expression for a_n, valid for $n \geqslant 2$.

(b) Show that the series $\sum_{n=1}^{\infty} a_n$ converges and find its value.

▸▸ Stage 2

Q[6](∗): Find the sum of the series $\sum_{n=2}^{\infty} \frac{3 \cdot 4^{n+1}}{8 \cdot 5^n}$. Simplify your answer completely.

Q[7](∗): Relate the number $0.2\bar{3} = 0.233333\ldots$ to the sum of a geometric series, and use that to represent it as a rational number (a fraction or combination of fractions, with no decimals).

Q[8](∗): Express $2.656565\ldots$ as a rational number, i.e. in the form p/q where p and q are integers.

Q[9](*): Express the decimal $0.\overline{321} = 0.321321321\ldots$ as a fraction.

Q[10](*): Find the *value* of the convergent series

$$\sum_{n=2}^{\infty}\left(\frac{2^{n+1}}{3^n}+\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$

Simplify your answer completely.

Q[11](*): Evaluate

$$\sum_{n=1}^{\infty}\left[\left(\frac{1}{3}\right)^n+\left(-\frac{2}{5}\right)^{n-1}\right]$$

Q[12](*): Find the sum of the series $\displaystyle\sum_{n=0}^{\infty}\frac{1+3^{n+1}}{4^n}$.

▸▸ Stage 3

3.3▴ Convergence Tests

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](*): Does the series $\displaystyle\sum_{n=2}^{\infty}\frac{n^2}{3n^2+\sqrt{n}}$ converge?

Q[2](*): Suppose that you want to use the Limit Comparison Test on the series $\sum_{n=0}^{\infty} a_n$ where $a_n = \frac{2^n+n}{3^n+1}$. Write down a sequence $\{b_n\}$ such that $\lim\limits_{n\to\infty}\frac{a_n}{b_n}$ exists and is nonzero. (You don't have to carry out the Limit Comparison Test)

Q[3](*): Decide whether each of the following statements is true or false. If false, provide a counterexample. If true provide a brief justification.

(a) If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

(b) If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ converges.

(c) If $0 \leqslant a_n \leqslant b_n$ and $\sum\limits_{n=1}^{\infty} b_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

▸▸ Stage 2

Q[4](*): Determine, with explanation, whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5^k}{4^k + 3^k}$ converges or diverges.

Q[5](*): Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n + \frac{1}{2}}$ is convergent or divergent. If it is convergent, find its value.

Q[6](*): Show that the series $\displaystyle\sum_{n=3}^{\infty} \frac{5}{n(\log n)^{3/2}}$ converges.

Q[7](*): Find the values of p for which the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges.

Q[8](*): Does $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converge or diverge?

Q[9](*): Use the comparison test (not the limit comparison test) to show whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{3n^2 - 7}}{n^3}$ converges or diverges.

Q[10](*): Determine whether the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^4 + 1}}{\sqrt{k^5 + 9}}$ concerges.

Q[11](*): Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 2^{n/3}}{(2n+7)^4}$ converge or diverge?

Q[12](*): Determine (with justication!) whether the series $\sum_{n=1}^{\infty} \frac{n^2 - \sin n}{n^6 + n^2}$ converges absolutely, converges but not absolutely, or diverges.

Q[13](*): Determine (with justication!) whether the series $\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(n^2+1)(n!)^2}$ converges absolutely, converges but not absolutely, or diverges.

Q[14](*): Determine (with justication!) whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^{101}}$ converges absolutely, converges but not absolutely, or diverges.

Q[15](*): Determine, with explanation, whether each of the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$

Q[16](*): Determine whether the series

$$\sum_{k=1}^{\infty} \frac{k^4 - 2k^3 + 2}{k^5 + k^2 + k}$$

converges or diverges.

Q[17](*): Determine whether each of the following series converge or diverge.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2 + n + 1}{n^5 - n}$

(b) $\displaystyle\sum_{m=1}^{\infty} \frac{3m + \sin\sqrt{m}}{m^2}$

Q[18](*): Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{6}{7^n}$ is convergent or divergent. If it is convergent, find its value.

Q[19](*): Determine, with explanation, whether each of the following series converge or diverge.

(a) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{2^{2n+1}}$

Q[20](*): Determine, with explanation, whether each of the following series converges or diverges.

(a) $\displaystyle\sum_{k=2}^{\infty} \frac{\sqrt[3]{k}}{k^2 - k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^{10} 10^k (k!)^2}{(2k)!}$.

(c) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k(\log k)(\log\log k)}$.

Q[21](*): Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 - 4}{2n^5 - 6n}$ is convergent or divergent.

Q[22](*): What is the smallest value of N such that the partial sum $\displaystyle\sum_{n=1}^{N} \frac{(-1)^n}{n \cdot 10^n}$ approximates $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 10^n}$ within an accuracy of 10^{-6}?

Q[23](*): It is known that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$ (you don't have to show this). Find N so that S_N, the N^{th} partial sum of the series, satisfies $|\frac{\pi^2}{12} - S_N| \leqslant 10^{-6}$. Be sure to say why your method can be applied to this particular series.

Q[24](*): The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^2}$ converges to some number S (you don't have to prove this). According to the Alternating Series Estimation Theorem, what is the smallest value of n for which the n^{th} partial sum of the series is at most $\frac{1}{100}$ away from S? For this value of n, write out the n^{th} partial sum of the series.

▸▸ Stage 3

Q[25](*): Determine, with explanation, whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{9^n n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\log n}}$

Q[26](*): (a) Prove that $\int_2^{\infty} \frac{x+\sin x}{1+x^2}\,\mathrm{d}x$ diverges.

(b) Explain why you *cannot* conclude that $\sum\limits_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$ diverges from part (a) and the Integral Test.

(c) Determine, with explanation, whether $\sum\limits_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$ converges or diverges.

Q[27](*): Show that $\sum\limits_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converges and find an interval of length 0.05 or less that contains its exact value.

Q[28](*): Suppose that the series $\sum\limits_{n=1}^{\infty} a_n$ converges and that $1 > a_n \geqslant 0$ for all n. Prove that the series $\sum\limits_{n=1}^{\infty} \frac{a_n}{1-a_n}$ also converges.

Q[29](*): Suppose that the series $\sum\limits_{n=0}^{\infty} (1 - a_n)$ converges, where $a_n > 0$ for $n = 0, 1, 2, 3, \cdots$. Determine whether the series $\sum\limits_{n=0}^{\infty} 2^n a_n$ converges or diverges.

Q[30](*): Assume that the series $\displaystyle\sum_{n=1}^{\infty} \frac{na_n - 2n + 1}{n+1}$ converges, where $a_n > 0$ for

$n = 1, 2, \cdots$. Is the following series

$$-\log a_1 + \sum_{n=1}^{\infty} \log\left(\frac{a_n}{a_{n+1}}\right)$$

convergent? If your answer is NO, justify your answer. If your answer is YES, evaluate the sum of the series $-\log a_1 + \sum\limits_{n=1}^{\infty} \log\left(\frac{a_n}{a_{n+1}}\right)$.

Q[31](*): Prove that if $a_n \geqslant 0$ for all n and if the series $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty} a_n^2$ also converges.

3.4▲ Absolute and Conditional Convergence

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](*): Decide whether the following statement is true or false. If false, provide a counterexample. If true provide a brief justification.

If $\sum\limits_{n=1}^{\infty}(-1)^{n+1}b_n$ converges, then $\sum\limits_{n=1}^{\infty} b_n$ also converges.

▸▸ Stage 2

Q[2](*): Determine whether the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{9n+5}$ is absolutely convergent, conditionally convergent, or divergent; justify your answer.

Q[3](*): Determine whether the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{2n+1}}{1+n}$ is absolutely convergent, conditionally convergent, or divergent.

Q[4](*): The series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{1+4^n}{3+2^{2n}}$ either: converges absolutely; converges conditionally; diverges; or none of the above. Determine which is correct.

Q[5](*): Does the series $\sum\limits_{n=5}^{\infty} \frac{\sqrt{n}\cos n}{n^2-1}$ converge conditionally, converge absolutely, or diverge?

▸▸ Stage 3

Q[6](*): Both parts of this question concern the series $S = \sum_{n=1}^{\infty} (-1)^{n-1} 24n^2 e^{-n^3}$.

(a) Show that the series S converges absolutely.

(b) Suppose that you approximate the series S by its fifth partial sum S_5. Give an upper bound for the error resulting from this approximation.

3.5▲ Power Series

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

▸▸ Stage 2

Q[1](*): (a) Find the radius of convergence of the series

$$\sum_{k=0}^{\infty} (-1)^k 2^{k+1} x^k$$

(b) You are given the formula for the sum of a geometric series, namely:

$$1 + r + r^2 + \cdots = \frac{1}{1-r}, \qquad |r| < 1$$

Use this fact to evaluate the series in part (a).

Q[2](*): Find the radius of convergence for the power series $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{10^{k+1}(k+1)!}$

Q[3](*): Find the radius of convergence for the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

Q[4](*): Consider the power series $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{\sqrt{n}}$, where x is a real number. Find the interval of convergence of this series.

Q[5](*): Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{x+1}{3} \right)^n$$

Q[6](*): Find the *interval of convergence* for the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^{4/5}(5^n-4)}.$$

Q[7](*): Find all values x for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^2}$ converges.

Q[8](*): Find the interval of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n}(x-1)^n$.

Q[9](*): Find, with explanation, the radius of convergence and the interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2^n(n+2)}$$

Q[10](*): Find the *interval of convergence* for the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^2 (x-a)^{2n}$ where a is a constant.

Q[11](*): Find the interval of convergence of the following series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(x+1)^k}{k^2 9^k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} a_k (x-1)^k$, where $a_k > 0$ for $k = 1, 2, \cdots$ and $\displaystyle\sum_{k=1}^{\infty} \left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \right) = \frac{a_1}{a_2}$.

Q[12](*): Find a power series representation for $\frac{x^3}{1-x}$.

▸▸ Stage 3

Q[13](*): Determine the values of x for which the series

$$\sum_{n=2}^{\infty} \frac{x^n}{3^{2n} \log n}$$

converges absolutely, converges conditionally, or diverges.

Q[14](*): (a) Find the power–series representation for $\int \frac{1}{1+x^3}\, \mathrm{d}x$ centred at 0 (i.e. in powers of x).

(b) The power series above is used to approximate $\int_0^{1/4} \frac{1}{1+x^3}\, \mathrm{d}x$. How many terms are required to guarantee that the resulting approximation is within 10^{-5} of the exact value? Justify your answer.

Q[15](*): (a) Show that $\sum_{n=0}^{\infty} nx^n = \dfrac{x}{(1-x)^2}$ for $-1 < x < 1$.

(b) Express $\sum_{n=0}^{\infty} n^2 x^n$ as a ratio of polynomials. For which x does this series converge?

Q[16](*): Suppose that you have a sequence $\{b_n\}$ such that the series $\sum_{n=0}^{\infty}(1-b_n)$ converges. Using the tests we've learned in class, prove that the radius of convergence of the power series $\sum_{n=0}^{\infty} b_n x^n$ is equal to 1.

Q[17](*): Assume $\{a_n\}$ is a sequence such that na_n decreases to C as $n \to \infty$ for some real number $C > 0$

(a) Find the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$. Justify your answer carefully.

(b) Find the interval of convergence of the above power series, that is, find all x for which the power series in (a) converges. Justify your answer carefully.

3.6▲ Taylor Series

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

▸▸ Stage 2

Q[1](*): Find the coefficient c_5 of the fifth degree term in the Maclaurin series $\sum_{n=0}^{\infty} c_n x^n$ for e^{3x}.

Q[2](*): The first two terms in the Maclaurin series for $x^2\sin(x^3)$ are $ax^5 + bx^{11}$, where a and b are constants. Find the values of a and b.

Q[3](*): Find the Maclaurin series for $f(x) = \frac{1}{2x-1}$.

Q[4](*): Let $\sum_{n=0}^{\infty} b_n x^n$ be the Maclaurin series for $f(x) = \frac{3}{x+1} - \frac{1}{2x-1}$, i.e. $\sum_{n=0}^{\infty} b_n x^n = \frac{3}{x+1} - \frac{1}{2x-1}$. Find b_n.

Q[5](*): Give the first two nonzero terms in the Maclaurin series for $\displaystyle\int \frac{e^{-x^2}-1}{x}\,\mathrm{d}x$.

Q[6](∗): Find the Maclaurin series for $\int x^4 \arctan(2x)\,\mathrm{d}x$.

Q[7](∗): Express the Taylor series of the function

$$f(x) = \log(1+2x)$$

about $x = 0$ in summation notation.

Q[8](∗): The Maclaurin series for $\arctan x$ is given by

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

which has radius of convergence equal to 1. Use this fact to compute the exact value of the series below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

Q[9](∗): Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.

Q[10](∗): Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{1}{e^k k!}$.

Q[11](∗): Evaluate the sum of the convergent series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\pi^k k!}$.

Q[12](∗): Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n}$.

Q[13](∗): Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n!} e^n$.

Q[14](∗): Suppose that $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{x}{1+3x^3}$ and $f(0) = 1$. Find the Maclaurin series for $f(x)$.

▸▸ Stage 3

Q[15](∗): Let $I(x) = \int_0^x \frac{1}{1+t^4}\,dt$.

(a) Find the Maclaurin series for $I(x)$.

(b) Approximate $I(1/2)$ to within ± 0.0001.

(c) Is your approximation in (b) larger or smaller than the true value of $I(1/2)$? Explain.

Q[16](∗): Using a Maclaurin series, the number $a = 1/5 - 1/7 + 1/18$ is found to be an approximation for $I = \int_0^1 x^4 e^{-x^2}\,\mathrm{d}x$. Give the best upper bound you can for $|I - a|$.

Q[17](*): Find an interval of length 0.0002 or less that contains the number

$$I = \int_0^{\frac{1}{2}} x^2 e^{-x^2}\,\mathrm{d}x$$

Q[18](*): Find the Taylor series for $f(x) = \log(x)$ centred at $a = 2$. Find the interval of convergence for this series.

Q[19](*): Let $I(x) = \int_0^x \frac{e^{-t}-1}{t}\,\mathrm{d}t$.

(a) Find the Maclaurin series for $I(x)$.

(b) Approximate $I(1)$ to within ± 0.01.

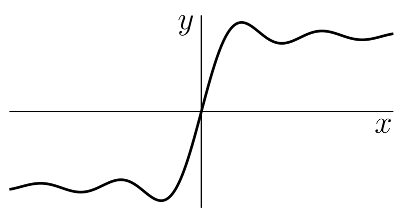
(c) Explain why your answer to part (b) has the desired accuracy.

Q[20](*): The function $\Sigma(x)$ is defined by $\Sigma(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t$.

(a) Find the Maclaurin series for $\Sigma(x)$.

(b) It can be shown that $\Sigma(x)$ has an absolute maximum which occurs at its smallest positive critical point (see the graph of $\Sigma(x)$ below). Find this critical point.

(c) Use the previous information to find the maximum value of $\Sigma(x)$ to within ± 0.01.



Q[21](*): Let $I(x) = \int_0^x \frac{\cos t - 1}{t^2}\,\mathrm{d}t$.

(a) Find the Maclaurin series for $I(x)$.

(b) Use this series to approximate $I(1)$ to within ± 0.01

(c) Is your estimate in (b) greater than $I(1)$? Explain.

Q[22](*): Let $I(x) = \int_0^x \frac{\cos t + t\sin t - 1}{t^2}\,\mathrm{d}t$

(a) Find the Maclaurin series for $I(x)$.

(b) Use this series to approximate $I(1)$ to within ± 0.001

(c) Is your estimate in (b) greater than or less than $I(1)$?

Q[23](*): Define $f(x) = \displaystyle\int_0^x \frac{1-e^{-t}}{t}\,\mathrm{d}t$.

(a) Show that the Maclaurin series for $f(x)$ is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} x^n$.

(b) Use the ratio test to determine the values of x for which the Maclaurin series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} x^n$ converges.

Q[24](∗): Show that $\int_0^1 \frac{x^3}{e^x-1}\,\mathrm{d}x \leqslant \frac{1}{3}$.

Q[25](∗): Use series to evaluate $\lim\limits_{x\to 0} \frac{1-\cos x}{1+x-e^x}$.

Q[26](∗): Evaluate $\lim\limits_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$.

Q[27](∗): Use power series to evaluate

$$\int_0^1 \frac{1-x^2-\cos x}{x^{5/2}}\,\mathrm{d}x$$

with an error less than 0.001.

Q[28](∗): (a) Show that the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ converges absolutely for all real numbers x.

(b) Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n)!}$.

Q[29](∗): Let $\cosh(x) = \frac{e^x+e^{-x}}{2}$.

(a) Find the power series expansion of $\cosh(x)$ about $x_0 = 0$ and determine its interval of convergence.

(b) Show that $3\frac{2}{3} \leqslant \cosh(2) \leqslant 3\frac{2}{3} + 0.1$.

(c) Show that $\cosh(t) \leqslant e^{\frac{1}{2}t^2}$ for all t.

Part II

HINTS TO PROBLEMS

Hints for Exercises 1.1. — Jump to TABLE OF CONTENTS.

H-2: Write out the general formula for the left Riemann sum (with the sum symbol being $\sum\limits_{k=1}^{n}$) and choose a, b and n to make it match the given sum.

H-3: Write out the given sum explicitly without using summation notation. Also write out the first few terms in the sums in the three bullets of Definition 1.1.11 in the CLP 101 notes. Then try to identify $b-a$, and n, followed by "right", "left" or "midpoint" and a.

H-4: The main step is to express the given sum as the right Riemann sum $\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$. Don't be afraid to guess Δx, a and $f(x)$ (review Definition 1.1.11 in the CLP 101 notes). Then write out explicitly $\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don't match.

H-5: The main step is to express the given sum as the right Riemann sum $\sum\limits_{k=1}^{n} f(a+k\Delta x)\Delta x$. Don't be afraid to guess Δx, a and $f(x)$ (review Definition 1.1.11 in the CLP 101 notes). Then write out explicitly $\sum\limits_{k=1}^{n} f(a+k\Delta x)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don't match.

H-6: Review Definition 1.1.11 in the CLP 101 notes.

H-8: The main step is to express the given sum in the form $\sum_{i=1}^{n} f(x_i)\Delta x$. Don't be afraid to guess Δx, x_i (for either a left or a right or a midpoint sum — review Definition 1.1.11 in the CLP 101 notes) and $f(x)$. Then write out explicitly $\sum_{i=1}^{n} f(x_i)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don't match.

H-9: The main step is to express the given sum in the form $\sum_{i=1}^{n} f(x_i)\Delta x$. Don't be afraid to guess Δx, x_i (probably, based on the symbol R_n, for a right Riemann sum — review Definition 1.1.11 in the CLP 101 notes) and $f(x)$. Then write out explicitly $\sum_{i=1}^{n} f(x_i)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don't match.

H-10: Try several different choices of Δx and x_i.

H-11: Sketch the graph of $f(x)$.

H-12: Draw a picture. See Example 1.1.15 in the CLP 101 notes.

H-13: At which time in the interval, for example, $0 \leqslant t \leqslant 0.5$, is the car moving the fastest?

H-14: Sure looks like a Riemann sum.

H-15: For part (b): don't panic!. Just take it one step at a time. The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given identity. The third step is to evaluate the limit $n \to \infty$.

H-16: Don't panic!. Just take it one step at a time. The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given formulas. The third step is to evaluate the limit $n \to \infty$.

H-17: Don't panic!. Just take it one step at a time. The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given formulas. The third step is to evaluate the limit $n \to \infty$.

H-18: You've probably seen this hint before. It is worth repeating. Don't panic!. Just take it one step at a time. The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given formula. The third step is to evaluate the limit $n \to \infty$.

Hints for Exercises 1.2. — Jump to TABLE OF CONTENTS.

H-2: Split the "target integral" up into pieces that can be evaluated using the given integrals.

H-3: Split the "target integral" up into pieces that can be evaluated using the given integrals.

H-4: Split the "target integral" up into pieces that can be evaluated using the given integrals.

H-5: The evaluation of this integral was also the subject of question 5 in Section 1.1. This time try using the method of Example 1.2.6 in the CLP 101 notes.

H-6: Split the integral into a sum of two integrals. Interpret each geometrically.

H-7: Hmmmm. Looks like a complicated integral. It's probably a trick question. Check for symmetries.

H-8: Check for symmetries again.

Hints for Exercises 1.3. — Jump to TABLE OF CONTENTS.

H-2: First find the general antiderivative by guessing and checking.

H-3: Be careful. Two of these make no sense at all.

H-4: Guess a function whose derivative is the integrand.

H-5: Split the given integral up into two integrals. Guess, for each of the two integrals, a function whose derivative is the integrand.

H-7: There is a good way to test where a function is increasing, decreasing, or constant, that also has something to do with topic of this section.

H-8: See Example 1.3.5 in the CLP 101 notes.

H-9: See Example 1.3.5 in the CLP 101 notes.

H-10: See Example 1.3.5 in the CLP 101 notes.

H-11: See Example 1.3.5 in the CLP 101 notes.

H-12: See Example 1.3.6 in the CLP 101 notes.

H-13: What is the title of this section?

H-14: See Example 1.3.6 in the CLP 101 notes.

H-15: See Example 1.3.6 in the CLP 101 notes.

H-16: See Example 1.3.6 in the CLP 101 notes.

H-17: See Example 1.3.6 in the CLP 101 notes.

H-18: Split up the domain of integration.

H-19: Apply $\frac{d}{dx}$ to both sides.

H-20: It is possible to guess an antiderivative for $f'(x)f''(x)$ that is expressed in terms of $f'(x)$.

H-21: When does the car stop? What is the relation between velocity and distance travelled?

H-22: See Example 1.3.5 in the CLP 101 notes. For the absolute maximum part of the question, study the sign of $f'(x)$.

H-23: See Example 1.3.5 in the CLP 101 notes. For the "minimum value" part of the question, study the sign of $f'(x)$.

H-24: See Example 1.3.5 in the CLP 101 notes. For the "maximum" part of the question, study the sign of $F'(x)$.

H-25: Review the definition of the definite integral and in particular Definitions 1.1.9 and 1.1.11 in the CLP 101 notes.

H-26: Review the definition of the definite integral and in particular Definitions 1.1.9 and 1.1.11 in the CLP 101 notes.

H-27: In general, the equation of the tangent line to the graph of $y = f(x)$ at $x = a$ is $y = f(a) + f'(a)(x - a)$.

Hints for Exercises 1.4. — Jump to TABLE OF CONTENTS.

H-2: What is the derivative of the argument of the cosine?

H-3: What is the title of the current section?

H-4: What is the derivative of $x^3 + 1$?

H-5: What is the derivative of $\log x$?

H-6: What is the derivative of $1 + \sin x$?

H-7: $\cos x$ is the derivative of what?

H-8: What is the derivative of the exponent?

H-9: What is the derivative of the argument of the square root?

H-10: There is a short slightly sneaky method — guess an antiderivative — and a really short still more sneaky method.

H-11: Review the definition of the definite integral and in particular Definitions 1.1.9 and 1.1.11 in the CLP 101 notes.

H-12: Review the definition of the definite integral and in particular Definitions 1.1.9 and 1.1.11 in the CLP 101 notes.

H-13: Review the definition of the definite integral and in particular Definitions 1.1.9 and 1.1.11 in the CLP 101 notes.

Hints for Exercises 1.5. — Jump to TABLE OF CONTENTS.

H-1: Draw a sketch first.

H-2: Draw a sketch first.

H-3: Draw a sketch first.

H-4: Draw a sketch first.

H-5: See Example 1.5.2 in the CLP 101 notes.

H-6: Part of the job is to determine whether $y = x$ lies above or below $y = 3x - x^2$.

H-7: Guess the intersection points by trying small integers.

H-8: Draw a sketch first. You can also exploit a symmetry of the region to simplify your solution.

H-9: Figure out where the two curves cross. To determine which curve is above the other, try evaluating $f(x)$ and $g(x)$ for some simple value of x. Alternatively, consider x very close to zero.

H-10: Think about whether it will easier to use vertical strips or horizontal strips.

H-11: You are asked for the area, not the signed area. Be very careful about signs.

H-12: You are asked for the area, not the signed area. Draw a sketch of the region and be very careful about signs.

H-13: You have to determine whether

- the curve $y = f(x) = x\sqrt{25 - x^2}$ lies above the line $y = g(x) = 3x$ for all $0 \leqslant x \leqslant 4$ or
- the curve $y = f(x)$ lies below the line $y = g(x)$ for all $0 \leqslant x \leqslant 4$ or
- $y = f(x)$ and $y = g(x)$ cross somewhere between $x = 0$ and $x = 4$.

One way to do so is to study the sign of $f(x) - g(x) = x\big(\sqrt{25 - x^2} - 3\big)$.

Hints for Exercises 1.6. — Jump to TABLE OF CONTENTS.

H-1: Draw sketchs. The mechanically easiest way to answer part (b) uses the method of cylindrical shells, which we have not covered. The method of washers, which we do know about also works, but requires you have more patience and also to have a good idea what the specified region looks like. Look at your sketch very careful when identifying the ends of your horizontal strips.

H-2: Draw sketchs.

H-3: Draw a sketch.

H-4: (a) Draw a sketch. (b) Draw a sketch. (c) Draw a sketch. See a pattern?

H-5: Sketch the region.

H-7: Sketch the region first.

H-8: You can save yourself quite a bit of work by interpretting the integral as the area of a known geometric figure.

H-9: See Example 1.6.3 in the CLP 101 notes.

H-10: See Example 1.6.5 in the CLP 101 notes.

H-11: Sketch the region. To find where the curves intersect, look at where $\cos(\frac{x}{2})$ and $x^2 - \pi^2$ both vanish.

H-12: See Example 1.6.6 in the CLP 101 notes.

H-13: See Example 1.6.6 in the CLP 101 notes.

H-14: See Example 1.6.1 in the CLP 101 notes.

H-17: The mechanically easiest way to answer part (b) uses the method of cylindrical shells, which we have not covered. The method of washers, which we do know about also works, but requires you have enough patience and also to have a good idea what $\mathcal{R}$ looks like. So it is crucial to first sketch $\mathcal{R}$. Then be very careful in identifying the left end of your horizontal strips.

H-18: The mechanically easiest way to answer part (c) uses the method of cylindrical shells, which we have not covered. The method of washers, which we do know about also works, but requires you have enough patience and also to have a good idea what R looks like. So look at the sketch in part (a) very careful when identifying the left end of your horizontal strips.

H-19: Note that the curves cross. Be very careful. The area of this region was found in Problem 11 of Section 1.5. It would be useful to review that problem.

Hints for Exercises 1.7. — Jump to TABLE OF CONTENTS.

H-1: What is the title of this section?

H-2: See Example 1.7.7 in the CLP 101 notes.

H-3: See Example 1.7.5 in the CLP 101 notes.

H-4: See Example 1.7.5 in the CLP 101 notes.

H-5: You know, or can easily look up, the derivative of arccos. You also know the title of this section.

H-6: See Example 1.7.9 in the CLP 101 notes.

H-7: What is the title of the current section?

H-8: See Examples 1.7.9 and 1.6.5 in the CLP 101 notes.

H-9: Think, first, about how to get rid of the square root in the argument of f'', and, second, how to convert f'' into f'. Note that you are told that $f'(2) = 4$ and $f(0) = 1$, $f(2) = 3$.

Hints for Exercises 1.8. — Jump to TABLE OF CONTENTS.

H-1: See Example 1.8.6 in the CLP 101 notes.

H-2: See Example 1.8.6 in the CLP 101 notes.

H-3: For practice, try doing this in two ways, with different substitutions.

H-4: See Example 1.8.14 in the CLP 101 notes.

H-5: See Example 1.8.7 in the CLP 101 notes.

H-6: See Example 1.8.16 in the CLP 101 notes.

Hints for Exercises 1.9. — Jump to TABLE OF CONTENTS.

H-1: What is this section about?

H-2: Do question 1 in this section first.

H-4: Do question 1 in this section first.

H-5: Do question 1 in this section first.

H-6: Do question 1 in this section first.

H-7: Do question 1 in this section first.

H-8: See Example 1.9.3 in the CLP 101 notes.

H-9: Complete the square.

H-10: In part (a) you are asked to integrate an even power of $\cos x$. For part (b) you can use a trigonometric substitution to reduce the integral of part (b) almost to the integral of part (a).

H-11: Do question 1 in this section first.

Hints for Exercises 1.10. — Jump to TABLE OF CONTENTS.

H-1: Review Example 1.10.1 in the CLP 101 notes. Is the "Algebraic Method" or the "Sneaky Method" going to be easier?

H-2: Review (1.10.7) through (1.10.11) of the CLP 101 notes. Be careful to *fully* factor the denominator.

H-3: What is the title of this section?

H-4: You can save yourself some work in developing your partial fraction expansion by renaming x^2 to y.

H-5: Review Steps 3 (particularly the "Sneaky Method") and 4 of Example 1.10.3 in the CLP 101 notes.

H-6: Review Steps 3 (particularly the "Sneaky Method") and 4 of Example 1.10.3 in the CLP 101 notes.

H-7: Fill in the blank: the integrand is a ____ function.

H-8: The integrand is yet another ____ function.

Hints for Exercises 1.11. — Jump to TABLE OF CONTENTS.

H-1: Draw a sketch.

H-2: See §1.11.1 in the CLP 101 notes.

H-3: See §1.11.3 in the CLP 101 notes.

H-4: See §1.11.2 in the CLP 101 notes. To set up the volume integral, see Example 1.6.6 in the CLP 101 notes.

H-5: See §1.11.3 in the CLP 101 notes. To set up the volume integral, see Example 1.6.6 in the CLP 101 notes.

H-8: The main step is to find an allowed K. It is *not necessary* to find the smallest possible allowed K.

H-9: The main step is to find M. This question is unusual in that its wording requires you to find the smallest possible allowed M.

H-10: The main steps in part (b) are to find the smallest possible values of K and L.

H-16: See §1.11.3 in the CLP 101 notes. To set up the volume integral, see Example 1.6.2 in the CLP 101 notes.

H-17: See Example 1.11.13 in the CLP 101 notes.

H-18: See Example 1.11.15 in the CLP 101 notes.

H-19: See Example 1.11.15 in the CLP 101 notes.

H-20: See Example 1.11.14 in the CLP 101 notes. You may also want to review the fundamental theorem of calculus and, in particular, Example 1.3.5 in the CLP 101 notes.

H-21: See Example 1.11.14 in the CLP 101 notes.

Hints for Exercises 1.12. — Jump to TABLE OF CONTENTS.

H-1: Read both the question and Theorem 1.12.17 in the CLP 101 notes *very* carefully.

H-2: Review Example 1.12.8 in the CLP 101 notes.

H-3: First: is the integrand unbounded, and if so, where? Second: when evaluating integrals, always check to see if you can use a simple substitution, before trying a complicated procedure, like partial fractions.

H-4: Is the integrand bounded?

H-5: See Example 1.12.21 in the CLP 101 notes.

H-6: Which of the two terms in the denominator is more important when $x \approx 0$? Which one is more important when x is very large?

H-7: Which of the two terms in the denominator is more important when x is very large?

H-8: Which of the two terms in the denominator is more important when $x \approx 0$? Which one is more important when x is very large?

H-9: What are the "problem x's" for this integral? Get a simple approximation to the integrand near each.

H-10: What is the limit of the integrand when $x \to 0$?

H-11: First find a t so that the error introduced by approximating $\int_0^\infty \frac{e^{-x}}{1+x}\, \mathrm{d}x$ by $\int_0^t \frac{e^{-x}}{1+x}\, \mathrm{d}x$ is at most $\frac{1}{2}10^{-4}$

Hints for Exercises 1.13. — Jump to TABLE OF CONTENTS.

H-1: The integrand is a rational function. So it is possible to use partial fractions. But there is a much easier way!

H-2: You should prepare your own personal internal list of integration techniques ordered from easiest to hardest. You should have associated to each technique your own personal list of signals that you use to decide when the technique is likely to be useful.

H-3: You should prepare your own personal internal list of integration techniques ordered from easiest to hardest. You should have associated to each technique your own personal list of signals that you use to decide when the technique is likely to be useful.

H-4: For the integral of secant, see See §1.8.3 or Example 1.10.5 in the CLP 101 notes.

H-5: Part (b) can be done by inspection – use a little highschool geometry! Part (d) is an improper integral.

H-6: Note that in part (d), $\tan^{-1} x$ means the arctangent of x, not 1 divided by $\tan x$. This is standard notation.

H-8: For part (a), see Example 1.7.11 in the CLP 101 notes. For part (d), see Example 1.10.4 in the CLP 101 notes.

H-9: For part (b), first complete the square in the denominator. For part (d) use a simple substitution.

H-10: For part (b), complete the square in the denominator. You can save some work by first comparing the derivative of the denominator with the numerator.

H-11: For part (b), the numerator is the derivative of a function that appears in the denominator.

H-12: For part (a), can you convert this into a partial fractions integral? For part (b), start by completing the square inside the square root.

H-13: For part (b), the numerator is the derivative of a function that is embedded in the denominator.

H-14: For part (a), split the integral in two. One part may be evaluated just by interpretting it appropriately, without doing any integration at all. For part (c), multiply both the numerator and denominator by e^x and then make a substitution.

H-15: For part (a), the substitution $u = \log x$ goves an integral that you have seen before.

Hints for Exercises 2.1. — Jump to TABLE OF CONTENTS.

H-1: See Example 2.1.2 in the CLP 101 notes.

H-2: Review Definition 2.1.1 in the CLP 101 notes.

H-3: Be careful about the units.

H-4: Suppose that the bucket is a distance y above the ground. How much work is required to raise it an additional height $\mathrm{d}y$.

H-5: See Example 2.1.4 in the CLP 101 notes.

Hints for Exercises 2.2. — Jump to TABLE OF CONTENTS.

H-1: Review the definition of "average value" in §2.2 of the CLP 101 notes.

H-5: Review the definition of "average value" in §2.2 of the CLP 101 notes.

H-6: Review the definition of "average value" in §2.2 of the CLP 101 notes.

H-7:

H-8: Review the definition of "average value" in §2.2 of the CLP 101 notes, and the trapezoidal rule in §1.11.2 of the CLP 101 notes

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Hints for Exercises 2.3. — Jump to TABLE OF CONTENTS.

H-1: Which method involves more work: horizontal strips or vertical strips?

H-2: Review (2.3.1) and (2.3.2) in the CLP 101 notes.

H-3: Review (2.3.1) and (2.3.2) in the CLP 101 notes. For practice, do the computation twice — once with horizontal strips and once with vertical strips. Watch for improper integrals.

H-4: See Example 2.3.2 in the CLP 101 notes.

H-5: See Example 2.3.1 in the CLP 101 notes.

H-6: See Example 2.3.2 in the CLP 101 notes.

H-7: See Example 2.3.2 in the CLP 101 notes.

H-8: See Example 2.3.1 in the CLP 101 notes.

H-9: You can save quite a bit of work by, firstly, exploiting symmetry and, secondly, thinking about whether it is more efficient to use vertical strips or horizontal strips.

H-10: Sketch the region. You can save quite a bit of work by exploiting symmetry.

H-11: Draw a sketch. In part (b) be careful about the equation of the right hand boundary of A.

H-12: Draw a sketch.

Hints for Exercises 2.4. — Jump to TABLE OF CONTENTS.

H-4: Simplify the equation.

H-5: Be careful with the arbitrary constant.

H-7: Be careful about signs.

H-8: Be careful about signs.

H-10: Move the y from the left hand side to the right hand side. Be careful about the signs. Remember that we need $y = -1$ when $x = 1$.

H-11: The unknown function $f(x)$ satisfies an equation that involves the derivative of f.

H-12: Try guessing the partial fractions expansion of $\frac{1}{x(x+1)}$.

H-14: The general solution to the differential equation will contain the constant k and one other constant. They are determined by the data given in the question.

H-15: What is the velocity of the mass at its highest point? The answer will depend on the (unspecified) constants v_0, m, g and k.

H-16: The general solution to the differential equation will contain the constant k and one other constant. They are determined by the data given in the question.

H-17: Review the method of integration by partial fractions and in particular Example 1.10.5 in the CLP 101 notes.

H-18: Review the method of integration by partial fractions and in particular Example 1.10.5 in the CLP 101 notes. Be careful about signs.

H-19: The general solution to the differential equation will contain a constant of proportionality and one other constant. They are determined by the data given in the question.

H-20: You do not need to know anything about investing or continuous compounding to do this problem. You are given the differential equation explicitly. The whole first sentence is just window dressing.

H-21: Again, you do not need to know anything about investing to do this problem. You are given the differential equation explicitly.

H-22: Differentiate the given integral equation.

H-23: Suppose that in a very short time interval $\mathrm{d}t$, the height of water in the tank changes by $\mathrm{d}h$ (which is negative). Express in two different ways the volume of water that has escaped during this time interval. Equating the two gives the needed differential equation.

H-24: Sketch the mercury in the tank at time t, when it has height h, and also at time $t + \mathrm{d}t$, when it has height $h + \mathrm{d}h$ (with $\mathrm{d}h < 0$). The difference between those two volumes is the volume of (essentially) a disk of thickness $-\mathrm{d}h$. Figure out the radius and then the volume of that disk. This volume has to be the same as the volume of mercury that left through the hole in the bottom of the sphere. Toricelli's law tells you what the volume is. Setting the two volumes equal to each other gives the differential equation which determines $h(t)$.

H-25: The fundamental theorem of calculus will be useful in part (b).

H-26: For any $p > 0$, determine first $y(t)$ and then the times at which $y = 2$, $y = 1$ and $y = 0$. The condition that "the top half takes exactly the same amount of time to drain as the bottom half" then gives an equation that determines p.

Hints for Exercises 3.1. — Jump to TABLE OF CONTENTS.

H-1: Simplify a_k. Also note that you are asked about the sequence $\{a_k\}$, not about the series $\sum_{k=1}^{\infty} a_k$.

H-2: What happens to $\frac{1}{n}$ as n gets very big?

H-3: What happens to $\frac{1}{n}$ as n gets very big?

H-4: This is trickier than it looks. Write $\frac{1}{n} = x$ and look at the limit as $x \to 0$.

H-5: Consider $a_n - L$.

Hints for Exercises 3.2. — Jump to TABLE OF CONTENTS.

H-1: You should recognize this series.

H-2: You should recognize this series.

H-3: When you see $\sum_k \Big(\cdots k \cdots \quad - \quad \cdots k+1 \cdots \Big)$, you should immediately think "telescoping series".

H-4: When you see $\sum_k \Big(\cdots n \cdots \quad - \quad \cdots n+1 \cdots \Big)$, you should immediately think "telescoping series". But be careful not to jump to conclusions — evaluate the n^{th} partial sum explicitly.

H-5: Review Definition 3.2.3 in the CLP 101 notes.

H-6: This is a special case of a general series whose sum we know.

H-7: Review Example 3.2.5 in the CLP 101 notes.

H-8: Review Example 3.2.5 in the CLP 101 notes.

H-9: Review Example 3.2.5 in the CLP 101 notes.

H-10: Split the series into two parts.

H-11: Split the series into two parts.

H-12: Split the series into two parts.

Hints for Exercises 3.3. — Jump to TABLE OF CONTENTS.

H-1: Always try the divergence test first (in your head).

H-2: Review Theorem 3.3.11 and Example 3.3.12 in the CLP 101 notes.

H-3: Don't jump to conclusions about properties of the a_n's.

H-4: Which test should you always try first (in your head)?

H-5: Review the integral test, which is Theorem 3.3.5 in the CLP 101 notes.

H-6: Review the integral test, which is Theorem 3.3.5 in the CLP 101 notes.

H-7: Review the integral test.

H-8: Try the integral test.

H-9: Review Example 3.3.9 in the CLP 101 notes.

H-10: What does the summand look like when k is very large?

H-11: What does the summand look like when n is very large?

H-12: What does the summand look like when n is very large?

H-13: This is a trick question. Be sure to verify *all* of the hypotheses of any convergence test you apply.

H-14: See Example 3.3.7 in the CLP101 notes.

H-15: $\cos(n\pi)$ is a sneaky way to write $(-1)^n$.

H-16: What is the behaviour for large k?

H-17: What is the behaviour for large n/m?

H-18: You should recognize this series.

H-19: Build up your own personal list of convergence tests ordered from easiest to hardest. With each test you should also build up a list of signals that you can use to guess whether or not it is useful to apply that test.

H-21: What does the summand look like when n is very large?

H-22: Review the alternating series test, which is given in Theorem 3.3.14 in the CLP 101 notes.

H-23: Review the alternating series test, which is given in Theorem 3.3.14 in the CLP 101 notes.

H-24: Review the alternating series test, which is given in Theorem 3.3.14 in the CLP 101 notes.

H-26: For part (a), see Example 1.12.23 in the CLP 101 notes.

For part (b), review Theorem 3.3.5 in the CLP 101 notes.

For part (c), see Example 3.3.12 in the CLP 101 notes.

H-27: Review the integral test, which is Theorem 3.3.5 in the CLP 101 notes.

H-28: What does the fact that the series $\sum\limits_{n=0}^{\infty} a_n$ converges guarantee about the behavior of a_n for large n?

H-29: What does the fact that the series $\sum\limits_{n=0}^{\infty} (1-a_n)$ converges guarantee about the behavior of a_n for large n?

H-30: What does the fact that the series $\sum\limits_{n=1}^{\infty} \dfrac{na_n - 2n + 1}{n+1}$ converges guarantee about the behavior of a_n for large n?

H-31: What does the fact that the series $\sum_{n=1}^{\infty} a_n$ converges guarantee about the behavior of a_n for large n? When is $x^2 \leqslant x$?

Hints for Exercises 3.4. — Jump to TABLE OF CONTENTS.

H-1: What is this section about?

H-3: Be careful about the signs.

H-4: Does the alternating series test really apply?

H-5: What does the summand look like when n is very large?

H-6: For part (a), replace n by x in the absolute value of the summand. Can you integrate the resulting function?

Hints for Exercises 3.5. — Jump to TABLE OF CONTENTS.

H-1: Review the discussion immediately following Definition 3.5.1 in the CLP 101 notes.

H-2: Review the discussion immediately following Definition 3.5.1 in the CLP 101 notes.

H-3: Review the discussion immediately following Definition 3.5.1 in the CLP 101 notes.

H-4: See Example 3.5.11 in the CLP 101 notes.

H-5: See Example 3.5.11 in the CLP 101 notes.

H-11: Start part (b) by computing the partial sums of $\sum_{k=1}^{\infty}\left(\frac{a_k}{a_{k+1}}-\frac{a_{k+1}}{a_{k+2}}\right)$

H-12: You should know a power series representation for $\frac{1}{1-x}$. Use it.

H-13: $n \geqslant \log n$ for all $n \geqslant 1$.

H-14: See Example 3.5.21 in the CLP 101 notes. For part (b), review §3.3.4 in the CLP 101 notes.

H-15: You know the geometric series expansion of $\frac{1}{1-x}$. What (calculus) operation(s) can you apply to that geometric series to convert it into the given series?

H-16: First show that the fact that the series $\sum_{n=0}^{\infty}(1-b_n)$ converges guarantees that $\lim_{n\to\infty} b_n = 1$.

H-17: What does a_n look like for large n?

Hints for Exercises 3.6. — Jump to TABLE OF CONTENTS.

H-1: You should know the Maclaurin series for e^x. Use it.

H-2: You should know the Maclaurin series for $\sin x$. Use it.

H-3: You should know the Maclaurin series for $\frac{1}{1-x}$. Use it.

H-4: You should know the Maclaurin series for $\frac{1}{1-x}$. Use it.

H-5: You should know the Maclaurin series for e^x. Use it.

H-6: You should know the Maclaurin series for $\arctan(x)$. Use it.

H-7: Review Example 3.5.20 in the CLP 101 notes.

H-8: $\frac{\pi}{2\sqrt{3}}$

H-9: There is an important Taylor series, one of the series in Theorem 3.6.5 of the CLP 101 notes, that looks a lot like the given series.

H-10: There is an important Taylor series, one of the series in Theorem 3.6.5 of the CLP 101 notes, that looks a lot like the given series.

H-11: There is an important Taylor series, one of the series in Theorem 3.6.5 of the CLP 101 notes, that looks a lot like the given series. Be careful about the limits of summation.

H-12: There is an important Taylor series, one of the series in Theorem 3.6.5 of the CLP 101 notes, that looks a lot like the given series.

H-13: Split the series into a sum of two series. There is an important Taylor series, one of the series in Theorem 3.6.5 of the CLP 101 notes, that looks a lot like each of the two series.

H-14: You should know the Maclaurin series for $\frac{1}{1-x}$. Use it.

H-15: See Example 3.5.21 in the CLP 101 notes. For parts (b) and (c), review §3.3.4 in the CLP 101 notes.

H-16: Look at the signs of successive terms in the series.

H-17: The magic word is "series".

H-18: You know the Maclaurin series for $\log(1+y)$. Use it! Remember that you are asked for a series expansion in powers of $x-2$. So you want y to be some constant times $x-2$.

H-19: See Example 3.6.10 in the CLP 101 notes. For parts (b) and (c), review §3.3.4 in the CLP 101 notes.

H-20: See Example 3.6.10 in the CLP 101 notes. For part (b), review the fundamental theorem of calculus in §1.3 of the CLP 101 notes. For part (c), review §3.3.4 in the CLP 101 notes.

H-21: See Example 3.6.10 in the CLP 101 notes. For parts (b) and (c), review §3.3.4 in the CLP 101 notes.

H-22: See Example 3.6.10 in the CLP 101 notes. For parts (b) and (c), review §3.3.4 in the CLP 101 notes.

H-24: Use the Maclaurin series for e^x.

H-25: See Example 3.6.17 in the CLP 101 notes

H-26: See Example 3.6.17 in the CLP 101 notes

H-28: Can you think of a way to eliminate the odd terms from $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$?

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Part III

ANSWERS TO PROBLEMS

Answers to Exercises 1.1 — Jump to TABLE OF CONTENTS

A-1: $\int_{-1}^{7} f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-1+\frac{8i}{n}\right)\frac{8}{n}$

A-2: $n = 4$, $a = 2$ and $b = 6$

A-3: It is a midpoint Riemann sum for f on the interval $[1, 5]$ with $n = 4$. It is also a left Riemann sum for f on the interval $[1.5, 5.5]$ with $n = 4$. It is also a right Riemann sum for f on the interval $[0.5, 4.5]$ with $n = 4$.

A-4: $f(x) = \sin^2(2 + x)$ and $b = 4$

A-5: $f(x) = x\sqrt{1-x^2}$

A-6: $\sum_{i=1}^{50}\left(5+\left(i-1/2\right)\frac{1}{5}\right)^8\frac{1}{5}$

A-7: 54

A-8: $\int_0^3 e^{-x/3}\cos(x)\,\mathrm{d}x$

A-9: $\int_0^1 xe^x\,\mathrm{d}x$

A-10: Here are three ways: $\int_0^2 e^{-1-x}\,\mathrm{d}x$, $\int_1^3 e^{-x}\,\mathrm{d}x$, $2\int_0^1 e^{-1-2x}\,\mathrm{d}x$.

A-11: $\int_0^3 f(x)\,\mathrm{d}x = 2.5$

A-12: 5

A-13: 53m

A-14: (a) There are many possible answers. Two are $\int_{-2}^{0}\sqrt{4-x^2}\,\mathrm{d}x$ and $\int_0^2\sqrt{4-(-2+x)^2}\,\mathrm{d}x$. (b) π

A-15: (a) 30 (b) $41\frac{1}{4}$

A-16: $\frac{56}{3}$

A-17: 6

A-18: 12

Answers to Exercises 1.2 — Jump to TABLE OF CONTENTS

A-1: (a) False. For example, the function

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geqslant 0 \end{cases}$$

provides a counterexample.

(b) False. For example, the functions $f(x) = g(x) = x$ provide a counterexample.

(c) False. For example, the functions $f(x) = g(x) = x$ provide a counterexample.

A-2: -21

A-3: -6

A-4: 20

A-5: 5

A-6: $20 + 2\pi$

A-7: 0

A-8: 0

Answers to Exercises 1.3 — Jump to TABLE OF CONTENTS

A-1: $e^2 - e^{-2}$

A-2: $F(x) = \frac{x^4}{4} + \frac{1}{2}\cos 2x + \frac{1}{2}$.

A-3: (a) True (b) False (c) False, unless $\int_a^b f(x)\,\mathrm{d}x = \int_a^b xf(x)\,\mathrm{d}x = 0$.

A-4: $5 - \cos 2$

A-5: 2

A-6: $F'(x) = \log(3)$ $G'(x) = -\log(3)$

A-7: $f(x)$ is increasing when $-\infty < x < 1$ and when $2 < x < \infty$.

A-8: $F'(x) = -\frac{\sin x}{\cos^3 x + 6}$

A-9: $4x^3 e^{(1+x^4)^2}$

A-10: $\left(\sin^6 x + 8\right)\cos x$

A-11: $F'(1) = 3e^{-1}$

A-12: $\dfrac{\sin u}{1 + \cos^3 u}$

A-13: $f(4) = 4\pi$

A-14: (a) $(2x+1)e^{-x^2}$ (b) $x = -1/2$

A-15: $e^{\sin x} - e^{\sin(x^4 - x^3)}\left(4x^3 - 3x^2\right)$

A-16: $-2x\cos\left(e^{-x^2}\right) - 5x^4\cos\left(e^{x^5}\right)$

A-17: $e^x\sqrt{\sin(e^x)} - \sqrt{\sin(x)}$

A-18: 14

A-19: $f(x) = 2x$

A-20: $\frac{5}{2}$

A-21: 45 m

A-22: $f'(x) = (2-2x)\log\left(1+e^{2x-x^2}\right)$ and $f(x)$ achieves its absolute maximum at $x=1$, because $f(x)$ is increasing for $x<1$ and decreasing for $x>1$.

A-23: The minimum is $\int_0^{-1}\frac{dt}{1+t^4}$. As x runs from $-\infty$ to ∞, the function $g(x)=\int_0^x\frac{dt}{1+t^4}$ decreases until x reaches 1 and then increases all $x>1$. So the minimum is achieved for $x=1$. At $x=1$, $x^2-2x=-1$.

A-24: F achieves its maximum value at $x=\pi$.

A-25: 2

A-26: $\log 2$

A-27: (a) $3x^2\int_0^{x^3+1} e^{t^3}\,dt + 3x^5 e^{(x^3+1)^3}$ (b) $y=-3(x+1)$

Answers to Exercises 1.4 — Jump to TABLE OF CONTENTS

A-1: $\int_0^1 \frac{f(u)}{\sqrt{1-u^2}}\,du$

A-2: $\frac{1}{2}\big(\sin(e)-\sin(1)\big)$

A-3: $\frac{1}{3}$

A-4: $-\frac{1}{300(x^3+1)^{100}}+C$

A-5: $\log 4$

A-6: $\log 2$

A-7: $4/3$

A-8: e^6-1

A-9: $\frac{1}{3}(4-x^2)^{3/2}+C$

A-10: 0

A-11: $\frac{1}{2}[\cos 1-\cos 2]\approx 0.478$

A-12: $\frac{1}{2}\sin 1$

A-13: $\frac{1}{3}[2\sqrt{2}-1]\approx 0.609$

Answers to Exercises 1.5 — Jump to TABLE OF CONTENTS

A-1: $\int_0^{\sqrt{2}}\left[x-(x^3-x)\right]\,dx$

A-2: $\int_{-3/2}^{4}\left[\frac{4}{5}(6-y^2)+2y\right]\,dy$

A-3: $\int_0^{4a}\left[\sqrt{4ax}-\frac{x^2}{4a}\right]\,dx$

A-4: $\int_1^{25} \left[-\frac{1}{12}(x+5) + \frac{1}{2}\sqrt{x} \right] \mathrm{d}x$

A-5: $\frac{1}{8}$

A-6: $\frac{4}{3}$

A-7: $\frac{5}{3} - \frac{1}{\log 2}$

A-8: $2\left[\frac{4}{\pi} - \frac{1}{2}\right]$

A-9: $\frac{20}{9}$

A-10: $\frac{1}{6}$

A-11: $2\left[\pi - \frac{1}{4}\pi^2\right]$

A-12: $\frac{31}{6} = 5.1\dot{6}$

A-13: $\frac{26}{3}$

Answers to Exercises 1.6 — Jump to TABLE OF CONTENTS

A-1: (a) $\pi \int_0^3 x e^{2x^2} \, \mathrm{d}x$

(b) $\int_0^1 \pi\left[\left(3+\sqrt{y}\right)^2 - \left(3-\sqrt{y}\right)^2\right]\mathrm{d}y + \int_1^4 \pi\left[\left(5-y\right)^2 - \left(3-\sqrt{y}\right)^2\right]\mathrm{d}y$

A-2: (a) $\int_{-1}^1 \pi\left[\left(5-4x^2\right)^2 - \left(2-x^2\right)^2\right] \mathrm{d}x$ (b) $\int_{-1}^0 \pi\left[\left(5+\sqrt{y+1}\right)^2 - \left(5-\sqrt{y+1}\right)^2\right] \mathrm{d}y$

A-3: $\pi \int_{-2}^2 \left[\left(9-x^2\right)^2 - \left(x^2+1\right)^2\right] \mathrm{d}x$

A-4: (a) $\int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a}\right] \mathrm{d}x$ (b) $\int_{-1}^1 \pi\left[\left(5-4x^2\right)^2 - \left(2-x^2\right)^2\right] \mathrm{d}x$

(c) $\int_{-1}^0 \pi\left[\left(5+\sqrt{y+1}\right)^2 - \left(5-\sqrt{y+1}\right)^2\right] \mathrm{d}y = \int_{-1}^0 20\pi\sqrt{y+1}\,\mathrm{d}y$

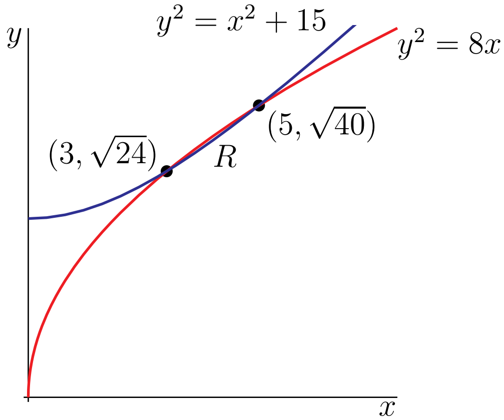
A-5: $\dfrac{\pi}{4}\left(e^{2a^2} - 1\right)$

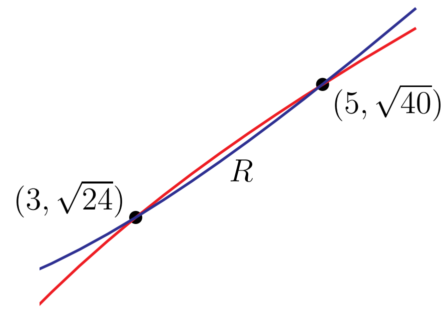
A-6: $\pi\left(\frac{17e^{18} - 4373}{36}\right)$

A-7: $\pi\left[\frac{38}{3} - \frac{514}{3^4}\right] = \pi\frac{512}{81}$

A-8: (a) $8\pi \int_{-1}^1 \sqrt{1-x^2}\,\mathrm{d}x$ (b) $4\pi^2$

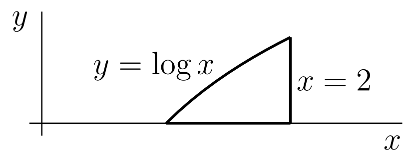
A-9: (a) The region R is the region between the blue and red curves, with $3 \leqslant x \leqslant 5$, in the figures below.





(b) $\frac{4}{3}\pi \approx 4.19$

A-10: (a) The region R is sketched below.



(b) $\pi\left[4\log 2 - \frac{3}{2}\right] \approx 3.998$

A-11: $\pi^2 + 8\pi^3 + \frac{8\pi^6}{5}$

A-12: $\frac{8}{3}$

A-13: $\frac{256\times 8}{15} = 136.5\dot{3}$

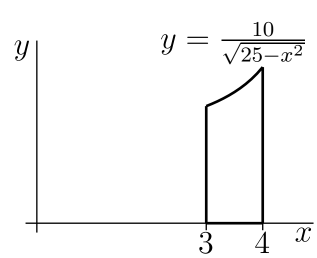
A-14: $\frac{28}{3}\pi h$

A-15: (a) $\frac{9}{2}$ (b) $\pi\int_{-1}^{2}\left[(4-x)^2 - \left(1+(x-1)^2\right)^2\right]\,\mathrm{d}x$

A-16: (a) $\frac{\pi}{2} - 1$ (b) $\frac{\pi^2}{2} - \pi \approx 1.793$

A-17: (a) $V_1 = \frac{4}{3}\pi c^2$ (b) $V_2 = \frac{\pi c}{3}\left[4\sqrt{2} - 2\right]$ (c) $c = 0$ or $c = \frac{1}{2}[2^{3/2} - 1]$

A-18: (a) The region R is



(b) $10\pi\log\frac{9}{4} = 20\pi\log\frac{3}{2}$ (c) 20π

A-19:
$\int_{\pi/2}^{\pi}\pi\left[(5+\pi\sin x)^2 - (5+2\pi-2x)^2\right]\,\mathrm{d}x + \int_{\pi}^{3\pi/2}\pi\left[(5+2\pi-2x)^2 - (5+\pi\sin x)^2\right]\,\mathrm{d}x$

Answers to Exercises 1.7 — Jump to TABLE OF CONTENTS

A-1: $\frac{x^2 \log x}{2} - \frac{x^2}{4} + C$

A-2: $-\frac{\log x}{6x^6} - \frac{1}{36x^6} + C$

A-3: π

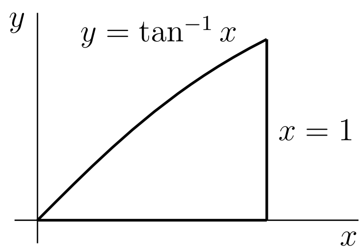
A-4: $\frac{\pi}{2} - 1$

A-5: $y \cos^{-1} y - \sqrt{1-y^2} + C$

A-6: $2y^2 \arctan(2y) - y + \frac{1}{2}\arctan(2y) + C$

A-7: (a) See the solution for the derivation. (b) $\frac{35}{256}\pi \approx 0.4295$

A-8: (a) $\frac{\pi}{4} - \frac{\ln 2}{2}$



(b) $\frac{\pi^2}{2} - \pi$

A-9: 12

Answers to Exercises 1.8 — Jump to TABLE OF CONTENTS

A-1: $\sin x - \frac{\sin^3 x}{3} + C$

A-2: $\frac{\sin^{37} t}{37} - \frac{\sin^{39} t}{39} + C$

A-3: $\frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C$

A-4: $\dfrac{\tan^{49} x}{49} + \dfrac{\tan^{47} x}{47} + C$

A-5: $\frac{\pi}{2}$

A-6: (a) See the solution for the derivation. (b) $\frac{13}{15} - \frac{\pi}{4} \approx 0.0813$

Answers to Exercises 1.9 — Jump to TABLE OF CONTENTS

A-1: (a) $x = \frac{4}{3}\sec\theta$ (b) $x = \frac{1}{2}\sin\theta$ (c) $x = 5\tan\theta$

A-2: $\frac{1}{4}\frac{x}{\sqrt{x^2+4}} + C$

A-3: $\frac{1}{2\sqrt{5}}$

A-4: $\log\left|\sqrt{1+\frac{x^2}{25}}+\frac{x}{5}\right|+C$

A-5: $-\frac{1}{16}\sqrt{1+\frac{16}{x^2}}+C$

A-6: $\frac{\pi}{6}$

A-7: $\frac{\sqrt{x^2-9}}{9x}+C=\frac{1}{9}\sqrt{1-\left(\frac{3}{x}\right)^2}+C$

A-8: $2\arcsin\frac{x}{2}+x\sqrt{1-\frac{x^2}{4}}+C$

A-9: $\arcsin\frac{x+1}{2}+C$

A-10: (a) See the solution. (b) $\frac{8+3\pi}{16}$

A-11: $\sqrt{25x^2-4}-2\,\mathrm{arcsec}\,\frac{5x}{2}+C$

Answers to Exercises 1.10 — Jump to TABLE OF CONTENTS

A-1: 3

A-2: $\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{Ex+F}{x^2+1}$

A-3: $\log\frac{4}{3}$

A-4: $-\frac{1}{x}-\arctan x+C$

A-5: $4\log|x-3|-2\log(x^2+1)+C$

A-6: $F(x)=\log|x-2|+\log|x^2+4|+2\arctan(x/2)+D$

A-7: $-2\log|x-3|+3\log|x+2|+C$

A-8: $-9\log|x+2|+14\log|x+3|+C$

Answers to Exercises 1.11 — Jump to TABLE OF CONTENTS

A-1: True. Because $f(x)$ is positive and concave up, the graph of $f(x)$ is always below the top of the trapezoids used in the trapezoidal rule.

A-2: $\frac{2\pi}{3}$

A-3: 4.377 m^3

A-4: 5403.5 cm^3

A-5: 0.6865 m^3

A-6: (a) 363,500 (b) 367,000

A-7: (a) $\frac{49}{2}$ (b) $\frac{77}{3}$

A-8: See the solution.

A-9: $\frac{3}{100}$

A-10: (a) $\frac{1/3}{3}\left((-3)^5+4\left(\frac{1}{3}-3\right)^5+2\left(\frac{2}{3}-3\right)^5+4(-2)^5+2\left(\frac{4}{3}-3\right)^5+4\left(\frac{5}{3}-3\right)^5+(-1)^5\right)$

(b) Simpson's Rule results in a smaller error bound.

A-11: Any $n \geqslant 68$ works.

A-12: $\frac{8}{15}$

A-13: $\frac{1}{180\times 3^4}=\frac{1}{14580}$

A-14: (a) $T_4=\frac{1}{8}\left[1+2\times\frac{4}{5}+2\times\frac{4}{6}+2\times\frac{4}{7}+\frac{1}{2}\right]$,

(b) $S_4=\frac{1}{12}\left[1+4\times\frac{4}{5}+2\times\frac{4}{6}+4\times\frac{4}{7}+\frac{1}{2}\right]$

(c) $\left|I-S_4\right| \leqslant \frac{24}{180\times 4^4}=\frac{3}{5760}$

A-15: (a) $T_4=8.03515$, $S_4=8.03509$

(b) $\left|\int_a^b f(x)\,dx-T_n\right| \leqslant \frac{2}{1000}\frac{8^3}{12(4)^2}=0.00533,\quad \left|\int_a^b f(x)\,dx-S_n\right| \leqslant \frac{4}{1000}\frac{8^5}{180(4)^4}=0.00284$

A-16: 494 ft^3

A-17: (a) 0.025635 (b) error $\leqslant 1.8\times 10^{-5}$

A-18: (a) 0.6931698 (b) $n \geqslant 12$ with n even

A-19: (a) 0.01345 (b) $n \geqslant 28$ with n even

A-20: $n \geqslant 259$

A-21: (a) 0.025635 (b) 1.8×10^{-5}

Answers to Exercises 1.12 — Jump to TABLE OF CONTENTS

A-1: False. For example, the functions $f(x)=e^{-x}$ and $g(x)=1$ provide a counterexample.

A-2: $q=\frac{1}{5}$

A-3: The integral diverges.

A-4: The integral diverges.

A-5: The integral *does not converge*.

A-6: The integral *converges*.

A-7: The integral *diverges*.

A-8: The integral *converges*.

A-9: The integral *converges*.

A-10: The integral *converges*.

A-11: $t = 10$ and $n = 2042$ will do the job. There are many other correct answers.

Answers to Exercises 1.13 — Jump to TABLE OF CONTENTS

A-1: $\frac{1}{2}\log|x^2-3|+C$

A-2: (a) 2 (b) $\frac{2}{15}$ (c) $\frac{3e^4}{16}+\frac{1}{16}$

A-3: (a) 1 (b) $\frac{8}{15}$

A-4: (a) e^2+1 (b) $\log(\sqrt{2}+1)$ (c) $\log\frac{15}{13}\approx 0.1431$

A-5: (a) $\frac{8}{15}\approx 0.53333$ (b) $\frac{9}{4}\pi$ (c) $\log 2-2+\frac{\pi}{2}\approx 0.264$ (d) $2\log 2-\frac{1}{2}\approx 0.886$

A-6: (a) $\frac{1}{15}$ (b) $\frac{1}{9}\frac{x}{\sqrt{x^2+9}}+C$ (c) $\frac{1}{2}\log|x-1|-\frac{1}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x+C$

(d) $\frac{1}{2}\left[x^2\tan^{-1}x-x+\tan^{-1}x\right]+C$

A-7: (a) $\frac{1}{12}$ (b) $2\sin^{-1}\frac{x}{2}+x\sqrt{1-\frac{x^2}{4}}+C$ (c) $x\log(1+x^2)-2x+2\tan^{-1}x+C$

(d) $-2\log|x|+\frac{1}{x}+2\log|x-1|+C$

A-8: (a) $\int_0^\infty e^{-x}\sin(2x)\,\mathrm{d}x=\frac{2}{5}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $\log 2-\frac{1}{2}\approx 0.193$

(d) $\log 2-\frac{1}{2}\approx 0.193$

A-9: (a) $\frac{1}{2}x^2\log x-\frac{1}{4}x^2+C$ (b) $\frac{1}{2}\log[(x+2)^2+1]-3\arctan(x+2)+C$

(c) $\frac{1}{2}\log|x-3|-\frac{1}{2}\log|x-1|+C$ (d) $\frac{1}{3}\arctan x^3+C$

A-10: (a) $\frac{\pi}{4}-\frac{1}{2}\log 2$ (b) $\log|x^2-2x+5|+\frac{1}{2}\tan^{-1}\frac{x-1}{2}+C$

A-11: (a) $\frac{x^2}{2}\log x-\frac{x^2}{4}+C$ (b) $-\frac{1}{300(x^3+1)^{100}}+C$ (c) $\frac{\sin^5 x}{5}-\frac{\sin^7 x}{7}+C$

(d) $2\arcsin\frac{x}{2}+\frac{x}{2}\sqrt{4-x^2}+C$

A-12: (a) $-\frac{1}{4}\log|e^x+1|+\frac{1}{4}\log|e^x-3|+C$ (b) $\frac{4\pi}{3}-2\sqrt{3}$

A-13: (a) $\frac{1}{2}\sec^2 x+\log|\cos x|+C$ (b) $\frac{1}{10}\tan^{-1}8\approx 0.1446$ (c) $\log 2-2+\frac{\pi}{2}\approx 0.2639$

A-14: (a) $\frac{9}{4}\pi+9$ (b) $2\log|x-2|-\log(x^2+4)+C$ (c) $\frac{\pi}{2}$

A-15: (a) $\frac{1}{2}x\left[\sin(\log x)-\cos(\log x)\right]+C$ (b) $\log|x-3|-\log|x-2|+C$

Answers to Exercises 2.1 — Jump to TABLE OF CONTENTS

A-1: $\frac{1}{4}$ J

A-2: $a=3$

A-3: 25

A-4: 196 J

A-5: 904,050π joules

A-6: $\int_0^3 9.8 \times 8000(2+z)(3-z)^2\,\mathrm{d}z$ joules

Answers to Exercises 2.2 — Jump to TABLE OF CONTENTS

A-1: 1

A-2: $\frac{1}{e-1}\left[\frac{2}{9}e^3 + \frac{1}{9}\right]$

A-3: $\frac{4}{\pi} + 1$

A-4: $\frac{2}{\pi}$

A-5: $\frac{10}{3}\log 7$

A-6: $\frac{1}{2(e-1)}$

A-7: $\frac{1}{2}$

A-8: (a) 130 km (b) 65 km/hr

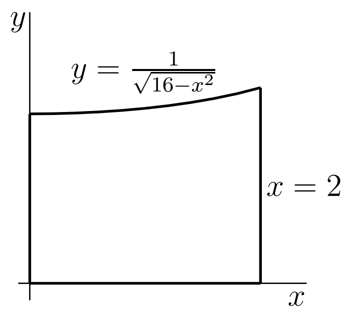
Answers to Exercises 2.3 — Jump to TABLE OF CONTENTS

A-1: $\bar{x} = -\frac{1}{3}\int_{-1}^{0} 6x^2\,\mathrm{d}x$

A-2: $\bar{y} = \frac{3}{4e} - \frac{e}{4}$

A-3: $\bar{y} = \frac{8}{5}$

A-4: (a)

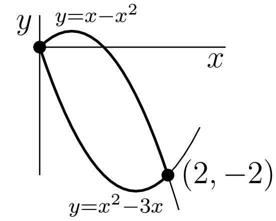


(b) $\frac{3\log 3}{8\pi}$

A-5: $\bar{x} = \frac{\frac{\pi}{4}\sqrt{2}-1}{\sqrt{2}-1}$ and $\bar{y} = \frac{1}{4(\sqrt{2}-1)}$

A-6: (a) $\bar{x} = \frac{k}{A}\left[\sqrt{2}-1\right]$, $\bar{y} = \frac{k^2\pi}{8A}$ (b) $k = \frac{8}{\pi}\left[\sqrt{2}-1\right]$

A-7: (a) The sketch is the figure below.



(b) $\frac{8}{3}$ (c) 1

A-8: $\frac{2}{\pi}\log 2 \approx 0.44127$

A-9: $\bar{x}=0$ and $\bar{y}=\frac{12}{24+9\pi}$

A-10: (a) $\frac{9}{4}\pi$ (b) $\bar{x}=0$ and $\bar{y}=\frac{4}{\pi}$

A-11: (a) $\bar{x}=\frac{8}{11}, \bar{y}=\frac{166}{55}$ (b) $\pi\int_0^4 y\,\mathrm{d}y+\pi\int_4^6 (6-y)^2\,\mathrm{d}y$

A-12: (a) $\bar{y}=\frac{e}{4}-\frac{3}{4e}$ (b) $\pi\left(\frac{e^2}{2}+2e-\frac{3}{2}\right)$

Answers to Exercises 2.4 — Jump to TABLE OF CONTENTS

A-1: $y=\log(x^2+2)$

A-2: $y(x)=3\sqrt{1+x^2}$

A-3: $y(t)=3\log\frac{-3}{C+\sin t}$.

A-4: $y=\sqrt[3]{\frac{3}{2}e^{x^2}+C}$.

A-5: $y=-\log\left(C-\frac{x^2}{2}\right)$. The solution only exists for $C-\frac{x^2}{2}>0$, i.e. for $C>0$ and $|x|<\sqrt{2C}$.

A-6: $y=(3e^x-3x^2+24)^{1/3}$

A-7: $y=f(x)=-\frac{1}{\sqrt{x^2+16}}$

A-8: $y=\sqrt{10x^3+4x^2+6x-4}$

A-9: $y(x)=e^{x^4/4}$

A-10: $y=\frac{1}{1-2x}$

A-11: $f(x)=e^{1+x^2/2}$

A-12: $y(x)=\sqrt{4+2\log\frac{2x}{x+1}}$. Note that, to satisfy $y(1)=2$, we need the positive square root.

A-13: $\frac{y^2}{2}+\frac{1}{3}\left[y^2-4\right]^{3/2}=\sec x+1$

A-14: 12 weeks

A-15: $t = \sqrt{\frac{m}{kg}} \tan^{-1}\left(\sqrt{\frac{k}{mg}}\, v_0\right)$

A-16: (a) $k = \frac{1}{400}$ (b) $t = 70\text{sec}$

A-17: (a) $x(t) = \frac{3-4e^{kt}}{1-2e^{kt}}$ (b) As $t \to \infty$, $x \to 2$.

A-18: (a) $P = \frac{4}{1+e^{-4t}}$ (b) At $t = \frac{1}{2}$, $P = 3.523$. As $t \to \infty$, $P \to 4$.

A-19: (a) $\frac{\mathrm{d}v}{\mathrm{d}t} = -kv^2$ (b) $v = \frac{400}{t+1}$ (c) $t = 7$

A-20: (a) $B(t) = C\, e^{0.06t - 0.02\cos t}$ with the arbitrary constant $C \geqslant 0$. (b) \$1159.89

A-21: (a) $B(t) = \{30000 - 50m\}\, e^{t/50} + 50m$ (b) \$600

A-22: $y(x) = \frac{4-e^{1-\cos x}}{2-e^{1-\cos x}}$. The largest allowed interval is

$$-\cos^{-1}(1-\log 2) < x < \cos^{-1}(1-\log 2) \approx 1.259$$

A-23: $180,000\sqrt{\frac{3}{g}} \approx 99,591\text{sec} \approx 27.66\text{hr}$

A-24: $t = \frac{4\times 144}{15}\sqrt{\frac{12^5}{2g}} \approx 2,394\,\text{sec} \approx 0.665\,\text{hr}$

A-25: (a) 3 (b) $y' = (y-1)(y-2)$ (c) $f(x) = \frac{4-e^x}{2-e^x}$

A-26: $p = \frac{1}{4}$

Answers to Exercises 3.1 — Jump to TABLE OF CONTENTS

A-1: $\lim\limits_{k\to\infty} a_k = 0$.

A-2: The sequence converges to 0.

A-3: 9

A-4: $\log 2$

A-5: See the solution.

Answers to Exercises 3.2 — Jump to TABLE OF CONTENTS

A-1: $\frac{1}{7\times 8^6}$

A-2: $\frac{3}{2}$

A-3: 6

A-4: $\cos\left(\frac{\pi}{3}\right) - \cos(0) = -\frac{1}{2}$

A-5: (a) $a_n \frac{11}{16n^2+24n+5}$ (b) $\frac{3}{4}$

A-6: $\frac{24}{5}$

A-7: $\frac{7}{30}$

A-8: $\frac{263}{99}$

A-9: $\frac{321}{999} = \frac{107}{333}$

A-10: 3

A-11: $\frac{1}{2} + \frac{5}{7} = \frac{17}{14}$

A-12: $\frac{40}{3} = 13\frac{1}{3}$

Answers to Exercises 3.3 — Jump to TABLE OF CONTENTS

A-1: No. It diverges.

A-2: $b_n = \dfrac{2^n}{3^n}$

A-3: (a) In general false. The harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ provides a counterexample.

(b) In general false. If $a_n = (-1)^n \frac{1}{n}$, then $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ is again the harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$, which diverges.

(c) In general false. Take, for example, $a_n = 0$ and $b_n = 1$.

A-4: It diverges.

A-5: The series diverges.

A-6: See the solution.

A-7: $p > 1$

A-8: It converges.

A-9: It converges.

A-10: The series converges.

A-11: It diverges.

A-12: It converges absolutely.

A-13: It diverges.

A-14: It converges absolutely.

A-15: (a) diverges (b) converges

A-16: diverges

A-17: (a) converges (b) diverges

A-18: $\frac{1}{7}$

A-19: (a) diverges by comparison with the harmonic series

(b) converges by the ratio test

A-20: (a) Converges by the limiting comparison test with $b = \frac{1}{k^{5/3}}$.

(b) Diverges by the ratio test.

(c) Diverges by the integral test.

A-21: It converges.

A-22: $N = 5$

A-23: $N \geqslant 999$

A-24: We need $n = 4$ and then $S_4 = \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{9^2}$

A-25: (a) converges (b) converges

A-26: (a) See the solution.

(b) $f(x) = \frac{x+\sin x}{1+x^2}$ is not a decreasing function.

(c) See the solution.

A-27: The sum is between 0.9035 and 0.9535.

A-28: See the solution.

A-29: It diverges.

A-30: It converges to $-\log 2 = \log \frac{1}{2}$,

A-31: See the solution.

Answers to Exercises 3.4 — Jump to TABLE OF CONTENTS

A-1: False. For example, $b_n = \frac{1}{n}$ provides a counterexample.

A-2: conditionally convergent

A-3: The series diverges.

A-4: It diverges.

A-5: It converges absolutely.

A-6: (a) See the solution. (b) $|S - S_5| \leqslant 24 \times 36e^{-6^3}$

Answers to Exercises 3.5 — Jump to TABLE OF CONTENTS

A-1: (a) $R = \frac{1}{2}$ (b) $\frac{2}{1+2x}$ for all $|x| < \frac{1}{2}$

A-2: $R = \infty$

A-3: 1

A-4: The interval of convergence is $-1 < x+2 \leqslant 1$ or $(-3,-1]$.

A-5: The interval of convergence is $-4 < x \leqslant 2$, or simply $(-4,2]$.

A-6: $-3 \leqslant x < 7$ or $[-3,7)$

A-7: The given series converges if and only if $-3 \leqslant x \leqslant -1$. Equivalently, the series has interval of convergence $[-3,-1]$.

A-8: The interval of convergence is $\frac{3}{4} \leqslant x < \frac{5}{4}$, or $\left[\frac{3}{4}, \frac{5}{4}\right)$.

A-9: The radius of convergence is 2. The interval of convergence is $-1 < x \leqslant 3$, or $\big(-1,3\big]$.

A-10: The interval of convergence is $a-1 < x < a+1$, or $\big(a-1, a+1\big)$.

A-11: (a) $|x+1| \leqslant 9$ or $-10 \leqslant x \leqslant 8$ or $[-10,8]$ (b) This series converges only for $x=1$.

A-12: $\sum\limits_{n=0}^{\infty} x^{n+3} = \sum\limits_{n=3}^{\infty} x^n$

A-13: The series converges absolutely for $|x| < 9$, converges conditionally for $x=-9$ and diverges otherwise.

A-14: (a) $\sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} + C$ (b) We need to keep two terms (the $n=0$ and $n=1$ terms).

A-15: (a) See the solution.

(b) $\sum\limits_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$. The series converges for $-1 < x < 1$.

A-16: See the solution.

A-17: (a) 1. (b) The series converges for $-1 \leqslant x < 1$, i.e. for the interval $[-1,1)$

Answers to Exercises 3.6 — Jump to TABLE OF CONTENTS

A-1: $c_5 = \frac{3^5}{5!}$

A-2: $a=1$, $b = -\frac{1}{3!} = -\frac{1}{6}$.

A-3: $-\sum\limits_{n=0}^{\infty} 2^n x^n$

A-4: $b_n = 3(-1)^n + 2^n$

A-5: $\int \frac{e^{-x^2}-1}{x}\,\mathrm{d}x = C - \frac{x^2}{2} + \frac{x^4}{8} + \cdots$. It is not clear from the wording of the question whether or not the arbitrary constant C is to be counted as one of the "first two nonzero terms".

A-6: $\sum\limits_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+6}}{(2n+1)(2n+6)} + C = \sum\limits_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n+6}}{(2n+1)(n+3)} + C$

A-7: $\sum_{n=0}^{\infty}(-1)^n \dfrac{2^{n+1}x^{n+1}}{n+1}$ for all $|x| < \frac{1}{2}$

A-8:

A-9: $\frac{1}{e}$

A-10: $e^{1/e}$

A-11: $e^{1/\pi} - 1$

A-12: $\log(3/2)$

A-13: $(e+2)e^e - 2$

A-14: $f(x) = 1 + \sum_{n=0}^{\infty}(-1)^n \frac{3^n}{3n+2}x^{3n+2}$

A-15: (a) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{4n+1}$ (b) 0.493967

(c) The approximate value of part (b) is larger than the true value of $I(1/2)$

A-16: $\frac{1}{66}$

A-17: Any interval of length 0.0002 that contains 0.03592 and 0.03600 is fine.

A-18: $\log(x) = \log 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n}(x-2)^n$. It converges when $0 < x \leqslant 4$.

A-19: (a) $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n\ n!}$ (b) -0.80 (c) See the solution.

A-20: (a) $\Sigma(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}$ (b) $x = \pi$ (c) 1.8525

A-21: (a) $I(x) = \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}}{(2n)!(2n-1)}$ (b) $I(1) = -\frac{1}{2} + \frac{1}{4!3} \pm \frac{1}{6!5} = -0.486 \pm 0.001$

(c) $I(1) < -\frac{1}{2} + \frac{1}{4!3}$

A-22: (a) $\frac{1}{2!}x - \frac{1}{4!}x^3 + \frac{1}{6!}x^5 - \frac{1}{8!}x^8 + \cdots$ (b) 0.460 (c) $I(1) < \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} < 0.460$

A-23: (a) See the solution. (b) The series converges for all x.

A-24: See the solution.

A-25: -1

A-26: $\frac{1}{5!} = \frac{1}{120}$

A-27: $-\frac{61}{60}$

A-28: (a) See the solution. (b) $\frac{1}{2}\left(e + \frac{1}{e}\right)$

A-29: (a) $\cosh(x) = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ for all x.

Part IV

SOLUTIONS TO PROBLEMS

Solutions to Exercises 1.1 — Jump to TABLE OF CONTENTS

S-1: In the given integral, the domain of integration runs from $a = -1$ to $b = 7$. So we have $\Delta x = (b-a)/n = (7-(-1))/n = 8/n$. The left hand end of the first subinterval is at $x_0 = a = -1$. So the right hand end of the i^{th} interval is at $x_i = -1 + 8i/n$. So

$$\int_{-1}^{7} f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-1+\frac{8i}{n}\right)\frac{8}{n}$$

S-2: In general the *left* Riemann sum for the integral $\int_a^b f(x)\,\mathrm{d}x$ is of the form

$$\sum_{k=1}^{n} f\left(a+(k-1)\frac{b-a}{n}\right)\frac{b-a}{n}$$

So

- To get the limits of summation to match the given sum, we need $n = 4$.
- Then to get the factor multiplying f to match that in the given sum, we need $\frac{b-a}{n} = 1$ or $b - a = 4$.
- , Finally, to get the argument of f to match that in the given sum, we need

  $$a+(k-1)\frac{b-a}{n} = a - \frac{b-a}{n} + k\frac{b-a}{n} = 1+k$$

  Subbing in $n = 4$ and $b - a = 4$, gives $a - 1 + k = 1 + k$, so $a = 2$ and $b = 6$.

S-3: $\sum\limits_{k=0}^{3} f(1.5+k)\cdot 1$ is a midpoint Riemann sum for f on the interval $[1, 5]$ with $n = 4$. It is also a left Riemann sum for f on the interval $[1.5, 5.5]$ with $n = 4$. It is also a right Riemann sum for f on the interval $[0.5, 4.5]$ with $n = 4$.

S-4: We identify the given sum as the right Riemann sum $\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$, with $a = 0$ (that's specified in the statement of the question), interval $\Delta x = 4/n$, $x_i = a + i\Delta x = 4i/n$ and $f(x) = \sin^2(2+x)$. So $b = a + n\Delta x = 4$.

S-5: The given sum is of the form

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}\sqrt{1-\frac{k^2}{n^2}} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n}\sqrt{1-\frac{k^2}{n^2}}\,\frac{1}{n} = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

with $\Delta x = \frac{1}{n}$, $a = 0$, $x_k^* = \frac{k}{n} = a + k\Delta x$ and $f(x) = x\sqrt{1-x^2}$. Since $x_0^* = 0$ and $x_n^* = 1$, the right hand side is the definition (using the right Riemann sum) of $\int_0^1 f(x)\,\mathrm{d}x$.

S-6: In general the midpoint Riemann sum is

$$\sum_{i=1}^{n} f\Big(a + \big(i - 1/2\big)\Delta x\Big)\ \Delta x \qquad \Delta x = \frac{b-a}{n}$$

In this problem we are told that $f(x) = x^8$, $a = 5$, $b = 15$ and $n = 50$, so that $\Delta x = \frac{b-a}{n} = \frac{1}{5}$ and the desired Riemann sum is

$$\sum_{i=1}^{50} \Big(5 + \big(i - 1/2\big)\frac{1}{5}\Big)^8\ \frac{1}{5}$$

S-7: The given integral has interval of integration going from $a = -1$ to $b = 5$. So when we use three approximating rectangles, all of the same width, the common width is $\Delta x = \frac{b-a}{n} = 2$. The first rectangle has left hand end point $x_0 = a = -1$, the second has left hand endpoint $x_1 = a + \Delta x = 1$, and the third has left hand end point $x_2 = a + 2\Delta x = 3$. So

$$\int_{-1}^{5} x^3\,\mathrm{d}x \approx \big[f(x_0) + f(x_1) + f(x_2)\big]\Delta x = \big[(-1)^3 + 1^3 + 3^3\big] \times 2 = 54$$

S-8: As i ranges from 1 to n, $3i/n$ range from $3/n$ to 3 with jumps of $\Delta x = 3/n$, so this is

$$\lim_{n\to\infty}\sum_{i=1}^{n} \frac{3}{n} e^{-i/n}\cos(3i/n) = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,\mathrm{d}x$$

where $x_i = 3i/n$, $f(x) = e^{-x/3}\cos(x)$, $a = x_0 = 0$ and $b = x_n = 3$. Thus

$$\lim_{n\to\infty}\sum_{i=1}^{n} \frac{3}{n} e^{-i/n}\cos(3i/n) = \int_0^3 e^{-x/3}\cos(x)\,\mathrm{d}x$$

S-9: As i ranges from 1 to n, the exponent i/n range from $1/n$ to 1 with jumps of $\Delta x = 1/n$. So let's try $x_i = i/n$, $\Delta x = \frac{1}{n}$. Then

$$R_n = \sum_{i=1}^{n} \frac{ie^{i/n}}{n^2} = \sum_{i=1}^{n} \frac{i}{n} e^{i/n}\frac{1}{n} = \sum_{i=1}^{n} x_i e^{x_i}\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

with $f(x) = xe^x$, and the limit

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,\mathrm{d}x$$

where $a = x_0 = 0$ and $b = x_n = 1$. Thus

$$\lim_{n\to\infty} R_n = \int_0^1 xe^x\,\mathrm{d}x$$

S-10: *choice #1:* If we set $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}$, i.e. $x_i = a + i\Delta x$ with $a = 0$, then

$$\begin{aligned}
\lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2i/n}\cdot\frac{2}{n}\right) &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-x_i}\Delta x\right) \\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\Delta x\right) && \text{with } f(x) = e^{-1-x} \\
&= \int_a^b f(x)\,\mathrm{d}x && \text{with } a = x_0 = 0 \text{ and } b = x_n = 2 \\
&= \int_0^2 e^{-1-x}\,\mathrm{d}x
\end{aligned}$$

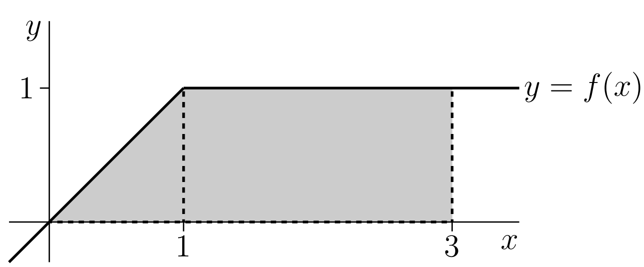
choice #2: If we set $\Delta x = \frac{2}{n}$ and $x_i = 1 + \frac{2i}{n}$, i.e. $x_i = a + i\Delta x$ with $a = 1$, then

$$\begin{aligned}
\lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2i/n}\cdot\frac{2}{n}\right) &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-x_i}\Delta x\right) \\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\Delta x\right) && \text{with } f(x) = e^{-x} \\
&= \int_a^b f(x)\,\mathrm{d}x && \text{with } a = x_0 = 1 \text{ and } b = x_n = 3 \\
&= \int_1^3 e^{-x}\,\mathrm{d}x
\end{aligned}$$

choice #3: If we set $\Delta x = \frac{1}{n}$ and $x_i = \frac{i}{n}$, i.e. $x_i = a + i\Delta x$ with $a = 0$, then

$$\begin{aligned}
\lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2i/n}\cdot\frac{2}{n}\right) &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2x_i}\,2\Delta x\right) \\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\Delta x\right) && \text{with } f(x) = 2e^{-1-2x} \\
&= \int_a^b f(x)\,\mathrm{d}x && \text{with } a = x_0 = 0 \text{ and } b = x_n = 1 \\
&= 2\int_0^1 e^{-1-2x}\,\mathrm{d}x
\end{aligned}$$

S-11: Here is a sketch the graph of $f(x)$.



There a linear increase from $x = 0$ to $x = 1$, followed by a constant. Using the interpretation of $\int_0^3 f(x)\,\mathrm{d}x$ as the area between $y = f(x)$ and the x–axis with x between 0 and 3, we can break this area into:

- $\int_0^1 f(x)\,\mathrm{d}x$: a right-angled triangle of height 1 and base 1 and hence area 0.5.
- $\int_1^3 f(x)\,\mathrm{d}x$: a rectangle of height 1 and base 2 and hence area 2.

Summing up: $\int_0^3 f(x)\,\mathrm{d}x = 2.5$.

S-12: Recall that

$$|x| = \begin{cases} -x & \text{if } x \leqslant 0 \\ x & \text{if } x \geqslant 0 \end{cases}$$

so that

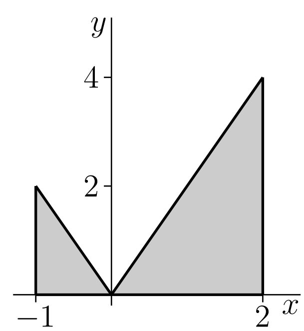
$$|2x| = \begin{cases} -2x & \text{if } x \leqslant 0 \\ 2x & \text{if } x \geqslant 0 \end{cases}$$

To picture the geometric figure whose area the integral represents observe that

- at the left hand end of the domain of integration $x = -1$ and the integrand $|2x| = |-2| = 2$ and
- as x increases from -1 towards 0, the integrand $|2x| = -2x$ decreases linearly, until
- when x hits 0 the integrand hits $|2x| = |0| = 0$ and then
- as x increases from 0, the integrand $|2x| = 2x$ increases linearly, until
- when x hits $+2$, the right hand end of the domain of integration, the integrand hits $|2x| = |4| = 4$.

So the integral $\int_{-1}^2 |2x|\,\mathrm{d}x$ is the area of the union of the two shaded triangles (one of base 1 and of height 2 and the other of base 2 and height 4) in the figure on the right below and

$$\int_{-1}^2 |2x|\,\mathrm{d}x = \frac{1}{2} \times 1 \times 2 + \frac{1}{2} \times 2 \times 4 = 5$$



S-13: The car's speed increases with time. So its highest speed on any time interval occurs at the right hand end of the interval and the best possible upper estimate for the distance traveled is given by the *right* Riemann sum with $\Delta x = 0.5$, which is

$$\big[v(0.5) + v(1.0) + v(1.5) + v(2.0)\big] \times 0.5 = \big[14 + 22 + 30 + 40\big] \times 0.5 = 53 \text{ m}$$

S-14: (a, solution #1) Set $x_i = -2 + \frac{2i}{n}$. Then $a = x_0 = -2$ and $b = x_n = 0$ and $\Delta x = \frac{2}{n}$. So

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\sqrt{4 - \left(-2 + \frac{2i}{n}\right)^2} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \qquad \text{with } f(x) = \sqrt{4 - x^2} \text{ and } \Delta x = \frac{2}{n}$$
$$= \int_{-2}^{0} \sqrt{4 - x^2}\, \mathrm{d}x$$

(a, solution #2) Set $x_i = \frac{2i}{n}$. Then $a = x_0 = 0$ and $b = x_n = 2$ and $\Delta x = \frac{2}{n}$. So

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\sqrt{4 - \left(-2 + \frac{2i}{n}\right)^2} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{with } f(x) = \sqrt{4 - (-2+x)^2},\ \Delta x = \frac{2}{n}$$
$$= \int_{0}^{2} \sqrt{4 - (-2+x)^2}\, \mathrm{d}x$$

(b) For the integral $\int_{-2}^{0} \sqrt{4 - x^2}\, \mathrm{d}x$, $y = \sqrt{4 - x^2}$ is equivalent to $x^2 + y^2 = 4$, $y \geqslant 0$. So the integral represents the area between the upper half of the circle $x^2 + y^2 = 4$ (which has radius 2) and the x-axis with $-2 \leqslant x \leqslant 0$, which is a quarter circle with area $\frac{1}{4}\pi 2^2 = \pi$.

S-15: (a) The left Riemann sum is defined as

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x \qquad \text{with } x_i = a + i\Delta x$$

We subdivide into $n = 3$ intervals, so that $\Delta x = \frac{b-a}{n} = \frac{3-0}{3} = 1$, $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$. The function $f(x) = 7 + x^3$ has the values $f(x_0) = 7 + 0^3 = 7$, $f(x_1) = 7 + 1^3 = 8$, and $f(x_2) = 7 + 2^3 = 15$, from which we evaluate

$$L_3 = \big[f(x_0) + f(x_1) + f(x_2)\big]\Delta x = \big[7 + 8 + 15\big] \times 1 = 30$$

(b) We divide into n intervals so that $\Delta x = \frac{b-a}{n} = \frac{3}{n}$ and $x_i = a + i\Delta x = \frac{3i}{n}$. The right Riemann sum is therefore:

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left[7 + \frac{(3i)^3}{n^3}\right]\frac{3}{n} = \sum_{i=1}^{n} \left[\frac{21}{n} + \frac{81\, i^3}{n^4}\right]$$

To calculate the sum:

$$R_n = \left(\frac{21}{n}\sum_{i=1}^{n} 1\right) + \left(\frac{81}{n^4}\sum_{i=1}^{n} i^3\right)$$
$$= \frac{21}{n} \times n + \frac{81}{n^4} \times \frac{n^4 + 2n^3 + n^2}{4} = 21 + \frac{81}{4}(1 + 2/n + 1/n^2)$$

To evaluate the limit exactly, we take $n \to \infty$. The expressions involving $1/n$ vanish leaving:

$$\int_0^3 (7 + x^3)\, \mathrm{d}x = \lim_{n\to\infty} R_n = 21 + \frac{81}{4} = 41\frac{1}{4}$$

S-16: In general, the n slice, the right–endpoint Riemann sum approximation to the integral $\int_a^b f(x)\,\mathrm{d}x$ is

$$\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$$

where $\Delta x = \frac{b-a}{n}$. In this problem, $a=2$, $b=4$, and $f(x)=x^2$, so that $\Delta x = \frac{2}{n}$ and the n slice, the right–endpoint Riemann sum approximation becomes

$$\begin{aligned}\sum_{i=1}^{n}\Big(2+\frac{2i}{n}\Big)^2\frac{2}{n} &= \sum_{i=1}^{n}\frac{8}{n}+\sum_{i=1}^{n}\frac{16i}{n^2}+\sum_{i=1}^{n}\frac{8i^2}{n^3}\\ &= \frac{8}{n}n+\frac{16}{n^2}\frac{n(n+1)}{2}+\frac{8}{n^3}n\frac{(n+1)(2n+1)}{6}\\ &= 8+8\Big(1+\frac{1}{n}\Big)+\frac{4}{3}\Big(1+\frac{1}{n}\Big)\Big(2+\frac{1}{n}\Big)\end{aligned}$$

So

$$\int_2^4 x^2\,\mathrm{d}x = \lim_{n\to\infty}\Big[8+8\Big(1+\frac{1}{n}\Big)+\frac{4}{3}\Big(1+\frac{1}{n}\Big)\Big(2+\frac{1}{n}\Big)\Big] = 8+8+\frac{4}{3}\times 2 = \frac{56}{3}$$

S-17: We'll use right Riemann sums with $a=0$ and $b=2$. When there are n strips $\Delta x = \frac{b-a}{n} = \frac{2}{n}$ and $x_i = a+i\Delta x = 2i/n$. So we need to evaluate

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\left(x_i^3+x_i\right)\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(\frac{2i}{n}\right)^3+\frac{2i}{n}\right)\frac{2}{n}\\ &= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(\frac{8i^3}{n^3}+\frac{2i}{n}\right)\\ &= \lim_{n\to\infty}\left(\frac{16}{n^4}\sum_{i=1}^{n}i^3+\frac{4}{n^2}\sum_{i=1}^{n}i\right)\\ &= \lim_{n\to\infty}\left(\frac{16(n^4+2n^3+n^2)}{n^4\cdot 4}+\frac{4(n^2+n)}{n^2\cdot 2}\right)\\ &= \lim_{n\to\infty}\left(\frac{16}{4}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)+\frac{4}{2}\left(1+\frac{1}{n}\right)\right)\\ &= \frac{16}{4}+\frac{4}{2} = 6.\end{aligned}$$

S-18: We'll use right Riemann sums with $a=1$, $b=4$ and $f(x)=2x-1$. When there are

n strips $\Delta x = \frac{b-a}{n} = \frac{3}{n}$ and $x_i = a + i\Delta x = 1 + 3i/n$. So we need to evaluate

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}(2x_i - 1)\,\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{6i}{n}-1\right)\frac{3}{n} \\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(\frac{6i}{n}+1\right) \\
&= \lim_{n\to\infty}\left(\frac{18}{n^2}\sum_{i=1}^{n} i + \frac{3}{n}\sum_{i=1}^{n} 1\right) \\
&= \lim_{n\to\infty}\left(\frac{18\cdot n(n+1)}{n^2\cdot 2} + \frac{3}{n}n\right) \\
&= \lim_{n\to\infty}\left(9\left(1+\frac{1}{n}\right)+3\right) \\
&= 9 + 3 = 12.
\end{aligned}$$

Solutions to Exercises 1.2 — Jump to TABLE OF CONTENTS

S-1: (a) False. For example if

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geqslant 0 \end{cases}$$

then $\int_{-3}^{-2} f(x)\mathrm{d}x = 0$ and $\int_{3}^{2} f(x)\mathrm{d}x = 1$.

(b) False. For example, if $f(x) = g(x) = x$, then

$$\int_{-3}^{-2} f(x)\,\mathrm{d}x = \int_{-3}^{-2} x\,\mathrm{d}x = \left[\frac{x^2}{2}\right]_{-3}^{-2} = \frac{4}{2} - \frac{9}{2} = -\frac{5}{2}$$

while

$$\int_{2}^{3} f(x)\,\mathrm{d}x = \int_{2}^{3} x\,\mathrm{d}x = \left[\frac{x^2}{2}\right]_{2}^{3} = \frac{9}{2} - \frac{4}{2} = \frac{5}{2}$$

(c) False. For example, if $f(x) = g(x) = x$, then $\int_0^1 f(x)\cdot g(x)\,\mathrm{d}x = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$ and $\int_0^1 f(x)\,\mathrm{d}x \cdot \int_0^1 g(x)\,\mathrm{d}x = \int_0^1 x\,\mathrm{d}x \cdot \int_0^1 x\,\mathrm{d}x = \frac{1}{2}\cdot\frac{1}{2}$.

S-2: The operation of integration is linear (that's part (d) of the "arithmetic of integration" Theorem 1.2.1 in the CLP 101 notes), so that:

$$\begin{aligned}
\int_2^3 [6f(x) - 3g(x)]\,\mathrm{d}x &= \int_2^3 6f(x)\,\mathrm{d}x - \int_2^3 3g(x)\,\mathrm{d}x \\
&= 6\int_2^3 f(x)\,\mathrm{d}x - 3\int_2^3 g(x)\,\mathrm{d}x = (6\times(-1)) - (3\times 5) = -21
\end{aligned}$$

S-3: The operation of integration is linear (that's part (d) of the "arithmetic of integration" Theorem 1.2.1 in the CLP 101 notes), so that:

$$\begin{aligned}\int_0^2 [2f(x) + 3g(x)]\,\mathrm{d}x &= \int_0^2 2f(x)\,\mathrm{d}x + \int_0^2 3g(x)\,\mathrm{d}x \\ &= 2\int_0^2 f(x)\,\mathrm{d}x + 3\int_0^2 g(x)\,\mathrm{d}x = (2\times 3) + (3\times(-4)) = -6\end{aligned}$$

S-4: Using part (d) of the "arithmetic of integration" Theorem 1.2.1, followed by parts (c) and (b) of the "arithmetic for the domain of integration" Theorem 1.2.3 in the CLP 101 notes,

$$\begin{aligned}\int_{-1}^2 \big[3g(x) - f(x)\big]\,\mathrm{d}x &= 3\int_{-1}^2 g(x)\,\mathrm{d}x - \int_{-1}^2 f(x)\,\mathrm{d}x \\ &= 3\int_{-1}^0 g(x)\,\mathrm{d}x + 3\int_0^2 g(x)\,\mathrm{d}x - \int_{-1}^0 f(x)\,\mathrm{d}x - \int_0^2 f(x)\,\mathrm{d}x \\ &= 3\int_{-1}^0 g(x)\,\mathrm{d}x + 3\int_0^2 g(x)\,\mathrm{d}x + \int_0^{-1} f(x)\,\mathrm{d}x - \int_0^2 f(x)\,\mathrm{d}x \\ &= 3\times 3 + 3\times 4 + 1 - 2 = 20\end{aligned}$$

S-5: Recall that

$$|x| = \begin{cases} -x & \text{if } x \leqslant 0 \\ x & \text{if } x \geqslant 0 \end{cases}$$

so that

$$|2x| = \begin{cases} -2x & \text{if } x \leqslant 0 \\ 2x & \text{if } x \geqslant 0 \end{cases}$$

Also recall, from Example 1.2.5 in the CLP 101 notes that

$$\int_a^b x\,\mathrm{d}x = \frac{b^2 - a^2}{2}$$

So

$$\begin{aligned}\int_{-1}^2 |2x|\,\mathrm{d}x &= \int_{-1}^0 |2x|\,\mathrm{d}x + \int_0^2 |2x|\,\mathrm{d}x = \int_{-1}^0 (-2x)\,\mathrm{d}x + \int_0^2 2x\,\mathrm{d}x \\ &= -2\int_{-1}^0 x\,\mathrm{d}x + 2\int_0^2 x\,\mathrm{d}x = -2\,\frac{0^2 - (-1)^2}{2} + \frac{2^2 - 0^2}{2} \\ &= 1 + 4 = 5\end{aligned}$$

S-6: We first use additivity:

$$\int_{-2}^{2}\left(5+\sqrt{4-x^2}\right)\mathrm{d}x = \int_{-2}^{2} 5\,\mathrm{d}x + \int_{-2}^{2}\sqrt{4-x^2}\,\mathrm{d}x$$

The first integral represents the area of a rectangle of height 5 and width 4 and so equals 20. The second integral represents the area above the x–axis and below the curve $y=\sqrt{4-x^2}$ or $x^2+y^2=4$. That is a semicircle of radius 2, which has area $\frac{1}{2}\pi 2^2$. So

$$\int_{-2}^{2}\left(5+\sqrt{4-x^2}\right)\mathrm{d}x = 20+2\pi$$

S-7: Note that the integrand $f(x)=\frac{\sin x}{\log(3+x^2)}$ is an odd function because

$$f(-x)=\frac{\sin(-x)}{\log(3+(-x)^2)}=\frac{-\sin x}{\log(3+x^2)}=-f(x)$$

The domain of integration $-2012\leqslant x\leqslant 2012$ is symmetric about $x=0$. So, by Theorem 1.2.11 of the CLP notes,

$$\int_{-2012}^{+2012}\frac{\sin x}{\log(3+x^2)}\mathrm{d}x=0$$

S-8: Note that the integrand $f(x)=x^{1/3}\cos x$ is an odd function because

$$f(-x)=(-x)^{1/3}\cos(-x)=-x^{1/3}\cos x=-f(x)$$

The domain of integration $-2012\leqslant x\leqslant 2012$ is symmetric about $x=0$. So, by Theorem 1.2.11 of the CLP notes,

$$\int_{-2012}^{+2012}x^{1/3}\cos x\,\mathrm{d}x=0$$

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S-1: The fundamental theorem of calculus tells us that

$$\begin{aligned}\int_{1}^{\sqrt{5}} f(x)\,\mathrm{d}x &= F(\sqrt{5})-F(1)\\ &= \left(e^{(\sqrt{5}^2-3}+1\right)-\left(e^{(1^2-3)}+1\right)\\ &= e^{5-3}-e^{1-3}=e^2-e^{-2}\end{aligned}$$

S-2: One function with derivative x^3 is $\frac{x^4}{4}$. One function with derivative $\sin 2x$ is $-\frac{1}{2}\cos 2x$. So the general antiderivative of $f(x)$ is $\frac{x^4}{4}+\frac{1}{2}\cos 2x + C$. To satisfy $F(0)=1$, we need[1]

$$\left[\frac{x^4}{4}+\frac{1}{2}\cos 2x + C\right]_{x=0} = 1 \iff \frac{1}{2}+C=1 \iff C=\frac{1}{2}$$

So $F(x)=\frac{x^4}{4}+\frac{1}{2}\cos 2x+\frac{1}{2}$.

S-3: (a) This is true, by part 2 of the fundamental theorem of calculus, Thereom 1.3.1 in the CLP 101 notes, with $G(x)=f(x)$ and $f(x)$ replaced by $f'(x)$.

(b) This is not only false, but it makes no sense at all. The integrand is strictly positive so the integral has to be strictly positive. In fact it's $+\infty$. The fundamental theorem of calculus does not apply because the integrand has a singularity at $x=0$.

(c) This is not only false, but it makes no sense at all, unless $\int_a^b f(x)\,\mathrm{d}x = \int_a^b xf(x)\,\mathrm{d}x = 0$. The left hand side is a number. The right hand side is a number times x. For example, if $a=0$, $b=1$ and $f(x)=1$, then the left hand side is $\int_0^1 x\,\mathrm{d}x = \frac{1}{2}$ and the right hand side is $x\int_0^1 \mathrm{d}x = x$.

S-4: By the fundamental theorem of calculus,

$$\int_0^2 (x^3+\sin x)\,\mathrm{d}x = \left(\frac{x^4}{4}-\cos x\right)\bigg|_0^2 = \frac{2^4}{4}-\cos 2+\cos 0 = 4-\cos 2+1 = 5-\cos 2.$$

S-5: By part (d) of our "Arithmetic of Integration" theorem, Theorem 1.2.1 in the CLP 101 notes,

$$\int_1^2 \frac{x^2+2}{x^2}\,\mathrm{d}x = \int_1^2 \left[1+\frac{2}{x^2}\right]\mathrm{d}x = \int_1^2 \mathrm{d}x + 2\int_1^2 \frac{1}{x^2}\,\mathrm{d}x$$

Then by the fundamental theorem of calculus,

$$\int_1^2 \mathrm{d}x + 2\int_1^2 \frac{1}{x^2}\,\mathrm{d}x = \Big[x\Big]_1^2 + 2\Big[-\frac{1}{x}\Big]_1^2 = [2-1]+2\Big[-\frac{1}{2}+1\Big] = 2$$

S-6: By the fundamental theorem of calculus,

$$\begin{aligned} F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \log(2+\sin t)\,\mathrm{d}t &&= \log(2+\sin x) \\ G'(y) &= \frac{\mathrm{d}}{\mathrm{d}y}\left[-\int_0^y \log(2+\sin t)\,\mathrm{d}t\right] &&= -\log(2+\sin y) \end{aligned}$$

1 The symbol $\iff$ is read "if and only if". This is used in mathematics to express the logical equivalence of two statements. To be more precise, the statement $P \iff Q$ tells us that P is true whenever Q is true and Q is true whenever P is true.

So

$$F'\left(\frac{\pi}{2}\right) = \log 3 \qquad G'\left(\frac{\pi}{2}\right) = -\log(3)$$

S-7: By the fundamental theorem of calculus,

$$f'(x) = 100(x^2 - 3x + 2)e^{-x^2} = 100(x-1)(x-2)e^{-x^2}$$

As $f(x)$ is increasing whenever $f'(x) > 0$ and $100e^{-x^2}$ is always strictly bigger than 0, we have $f(x)$ increasing if and only if $(x-1)(x-2) > 0$, which is the case if and only if $(x-1)$ and $(x-2)$ are of the same sign. Both are positive when $x > 2$ and both are negative when $x < 1$. So $f(x)$ is increasing when $-\infty < x < 1$ and when $2 < x < \infty$.

S-8: Write $G(x) = \int_0^x \frac{1}{t^3+6}\,\mathrm{d}t$. By the fundamental theorem of calculus, $G'(x) = \frac{1}{x^3+6}$. Since $F(x) = G(\cos x)$, the chain rule gives

$$F'(x) = G'(\cos x)\cdot(-\sin x) = -\frac{\sin x}{\cos^3 x + 6}$$

S-9: Define $g(x) = \int_0^x e^{t^2}\,\mathrm{d}t$. By the fundamental theorem of calculus, $g'(x) = e^{x^2}$. As $f(x) = g(1+x^4)$ the chain rule gives

$$f'(x) = 4x^3 g'(1+x^4) = 4x^3 e^{(1+x^4)^2}$$

S-10: Define $g(x) = \int_0^x (t^6+8)\mathrm{d}t$. By the fundamental theorem of calculus, $g'(x) = x^6 + 8$. We are to compute the derivative of $f(x) = g(\sin x)$. The chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{\sin x} (t^6+8)\mathrm{d}t\right) = g'(\sin x)\cos x = \left(\sin^6 x + 8\right)\cos x$$

S-11: Let $G(x) = \int_0^x e^{-t}\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t$. By the fundamental theorem of calculus, $G'(x) = e^{-x}\sin\left(\frac{\pi x}{2}\right)$ and, since $F(x) = G(x^3)$, $F'(x) = 3x^2 G'(x^3) = 3x^2 e^{-x^3}\sin\left(\frac{\pi x^3}{2}\right)$ and $F'(1) = 3e^{-1}\sin\left(\frac{\pi}{2}\right) = 3e^{-1}$.

S-12: Define $g(x) = \int_x^0 \frac{\mathrm{d}t}{1+t^3} = -\int_0^x \frac{1}{1+t^3}\,\mathrm{d}t$, so that $g'(x) = -\frac{1}{1+x^3}$ by the fundamental theorem of calculus. Then by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\int_{\cos u}^0 \frac{\mathrm{d}t}{1+t^3}\right) = \frac{\mathrm{d}}{\mathrm{d}u} g(\cos u) = g'(\cos u)\cdot\frac{\mathrm{d}}{\mathrm{d}u}\cos u = -\frac{1}{1+\cos^3 u}\cdot(-\sin u).$$

S-13: Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of $x\sin(\pi x) = \int_0^x f(t)\,\mathrm{d}t$. Then, by the fundamental theorem of calculus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\{x\sin(\pi x)\} = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(t)\,\mathrm{d}t = f(x) &\implies f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\{x\sin(\pi x)\} = \sin(\pi x) + \pi x\cos(\pi x)\\ &\implies f(4) = \sin(4\pi) + 4\pi\cos(4\pi) = 4\pi\end{aligned}$$

S-14: (a) Write

$$F(x) = G(x^2) - H(-x) \qquad \text{with} \quad G(y) = \int_0^y e^{-t}\,\mathrm{d}t,\ H(y) = \int_0^y e^{-t^2}\,\mathrm{d}t$$

By the Fundamental Theorem of Calculus,

$$G'(y) = e^{-y} \qquad H'(y) = e^{-y^2}$$

Hence, by the chain rule,

$$F'(x) = 2xG'(x^2) - (-1)H'(-x) = 2xe^{-(x^2)} + e^{-(-x)^2} = (2x+1)e^{-x^2}$$

(b) Observe that $F'(x) < 0$ for $x < -1/2$ and $F'(x) > 0$ for $x > -1/2$. Hence $F(x)$ is decreasing for $x < -1/2$ and increasing for $x > -1/2$, and $F(x)$ must take its minimum value when $x = -1/2$.

S-15: Write

$$\begin{aligned}F(x) &= \int_0^x e^{\sin t}\,\mathrm{d}t + \int_{x^4-x^3}^0 e^{\sin t}\,\mathrm{d}t = \int_0^x e^{\sin t}\,\mathrm{d}t - \int_0^{x^4-x^3} e^{\sin t}\,\mathrm{d}t\\ &= G(x) - G(x^4 - x^3)\end{aligned}$$

with

$$G(y) = \int_0^y e^{\sin t}\,\mathrm{d}t$$

By the Fundamental Theorem of Calculus,

$$G'(y) = e^{\sin y}$$

Hence, by the chain rule,

$$\begin{aligned}F'(x) &= G'(x) - G'(x^4 - x^3)\,\frac{\mathrm{d}}{\mathrm{d}x}(x^4 - x^3)\\ &= G'(x) - G'(x^4 - x^3)\,(4x^3 - 3x^2)\\ &= e^{\sin x} - e^{\sin(x^4 - x^3)}(4x^3 - 3x^2)\end{aligned}$$

S-16: Write

$$\begin{aligned} F(x) &= \int_{x^5}^{-x^2} \cos\left(e^t\right) \mathrm{d}t = \int_0^{-x^2} \cos\left(e^t\right) \mathrm{d}t + \int_{x^5}^{0} \cos\left(e^t\right) \mathrm{d}t \\ &= \int_0^{-x^2} \cos\left(e^t\right) \mathrm{d}t - \int_0^{x^5} \cos\left(e^t\right) \mathrm{d}t \\ &= G(-x^2) - G(x^5) \end{aligned}$$

with

$$G(y) = \int_0^{y} \cos\left(e^t\right) \mathrm{d}t$$

By the Fundamental Theorem of Calculus,

$$G'(y) = \cos\left(e^y\right)$$

Hence, by the chain rule,

$$\begin{aligned} F'(x) &= G'(-x^2)\, \frac{\mathrm{d}}{\mathrm{d}x}\big(-x^2\big) - G'(x^5)\, \frac{\mathrm{d}}{\mathrm{d}x}\big(x^5\big) \\ &= G'(-x^2)\, (-2x) - G'(x^5)\, (5x^4) \\ &= -2x\cos\left(e^{-x^2}\right) - 5x^4\cos\left(e^{x^5}\right) \end{aligned}$$

S-17: Write

$$\begin{aligned} F(x) &= \int_{x}^{e^x} \sqrt{\sin t}\, \mathrm{d}t \\ &= \int_0^{e^x} \sqrt{\sin t}\, \mathrm{d}t + \int_x^0 \sqrt{\sin t}\, \mathrm{d}t = \int_0^{e^x} \sqrt{\sin t}\, \mathrm{d}t - \int_0^{x} \sqrt{\sin t}\, \mathrm{d}t \\ &= G(e^x) - G(x) \end{aligned}$$

with

$$G(y) = \int_0^{y} \sqrt{\sin t}\, \mathrm{d}t$$

By the Fundamental Theorem of Calculus,

$$G'(y) = \sqrt{\sin y}$$

Hence, by the chain rule,

$$\begin{aligned} F'(x) &= G'(e^x)\, \frac{\mathrm{d}}{\mathrm{d}x}\big(e^x\big) - G'(x) \\ &= e^x G'(e^x) - G'(x) \\ &= e^x\sqrt{\sin(e^x)} - \sqrt{\sin(x)} \end{aligned}$$

S-18: Splitting up the domain of integration

$$\begin{aligned}\int_1^5 f(x)\,\mathrm{d}x &= \int_1^3 f(x)\,\mathrm{d}x + \int_3^5 f(x)\,\mathrm{d}x \\ &= \int_1^3 3\,\mathrm{d}x + \int_3^5 x\,\mathrm{d}x \\ &= 3x\Big|_{x=1}^{x=3} + \frac{x^2}{2}\Big|_{x=3}^{x=5} \\ &= 14\end{aligned}$$

S-19: Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of $x^2 = 1 + \int_1^x f(t)\,dt$ gives, by the Fundamental Theorem of Calculus, $2x = f(x)$.

S-20: By the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f'(x)\big)^2 = 2f'(x)\,f''(x)$$

so $\frac{1}{2}f'(x)^2$ is an antiderivative for $f'(x)\,f''(x)$ and, by the fundamental theorem of calculus,

$$\int_1^2 f'(x)f''(x)\,\mathrm{d}x = \frac{1}{2}f'(x)^2\Big|_{x=1}^{x=2} = \frac{1}{2}f'(2)^2 - \frac{1}{2}f'(1)^2 = \frac{5}{2}$$

S-21: The car stops when $v(t) = 30 - 10t = 0$, which occurs at time $t = 3$. The distance covered up to that time is

$$\int_0^3 v(t)\,\mathrm{d}t = (30t - 5t^2)\Big|_0^3 = (90-45) - 0 = 45\,\mathrm{m}.$$

S-22: Define $g(x) = \int_0^x \log\left(1+e^t\right)\mathrm{d}t$. By the fundamental theorem of calculus, $g'(x) = \log\left(1+e^x\right)$. But $f(x) = g(2x - x^2)$, so, by the chain rule,

$$f'(x) = g'(2x-x^2)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(2x-x^2) = (2-2x)\cdot\log\left(1+e^{2x-x^2}\right)$$

Observe that $e^{2x-x^2} > 0$ for all x so that $1+e^{2x-x^2} > 1$ for all x and $\log\left(1+e^{2x-x^2}\right) > 0$ for all x. Since $2-2x$ is positive for $x<1$ and negative for $x>1$, $f'(x)$ is also positive for $x<1$ and negative for $x>1$. That is, $f(x)$ is increasing for $x<1$ and decreasing for $x>1$. So $f(x)$ achieves its absolute maximum at $x=1$.

S-23: Let $f(x) = \int_0^{x^2-2x} \frac{\mathrm{d}t}{1+t^4}$ and $g(x) = \int_0^x \frac{\mathrm{d}t}{1+t^4}$. Then $g'(x) = \frac{1}{1+x^4}$ and, since $f(x) = g(x^2-2x)$, $f'(x) = (2x-2)g'(x^2-2x) = 2\frac{x-1}{1+(x^2-2x)^4}$. This is zero for $x = 1$, negative for $x < 1$ and positive for $x > 1$. Thus as x runs from $-\infty$ to ∞, $f(x)$ decreases until x reaches 1 and then increases all $x > 1$. So the minimum of $f(x)$ is achieved for $x = 1$. At $x = 1$, $x^2 - 2x = -1$ and $f(1) = \int_0^{-1} \frac{\mathrm{d}t}{1+t^4}$.

S-24: Define $G(x) = \int_0^x \sin(\sqrt{t})\,\mathrm{d}t$. By the fundamental theorem of calculus $G'(x) = \sin(\sqrt{x})$. Since $F(x) = G(x^2)$ we have $F'(x) = 2xG'(x^2) = 2x \sin x$. Thus F increases as x runs from to 0 to π (since $F'(x) > 0$ there) and decreases as x runs from π to $4 < 2\pi$ (since $F'(x) < 0$ there). Thus F achieves its maximum value at $x = \pi$.

S-25: The given sum is of the form

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{\pi}{n} \sin\left(\frac{j\pi}{n}\right) = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\Delta x$$

with $\Delta x = \frac{\pi}{n}$, $x_j^* = \frac{j\pi}{n}$ and $f(x) = \sin(x)$. Since $x_0^* = 0$ and $x_n^* = \pi$, the right hand side is the definition (using the right Riemann sum) of

$$\int_0^{\pi} f(x)\,\mathrm{d}x = \int_0^{\pi} \sin(x)\,\mathrm{d}x = -\cos(x)\Big|_0^{\pi} = 2$$

S-26: The given sum is of the form

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{1+\frac{j}{n}} = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\Delta x$$

with $\Delta x = \frac{1}{n}$, $x_j = \frac{j}{n}$ and $f(x) = \frac{1}{1+x}$. The right hand side is the definition (using the right Riemann sum) of

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{1}{1+x}\,\mathrm{d}x = \log|1+x|\Big|_0^1 = \log 2$$

S-27: (a) Using the product rule, followed by the chain rule, followed by the fundamental theorem of calculus,

$$\begin{aligned}
f'(x) &= 3x^2 \int_0^{x^3+1} e^{t^3}\,\mathrm{d}t + x^3 \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{x^3+1} e^{t^3}\,\mathrm{d}t \\
&= 3x^2 \int_0^{x^3+1} e^{t^3}\,\mathrm{d}t + x^3\ \left[3x^2\right] \left[\frac{d}{dy}\int_0^y e^{t^3}\,\mathrm{d}t\right]_{y=x^3+1} \\
&= 3x^2 \int_0^{x^3+1} e^{t^3}\,\mathrm{d}t + x^3\ \left[3x^2\right] \left[e^{y^3}\right]_{y=x^3+1} \\
&= 3x^2 \int_0^{x^3+1} e^{t^3}\,\mathrm{d}t + x^3\ \left[3x^2\right] e^{(x^3+1)^3} \\
&= 3x^2 \int_0^{x^3+1} e^{t^3}\,\mathrm{d}t + 3x^5 e^{(x^3+1)^3}
\end{aligned}$$

(b) In general, the equation of the tangent line to the graph of $y = f(x)$ at $x = a$ is

$$y = f(a) + f'(a)\,(x - a)$$

Substituting in the given $f(x)$ and

$$a = -1 \qquad\qquad a^3 + 1 = (-1)^3 + 1 = 0$$

$$f(-1) = (-1)^3 \int_0^0 e^{t^3}\,\mathrm{d}t = 0 \qquad\qquad f'(-1) = 3(-1)^2 \times 0 + 3(-1)^5 e^{0^3} = -3$$

the equation of the tangent line is $y = -3(x + 1)$

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S-1: We substitute $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$, $\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - u^2}$, $\mathrm{d}x = \frac{\mathrm{d}u}{\cos x} = \frac{\mathrm{d}u}{\sqrt{1-u^2}}$. When $x = 0$ we have $u = \sin 0 = 0$ and when $x = \frac{\pi}{2}$ we have $u = \sin\frac{\pi}{2} = 1$ so that

$$\int_{x=0}^{x=\pi/2} f(\sin x)\,\mathrm{d}x = \int_{u=0}^{u=1} f(u)\,\frac{\mathrm{d}u}{\sqrt{1 - u^2}}$$

S-2: We write $u(x) = e^{x^2}$ and find $\mathrm{d}u = u'(x)\,\mathrm{d}x = 2xe^{x^2}\mathrm{d}x$. Note that $u(1) = e^{1^2} = e$ when $x = 1$, and $u(0) = e^{0^2} = 1$ when $x = 0$. Therefore:

$$\begin{aligned}
\int_0^1 xe^{x^2}\cos(e^{x^2})\,\mathrm{d}x &= \frac{1}{2}\int_{x=0}^{x=1} \cos(u(x))u'(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{u=1}^{u=e} \cos(u)\,\mathrm{d}u \\
&= \frac{1}{2}\sin(u)\big|_1^e = \frac{1}{2}\big(\sin(e) - \sin(1)\big).
\end{aligned}$$

S-3: Substituting $y = x^3$, $\mathrm{d}y = 3x^2$

$$\int_1^2 x^2 f(x^3)\,\mathrm{d}x = \frac{1}{3}\int_1^8 f(y)\,\mathrm{d}y = \frac{1}{3}$$

S-4: Setting $u = x^3 + 1$, we have $\mathrm{d}u = 3x^2\,\mathrm{d}x$ and so

$$\int \frac{x^2\,\mathrm{d}x}{(x^3+1)^{101}} = \int \frac{\mathrm{d}u/3}{u^{101}} = -\frac{1}{3\times 100u^{100}} + C = -\frac{1}{300(x^3+1)^{100}} + C$$

S-5: Setting $u = \log x$, we have $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and so

$$\int_e^{e^4} \frac{\mathrm{d}x}{x\log x} = \int_{x=e}^{x=e^4} \frac{1}{\log x}\cdot\frac{1}{x}\,\mathrm{d}x = \int_{u=1}^{u=4} \frac{1}{u}\,\mathrm{d}u,$$

since $u = \log(e) = 1$ when $x = e$ and $u = \log(e^4) = 4$ when $x = e^4$. Then, by the fundamental theorem of calculus,

$$\int_1^4 \frac{1}{u}\,\mathrm{d}u = (\log|u|)\Big|_1^4 = \log 4 - \log 1 = \log 4.$$

S-6: Setting $u = 1 + \sin x$, we have $\mathrm{d}u = \cos x\,\mathrm{d}x$ and so

$$\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\mathrm{d}x = \int_{x=0}^{x=\pi/2} \frac{1}{1+\sin x}\cos x\,\mathrm{d}x = \int_{u=1}^{u=2} \frac{\mathrm{d}u}{u}$$

since $u = 1 + \sin 0 = 1$ when $x = 0$ and $u = 1 + \sin(\pi/2) = 2$ when $x = \pi/2$. Then, by the fundamental theorem of calculus,

$$\int_{u=1}^{u=2} \frac{\mathrm{d}u}{u} = \log|u|\Big|_1^2 = \log 2$$

S-7: Setting $u = \sin x$, we have $\mathrm{d}u = \cos x\,\mathrm{d}x$ and so

$$\int_0^{\pi/2} \cos x\cdot(1+\sin^2 x)\mathrm{d}x = \int_{x=0}^{x=\pi/2} (1+\sin^2 x)\cdot\cos x\,\mathrm{d}x = \int_{u=0}^{u=1} (1+u^2)\,\mathrm{d}u,$$

since $u = \sin 0 = 0$ when $x = 0$ and $u = \sin(\pi/2) = 1$ when $x = \pi/2$. Then, by the fundamental theorem of calculus,

$$\int_0^1 (1+u^2)\,\mathrm{d}u = (u + u^3/3)\Big|_0^1 = (1+1/3) - 0 = 4/3.$$

S-8: Substituting $t = x^2 - x$, $\mathrm{d}t = (2x-1)\,\mathrm{d}x$ and noting that $t = 0$ when $x = 1$ and $t = 6$ when $x = 3$,

$$\int_1^3 (2x-1)e^{x^2-x}\,\mathrm{d}x = \int_0^6 e^t\,\mathrm{d}t = \big[e^t\big]_0^6 = e^6 - 1$$

S-9: We use the substitution $u = 4 - x^2$, for which $\mathrm{d}u = -2x\,\mathrm{d}x$,:

$$\begin{aligned}
\int \frac{x^2-4}{\sqrt{4-x^2}}\,x\,\mathrm{d}x &= \int \frac{1}{2}\frac{4-x^2}{\sqrt{4-x^2}}(-2x)\,\mathrm{d}x \\
&= \frac{1}{2}\int \frac{u}{\sqrt{u}}\,\mathrm{d}u \\
&= \frac{1}{2}\int \sqrt{u}\,\mathrm{d}u \\
&= \frac{1}{2}\frac{u^{3/2}}{3/2} + C \\
&= \frac{1}{3}(4-x^2)^{3/2} + C
\end{aligned}$$

S-10: *The slightly sneaky method:* We note that $\frac{\mathrm{d}}{\mathrm{d}x}e^{x^2} = 2x\,e^{x^2}$, so that $\frac{1}{2}e^{x^2}$ is a antiderivative for the integrand xe^{x^2}. So

$$\int_{-2}^2 xe^{x^2}\,dx = \left[\frac{1}{2}e^{x^2}\right]_{-2}^2 = \frac{1}{2}e^4 - \frac{1}{2}e^4 = 0$$

The really sneaky method: The integrand $f(x) = xe^{x^2}$ is an odd function (meaning that $f(-x) = -f(x)$). So by Theorem 1.2.11 in the CLP 101 notes every integral of the form $\int_{-a}^a xe^{x^2}\,\mathrm{d}x$ is zero.

S-11: The given sum is of the form

$$\lim_{n\to\infty}\sum_{j=1}^n \frac{j}{n^2}\sin\left(1+\frac{j^2}{n^2}\right) = \lim_{n\to\infty}\sum_{j=1}^n f(x_j^*)\Delta x$$

with $\Delta x = \frac{1}{n}$, $x_j^* = \frac{j}{n}$ and $f(x) = x\sin(1+x^2)$. Since $x_0^* = 0$ and $x_n^* = 1$, the right hand side is the definition (using the right Riemann sum) of

$$\begin{aligned}
\int_0^1 f(x)\,\mathrm{d}x &= \int_0^1 x\sin(1+x^2)\,\mathrm{d}x \\
&= \frac{1}{2}\int_1^2 \sin(y)\,\mathrm{d}y \qquad \text{with } y = 1+x^2,\ \mathrm{d}y = 2x\,\mathrm{d}x \\
&= \frac{1}{2}\big[-\cos(y)\big]_1^2 \\
&= \frac{1}{2}[\cos 1 - \cos 2] \approx 0.478
\end{aligned}$$

S-12: The given sum is of the form

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2}\cos\left(\frac{j^2}{n^2}\right)=\lim_{n\to\infty}\sum_{j=1}^{n}f(x_j^*)\Delta x$$

with $\Delta x=\frac{1}{n}$, $x_j^*=\frac{j}{n}$ and $f(x)=x\cos(x^2)$. Since $x_0^*=0$ and $x_n^*=1$, the right hand side is the definition (using the right Riemann sum) of

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x&=\int_0^1 x\cos(x^2)\,\mathrm{d}x\\&=\frac{1}{2}\int_0^1\cos(y)\,\mathrm{d}y\qquad\text{with } y=x^2,\ \mathrm{d}y=2x\,\mathrm{d}x\\&=\frac{1}{2}\Big[\sin(y)\Big]_0^1\\&=\frac{1}{2}\sin 1\end{aligned}$$

S-13: The given sum is of the form

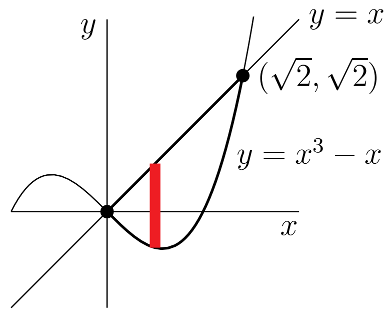
$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2}\sqrt{1+\frac{j^2}{n^2}}=\lim_{n\to\infty}\sum_{j=1}^{n}f(x_j^*)\Delta x$$

with $\Delta x=\frac{1}{n}$, $x_j^*=\frac{j}{n}$ and $f(x)=x\sqrt{1+x^2}$. Since $x_0^*=0$ and $x_n^*=1$, the right hand side is the definition (using the right Riemann sum) of

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x&=\int_0^1 x\sqrt{1+x^2}\,\mathrm{d}x\\&=\frac{1}{2}\int_1^2\sqrt{y}\,\mathrm{d}y\qquad\text{with } y=1+x^2,\ \mathrm{d}y=2x\,\mathrm{d}x\\&=\frac{1}{2}\left[\frac{y^{3/2}}{3/2}\right]_1^2\\&=\frac{1}{3}[2\sqrt{2}-1]\approx 0.609\end{aligned}$$

Solutions to Exercises 1.5 — Jump to TABLE OF CONTENTS

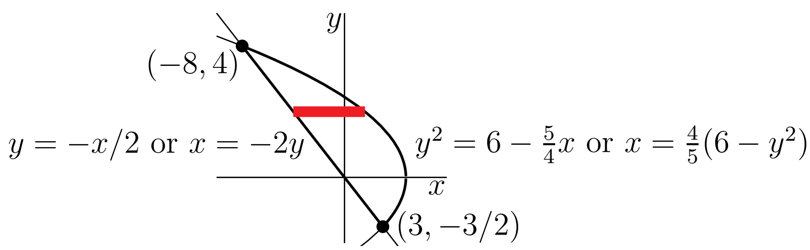
S-1: The curves intersect when $y=x$ and $y=x^3-x$ so that $x=x^3-x$ or $x(x^2-2)=0$. For $x\geqslant 0$, the curves intersect at $(0,0)$ and $(\sqrt{2},\sqrt{2})$. Using vertical strips as in the sketch



the top and bottom boundaries of the specified region are $y = T(x) = x$ and $y = B(x) = x^3 - x$, respectively. So

$$\text{Area} = \int_0^{\sqrt{2}} \big[T(x) - B(x)\big]\,\mathrm{d}x = \int_0^{\sqrt{2}} \big[x - (x^3 - x)\big]\,\mathrm{d}x$$

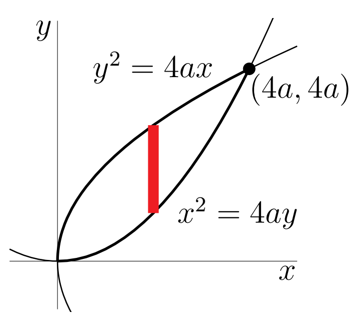
S-2: The curves intersect at $(-8, 4)$ and $(3, -\frac{3}{2})$. Using horizontal strips as in the sketch



we have

$$\text{Area} = \int_{-3/2}^{4} \left[\frac{4}{5}(6 - y^2) + 2y\right]\mathrm{d}y$$

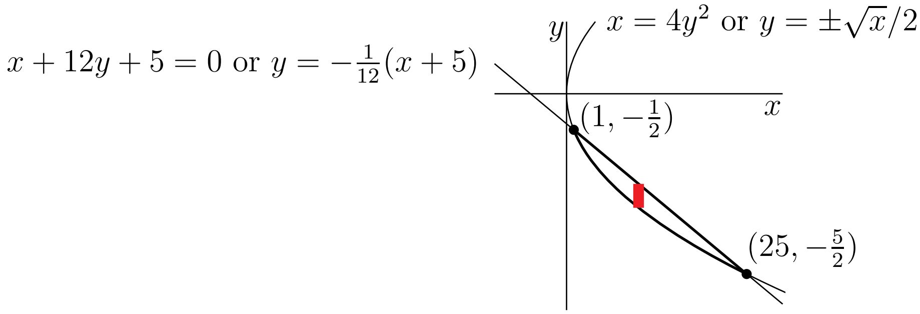
S-3: The curves intersect at $(0, 0)$ and $(4a, 4a)$. Using vertical strips as in the sketch



we have

$$\text{Area} = \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a}\right]\mathrm{d}x$$

S-4: The curves intersect when $x = 4y^2$ and $0 = 4y^2 + 12y + 5 = (2y+5)(2y+1)$. So the curves intersect at $(1, -\frac{1}{2})$ and $(25, -\frac{5}{2})$. Using vertical strips as in the sketch



we have

$$\text{Area} = \int_1^{25} \Big[-\frac{1}{12}(x+5) + \frac{1}{2}\sqrt{x} \Big]\, dx$$

S-5: We are asked for the area between the top curve $y = T(x) = \frac{1}{(2x-4)^2}$ and the bottom curve $y = B(x) = 0$ with x running from $a = 0$ to $b = 1$. So, by (1.5.1) in the CLP 101 notes, the area is

$$\int_a^b \big[T(x) - B(x)\big] dx = \int_0^1 \frac{dx}{(2x-4)^2} = \Big[-\frac{1}{2} \cdot \frac{1}{2x-4} \Big]_0^1 = \Big[\frac{1}{4} - \frac{1}{8} \Big] = \frac{1}{8}$$

S-6: The two curves $y = f(x) = x$ and $y = g(x) = 3x - x^2$, intersect when

$$\begin{aligned} f(x) = g(x) \quad &\iff \quad x = 3x - x^2 \quad \iff \quad 2x - x^2 = 0 \quad \iff \quad x(2-x) = 0 \\ &\iff \quad x = 0, 2 \end{aligned}$$

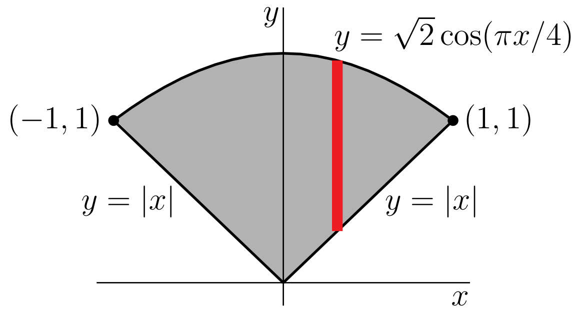
Furthermore $g(x) - f(x) = 2x - x^2 = x(2-x)$ is positive for all $0 \leqslant x \leqslant 2$. That is, the curve $y = 3x - x^2$ lies above the line $y = x$ for all $0 \leqslant x \leqslant 2$. We therefore evaluate the integral:

$$\int_0^2 \big[(3x - x^2) - x\big]\, dx = \int_0^2 [2x - x^2]\, dx = \Big[x^2 - \frac{x^3}{3} \Big]_0^2 = \Big[4 - \frac{8}{3} \Big] - 0 = \frac{4}{3}$$

S-7: The two curves cross when $x = 0$ ($y = 1$ for both curves) and $x = 1$ ($y = 2$ for both curves). Since $2^x = \big(e^{\log 2}\big)^x = e^{x \log 2}$, the area is

$$\begin{aligned} \text{Area} &= \int_0^1 \big[(1 + \sqrt{x}) - e^{x \log 2}\big]\, dx = \Big[x + \frac{2}{3}x^{3/2} - \frac{1}{\log 2} 2^x \Big]_0^1 \\ &= 1 + \frac{2}{3} - \frac{1}{\log 2}[2 - 1] = \frac{5}{3} - \frac{1}{\log 2} \end{aligned}$$

S-8: Here is a sketch of the specified region.



It is symmetric about the y–axis. So we will compute the area of the part with $x \geqslant 0$ and multiply by 2. The curves $y = \sqrt{2}\cos(\pi x/4)$ and $y = x$ intersect when $x = \sqrt{2}\cos(\pi x/4)$ or $\cos(\pi x/4) = \frac{x}{\sqrt{2}}$, which is the case[2] when $x = 1$. So, using vertical strips as in the figure above, the area (including the multiplication by 2) is

$$2\int_0^1 \left[\sqrt{2}\cos(\pi x/4) - x\right] \mathrm{d}x = 2\left[\sqrt{2}\,\frac{4}{\pi}\sin(\pi x/4) - \frac{x^2}{2}\right]_0^1 = \boxed{2\left[\frac{4}{\pi} - \frac{1}{2}\right]}$$

S-9: For our computation, we will need an antiderivative of $x^2\sqrt{x^3+1}$, which can be found using the substitution $u = x^3 + 1$, $\mathrm{d}u = 3x^2\,\mathrm{d}x$:

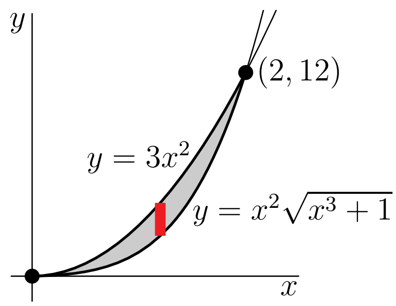
$$\int x^2\sqrt{x^3+1}\,\mathrm{d}x = \int \sqrt{u}\cdot\frac{1}{3}\,\mathrm{d}u = \frac{1}{3}\int u^{1/2}\,\mathrm{d}u = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \frac{2}{9}(x^3+1)^{3/2} + C.$$

The two functions $f(x)$ and $g(x)$ are clearly equal at $x = 0$. If $x \neq 0$, then the functions are equal when

$$\begin{aligned} 3x^2 &= x^2\sqrt{x^3+1} \\ 3 &= \sqrt{x^3+1} \\ 9 &= x^3+1 \\ 8 &= x^3 \\ 2 &= x. \end{aligned}$$

The function $g(x) = 3x^2$ is the larger of the two on the interval $[0,2]$, as can be seen by plugging in $x = 1$, say, or by observing that when x is very small $f(x) = x^2\sqrt{x^3+1} \approx x^2$ and $g(x) = 3x^2$.

2 The solution $x = 1$ was found by guessing. To guess a solution to $\cos(\pi x/4) = \frac{x}{\sqrt{2}}$ just ask yourself what simple angle has a cosine that involves $\sqrt{2}$. This guessing strategy is essentially useless in the real world, but works great on problem sets and exams.

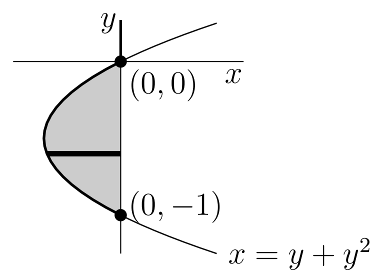


The area in question is therefore

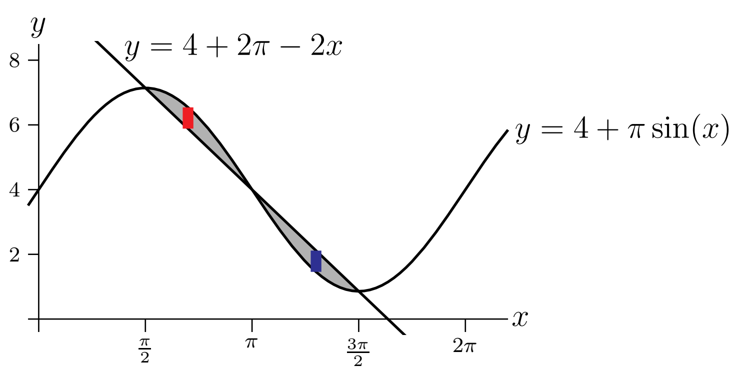
$$\begin{aligned}\int_0^2 \left(3x^2 - x^2\sqrt{x^3+1}\right) \mathrm{d}x &= \left(x^3 - \frac{2}{9}(x^3+1)^{3/2}\right)\Big|_0^2 \\ &= \left(2^3 - \frac{2}{9}(2^3+1)^{3/2}\right) - \left(0^3 - \frac{2}{9}(0^3+1)^{3/2}\right) \\ &= \left(8-6\right) - \left(0 - \frac{2}{9}\right) = \frac{20}{9}.\end{aligned}$$

S-10: A point (x, y) on the curve $x = y^2 + y = y(y+1)$ has $x = 0$ for $y = -1, 0$, has $x < 0$ for $-1 < y < 0$ (the factors y and $y+1$ have opposite signs) and has $x > 0$ for $y < -1$ and $y > 0$ (the factors y and $y+1$ are either both positive or both negative). This leads to the figure below. So, using horizontal slices,

$$\text{area} = \int_{-1}^{0} \left(0 - (y^2+y)\right) \mathrm{d}y = -\left[\frac{y^3}{3} + \frac{y^2}{2}\right]_{-1}^{0} = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$$



S-11: We will compute the area by using thin vertical strips as in the sketch



The line $y = 4 + 2\pi - 2x$ intersects the curve $y = 4 + \pi \sin x$ when

$$4 + 2\pi - 2x = 4 + \pi \sin x \iff \sin x = 2 - \frac{2}{\pi}x \iff x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

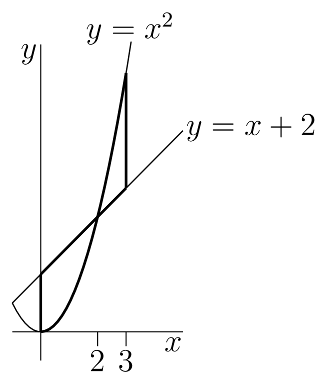
These solutions were guessed by looking at the sketch above, but then verified by substituting them back into the equation. From the sketch we see that

- when $\frac{\pi}{2} \leqslant x \leqslant \pi$, the top of the strip is at $y = 4 + \pi \sin x$ and the bottom of the strip is at $y = 4 + 2\pi - 2x$. So the strip has height $\big[(4 + \pi \sin x) - (4 + 2\pi - 2x)\big]$ and width $\mathrm{d}x$ and hence area $\big[(4 + \pi \sin x) - (4 + 2\pi - 2x)\big]\mathrm{d}x$.
- when $\pi \leqslant x \leqslant \frac{3\pi}{2}$, the top of the strip is at $y = 4 + 2\pi - 2x$ and the bottom of the strip is at $y = 4 + \pi \sin x$. So the strip has height $\big[(4 + 2\pi - 2x) - (4 + \pi \sin x)\big]$ and width $\mathrm{d}x$ and hence area $\big[(4 + 2\pi - 2x) - (4 + \pi \sin x)\big]\mathrm{d}x$.

So the total

$$\begin{aligned}
\text{Area} &= \int_{\pi/2}^{\pi} \big[(4 + \pi \sin x) - (4 + 2\pi - 2x)\big]\, \mathrm{d}x + \int_{\pi}^{3\pi/2} \big[(4 + 2\pi - 2x) - (4 + \pi \sin x)\big]\, \mathrm{d}x \\
&= \int_{\pi/2}^{\pi} \big[\pi \sin x - 2\pi + 2x\big]\, \mathrm{d}x + \int_{\pi}^{3\pi/2} \big[2\pi - 2x - \pi \sin x\big]\, \mathrm{d}x \\
&= \big[-\pi \cos x - 2\pi x + x^2\big]_{\pi/2}^{\pi} + \big[2\pi x - x^2 + \pi \cos x\big]_{\pi}^{3\pi/2} \\
&= \big[\pi - \pi^2 + \frac{3}{4}\pi^2\big] + \big[\pi^2 - \frac{5}{4}\pi^2 + \pi\big] \\
&= 2\Big[\pi - \frac{1}{4}\pi^2\Big]
\end{aligned}$$

S-12: First, here is a sketch of the region. We are not asked for it, but it is still a crucial for understanding the question.



The two curves $y = x + 2$ and $y = x^2$ cross at $x = 2$, $y = 4$. The area of the part between them with $0 \leqslant x \leqslant 2$ is

$$\int_0^2 \big[x + 2 - x^2\big]\, \mathrm{d}x = \Big[\tfrac{1}{2}x^2 + 2x - \frac{1}{3}x^3\Big]_0^2 = 2 + 4 - \frac{8}{3} = \frac{10}{3}$$

The area of the part between them with $2 \leqslant x \leqslant 3$ is

$$\int_2^3 \big[x^2 - (x + 2)\big]\, \mathrm{d}x = \Big[\frac{1}{3}x^3 - \tfrac{1}{2}x^2 - 2x\Big]_2^3 = 9 - \frac{9}{2} - 6 - \frac{8}{3} + 2 + 4 = \frac{11}{6}$$

The total area is $\frac{10}{3} + \frac{11}{6} = \frac{31}{6} = 5.1\dot{6}$.

S-13: The curve $y = f(x) = x\sqrt{25 - x^2}$ lies above the line $y = g(x) = 3x$ at all values of x for which $f(x) \geqslant g(x)$, i.e. $f(x) - g(x) \geqslant 0$. Now

$$f(x) - g(x) = x\sqrt{25 - x^2} - 3x = x\big(\sqrt{25 - x^2} - 3\big)$$

The first factor is positive for all $x \geqslant 0$. The second factor is positive whenever

$$\sqrt{25 - x^2} - 3 \geqslant 0 \quad \Longleftrightarrow \quad \sqrt{25 - x^2} \geqslant 3 \quad \Longleftrightarrow \quad 25 - x^2 \geqslant 9 \quad \Longleftrightarrow \quad x^2 \leqslant 16 \quad \Longleftrightarrow \quad |x| \leqslant 4$$

So $y = f(x) = x\sqrt{25 - x^2}$ lies above $y = g(x) = 3x$ for all $0 \leqslant x \leqslant 4$. The area we need to calculate is therefore:

$$A = \int_0^4 [x\sqrt{25 - x^2} - 3x]\,\mathrm{d}x = \int_0^4 x\sqrt{25 - x^2}\,\mathrm{d}x - \int_0^4 3x\,\mathrm{d}x = A_1 - A_2.$$

To evaluate A_1, we use the substitution $u(x) = 25 - x^2$, for which $\mathrm{d}u = u'(x)\,\mathrm{d}x = -2x\,\mathrm{d}x$; and $u(4) = 25 - 4^2 = 9$ when $x = 4$, while $u(0) = 25 - 0^2 = 25$ when $x = 0$. Therefore

$$A_1 = \int_{x=0}^{x=4} x\sqrt{25 - x^2}\,\mathrm{d}x = -\frac{1}{2}\int_{u=25}^{u=9} \sqrt{u}\,\mathrm{d}u = -\frac{1}{3}u^{3/2}\Big|_{25}^{9} = \frac{125 - 27}{3} = \frac{98}{3}$$

For A_2 we use the antiderivative directly:

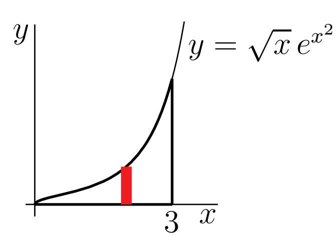
$$A_2 = \int_0^4 3x\,\mathrm{d}x = \frac{3x^2}{2}\Big|_0^4 = 24$$

Therefore the total area is:

$$A = \frac{98}{3} - 24 = \frac{26}{3}$$

Solutions to Exercises 1.6 — Jump to TABLE OF CONTENTS

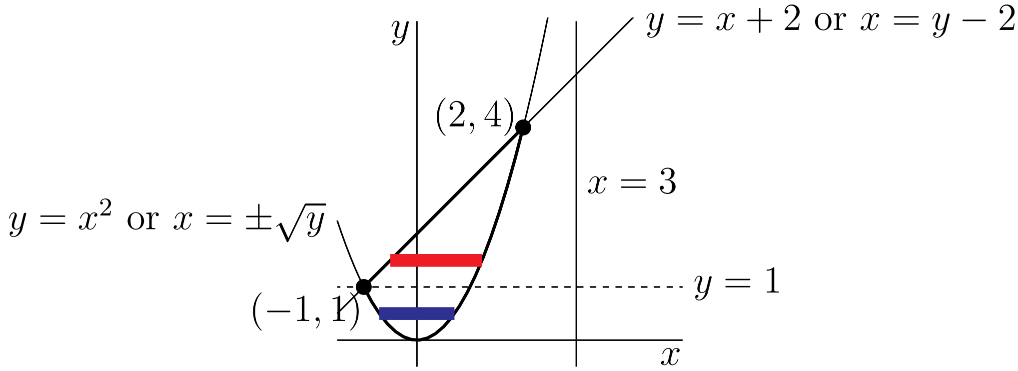
S-1: (a) When the strip shown in the figure



is rotated about the x–axis, it forms a thin disk of radius $\sqrt{x}e^{x^2}$ and thickness dx and hence of cross sectional area $\pi x e^{2x^2}$ and volume $\pi x e^{2x^2}$ dx So the volume of the solid is

$$\pi \int_0^3 x e^{2x^2} \, dx$$

(b) The curves intersect at $(-1, 1)$ and $(2, 4)$.

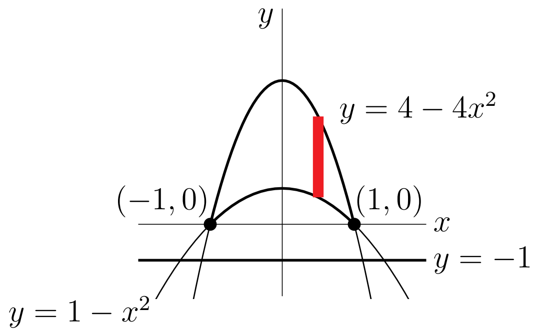


We'll use horizontal washers as in Example 1.6.5 of the CLP 101 notes.

- We use thin horizontal strips of width dy as in the figure above.
- When we rotate about the line $x = 3$, each strip sweeps out a thin washer
  - whose inner radius is $r_{in} = 3 - \sqrt{y}$, and
  - whose outer radius is $r_{out} = 3 - (y - 2) = 5 - y$ when $y \geqslant 1$ (see the red strip in the figure on the right above), and whose outer radius is $r_{out} = 3 - (-\sqrt{y}) = 3 + \sqrt{y}$ when $y \leqslant 1$ (see the blue strip in the figure on the right above) and
  - whose thickness is dy and hence
  - whose volume is $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi\big[(5-y)^2 - (3-\sqrt{y})^2\big]\mathrm{d}y$ when $y \geqslant 1$ and whose volume is $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi\big[(3+\sqrt{y})^2 - (3-\sqrt{y})^2\big]\mathrm{d}y$ when $y \leqslant 1$ and
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = 4$, the total volume is

$$\int_0^1 \pi\big[(3+\sqrt{y})^2 - (3-\sqrt{y})^2\big]\mathrm{d}y + \int_1^4 \pi\big[(5-y)^2 - (3-\sqrt{y})^2\big]\mathrm{d}y$$

S-2: (a) The curves intersect at $(1, 0)$ and $(-1, 0)$. When the strip shown in the figure



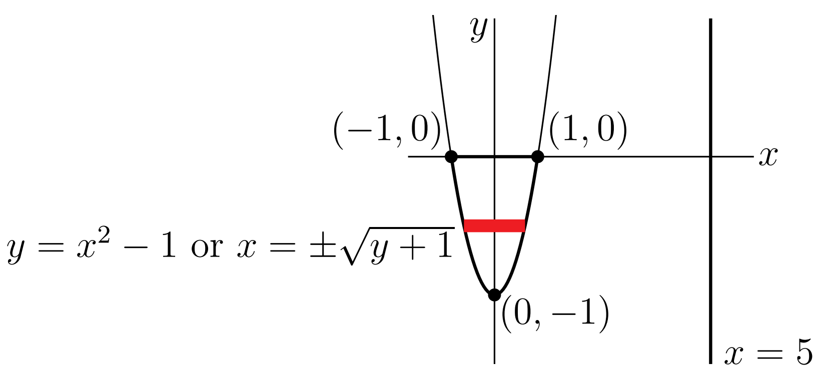
is rotated about the line $y = -1$, it forms a thin washer of

- inner radius $(1 - x^2) - (-1) = 2 - x^2$,
- outer radius $(4 - 4x^2) - (-1) = 5 - 4x^2$ and
- thickness $\mathrm{d}x$ and hence of
- cross sectional area $\pi\big[(5 - 4x^2)^2 - (2 - x^2)^2\big]$ and
- volume $\pi\big[(5 - 4x^2)^2 - (2 - x^2)^2\big]\,\mathrm{d}x$.

So the volume of the solid is

$$\int_{-1}^{1} \pi\big[(5 - 4x^2)^2 - (2 - x^2)^2\big]\,\mathrm{d}x$$

(b) The curve $y = x^2 - 1$ intersects $y = 0$ at $(1, 0)$ and $(-1, 0)$.



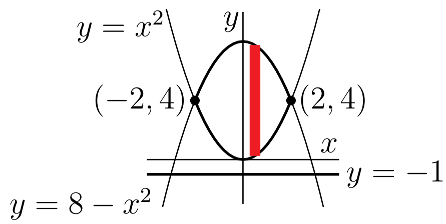
We'll use horizontal washers.

- We use thin horizontal strips of width $\mathrm{d}y$ as in the figure above.
- When we rotate about the line $x = 5$, each strip sweeps out a thin washer
  - whose inner radius is $r_{in} = 5 - \sqrt{y+1}$, and
  - whose outer radius is $r_{out} = 5 - (-\sqrt{y+1}) = 5 + \sqrt{y+1}$ and
  - whose thickness is $\mathrm{d}y$ and hence
  - whose volume is $\pi(r_{out}^2 - r_{in}^2)\,\mathrm{d}y = \pi\big[(5 + \sqrt{y+1})^2 - (5 - \sqrt{y+1})^2\big]\,\mathrm{d}y$

- As our topmost strip is at $y = 0$ and our bottommost strip is at $y = -1$ (when $x = 0$), the total volume is

$$\int_{-1}^{0} \pi\big[\big(5+\sqrt{y+1}\big)^2 - \big(5-\sqrt{y+1}\big)^2\big]\,\mathrm{d}y$$

S-3: (a) The curves intersect at $(-2, 4)$ and $(2, 4)$. When the strip shown in the figure



is rotated about the line $y = -1$, it forms a thin washer (punctured disc) of

- inner radius $x^2 + 1$,
- outer radius $9 - x^2$ and
- thickness $\mathrm{d}x$ and hence of
- cross sectional area $\pi\big[(9-x^2)^2 - (x^2+1)^2\big]$ and
- volume $\pi\big[(9-x^2)^2 - (x^2+1)^2\big]\,\mathrm{d}x$.

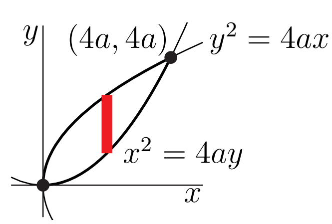
So the volume of the solid is

$$\pi\int_{-2}^{2} \big[(9-x^2)^2 - (x^2+1)^2\big]\,\mathrm{d}x$$

S-4: (a) The curves intersect at points (x, y) which satisfy both $y^2 = 4ax$ and $x^2 = 4ay$. Substituting $y = \frac{x^2}{4a}$ (from the second equation) into the first equation gives

$$\frac{x^4}{4^2a^2} = 4ax \iff x^4 = 4^3a^3x \iff x(x^3 - 4^3a^3) = 0$$

This has two solutions: $x = 0$ and $x = 4a$. The corresponding values of y are $y = 0$ and $y = 4a$. So the curves intesect at $(0, 0)$ and $(4a, 4a)$. The strip shown in the figure



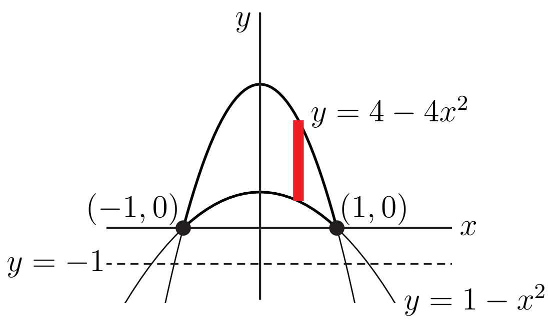
runs from $y = B(x) = \frac{x^2}{4a}$ (gotten by solving $x^2 = 4ay$ for y) to $y = T(x) = \sqrt{4ax}$ (gotten by solving $y^2 = 4ax$ for y) and hence has height $T(x) - B(x) = \sqrt{4ax} - \frac{x^2}{4a}$ and width $\mathrm{d}x$. So the desired

$$\text{Area} = \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a}\right] \mathrm{d}x$$

(b) The curves intersect at points (x, y) which satisfy both $y = 1 - x^2$ and $y = 4(1 - x^2)$. That is, where

$$1 - x^2 = 4(1 - x^2) \iff 3(1 - x^2) = 0 \iff x = \pm 1$$

Thus the curves intersect at $(1, 0)$ and $(-1, 0)$. When the strip shown in the figure



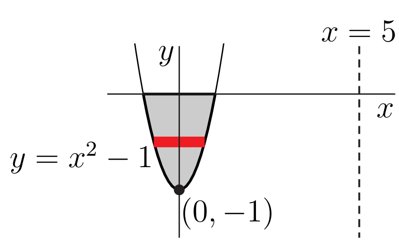
is rotated about the line $y = -1$, it forms a thin washer (punctured disc) of

- inner radius $(1 - x^2) - (-1) = 2 - x^2$,
- outer radius $(4 - 4x^2) - (-1) = 5 - 4x^2$ and
- thickness $\mathrm{d}x$ and hence of
- cross sectional area $\pi\big[(5 - 4x^2)^2 - (2 - x^2)^2\big]$ and
- volume $\pi\big[5 - 4x^2)^2 - (2 - x^2)^2\big]\,\mathrm{d}x$.

So the volume of the solid is

$$\int_{-1}^{1} \pi\big[(5 - 4x^2)^2 - (2 - x^2)^2\big]\,\mathrm{d}x$$

(c) Note that solving $y = x^2 - 1$ for x gives $x = \pm\sqrt{y + 1}$. When the strip shown in the figure



is rotated about the line $x = 5$, it forms a thin washer (punctured disc) of

- inner radius $5 - \sqrt{y + 1}$,
- outer radius $5 + \sqrt{y + 1}$ and

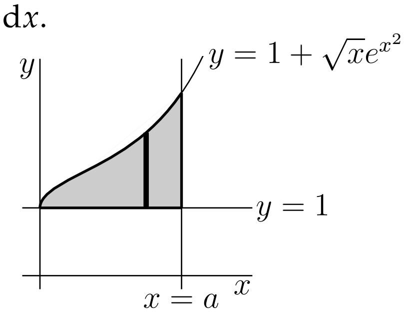
- thickness $\mathrm{d}y$ and hence of
- cross sectional area $\pi\big[(5+\sqrt{y+1})^2 - (5-\sqrt{y+1})^2\big] = 20\pi\sqrt{y+1}$ and
- volume $\pi\big[(5+\sqrt{y+1})^2 - (5-\sqrt{y+1})^2\big]\,\mathrm{d}y = 20\pi\sqrt{y+1}\,\mathrm{d}y$.

So the volume of the solid is

$$\int_{-1}^{0} \pi\big[(5+\sqrt{y+1})^2 - (5-\sqrt{y+1})^2\big]\,\mathrm{d}y = \int_{-1}^{0} 20\pi\sqrt{y+1}\,\mathrm{d}y$$

S-5: Let $f(x) = 1+\sqrt{x}e^{x^2}$. On the vertical slice a distance x from the y-axis, sketched in the figure below, y runs from 1 to $f(x)$. Upon rotation about the line $y=1$, this thin slice sweeps out a cylinder of thickess $\mathrm{d}x$ and radius $f(x)-1$ and hence of volume $\pi[f(x)-1]^2\,\mathrm{d}x$. The full volume generated (for any fixed $a>0$) is

$$\int_0^a \pi[f(x)-1]^2\,\mathrm{d}x = \pi\int_0^a xe^{2x^2}\,\mathrm{d}x.$$



Using the substitution $u = 2x^2$, so that $\mathrm{d}u = 4x\,\mathrm{d}x$:

$$\text{Volume} = \pi\int_0^{2a^2} e^u\,\frac{\mathrm{d}u}{4} = \frac{\pi}{4}e^u\Big|_0^{2a^2} = \frac{\pi}{4}\left(e^{2a^2}-1\right)$$

S-6: For a fixed value of x, if we rotate about the x-axis, we form a washer of inner radius $B(x)$ and outer radius $T(x)$ and hence of area $\pi[T(x)^2 - B(x)^2]$. We integrate this function from $x=0$ to $x=3$ to find the total volume V:

$$\begin{aligned}
V &= \int_0^3 \pi[T(x)^2 - B(x)^2]\,\mathrm{d}x \\
&= \pi\int_0^3 (\sqrt{x}e^{3x})^2 - (\sqrt{x}(1+2x))^2\,\mathrm{d}x \\
&= \pi\int_0^3 \left(xe^{6x} - (x+4x^2+4x^3)\right)\mathrm{d}x \\
&= \pi\int_0^3 xe^{6x}\,\mathrm{d}x - \pi\left[\frac{x^2}{2}+\frac{4x^3}{3}+x^4\right]_0^3 \\
&= \pi\int_0^3 xe^{6x}\,\mathrm{d}x - \pi\left[\frac{3^2}{2}+\frac{4\cdot 3^3}{3}+3^4\right]
\end{aligned}$$

For the first integral, we use integration by parts with $u(x) = x$, $\mathrm{d}v = e^{6x}\mathrm{d}x$, so that $\mathrm{d}u = \mathrm{d}x$ and $v(x) = \frac{1}{6}e^{6x}$:

$$\begin{aligned}
\int_0^3 xe^{6x}\,\mathrm{d}x &= \frac{xe^{6x}}{6}\Big|_0^3 - \int_0^3 \frac{1}{6}e^{6x}\,\mathrm{d}x \\
&= \frac{3e^{18}}{6} - 0 - \frac{1}{36}e^{6x}\Big|_0^3 = \frac{e^{18}}{2} - \left(\frac{e^{18}}{36} - \frac{1}{36}\right).
\end{aligned}$$

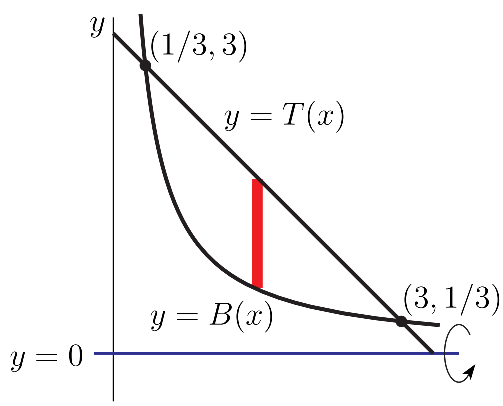
Therefore, the total volume is

$$V = \pi\left[\frac{e^{18}}{2} - \left(\frac{e^{18}}{36} - \frac{1}{36}\right)\right] - \pi\left[\frac{3^2}{2} + \frac{4\cdot 3^3}{3} + 3^4\right] = \pi\left(\frac{17e^{18} - 4373}{36}\right).$$

S-7:

The curves $y = 1/x$ and $3x + 3y = 10$, i.e. $y = \frac{10}{3} - x$ intersect when

$$\begin{aligned}\frac{1}{x} = \frac{10}{3} - x &\iff 3 = 10x - 3x^2 \iff 3x^2 - 10x + 3 = 0\\ &\iff (3x-1)(x-3) = 0\\ &\iff x = 3\,,\ \frac{1}{3}\end{aligned}$$



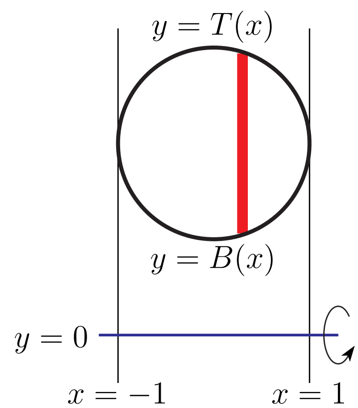
When the region is rotated about the x–axis, the vertical strip in the figure above sweeps out a washer with thickness dx, outer radius $T(x) = \frac{10}{3} - x$ and inner radius $B(x) = \frac{1}{x}$. This washer has volume

$$\pi\big(T(x)^2 - B(x)^2\big)\,\mathrm{d}x = \pi\Big(\frac{100}{9} - \frac{20}{3}x + x^2 - \frac{1}{x^2}\Big)\,\mathrm{d}x$$

Hence the volume of the solid is

$$\begin{aligned}\pi\int_{1/3}^{3}\Big(\frac{100}{9} - \frac{20}{3}x + x^2 - \frac{1}{x^2}\Big)\,\mathrm{d}x &= \pi\Big[\frac{100x}{9} - \frac{10}{3}x^2 + \frac{1}{3}x^3 + \frac{1}{x}\Big]_{1/3}^{3}\\ &= \pi\Big[\frac{38}{3} - \frac{514}{3^4}\Big] = \pi\frac{512}{81}\end{aligned}$$

S-8: (a) The top and the bottom of the circle have equations $y = T(x) = 2 + \sqrt{1-x^2}$ and $y = B(x) = 2 - \sqrt{1-x^2}$, respectively.



When R is rotated about the x–axis, the vertical strip of R in the figure above sweeps out a washer with thickness $\mathrm{d}x$, outer radius $T(x)$ and inner radius $B(x)$. This washer has volume

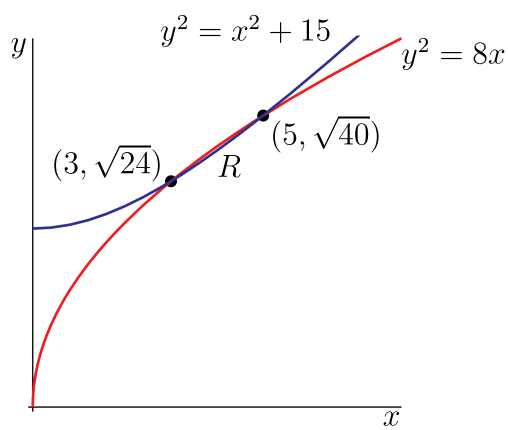
$$\pi\big(T(x)^2 - B(x)^2\big)\,\mathrm{d}x = \pi\big(T(x)+B(x)\big)\big(T(x)-B(x)\big)\,\mathrm{d}x = \pi \times 4 \times 2\sqrt{1-x^2}\,\mathrm{d}x$$

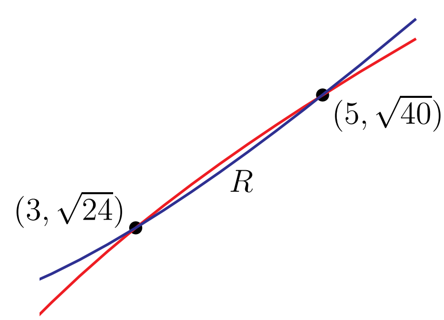
Hence the volume of the solid is

$$8\pi \int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x$$

(b) Since $y = \sqrt{1-x^2}$ is equivalent to $x^2 + y^2 = 1$, $y \geqslant 0$, the integral is 8π times the area of the upper half of the circle $x^2 + y^2 = 1$ and hence is $8\pi \times \frac{1}{2}\pi 1^2 = 4\pi^2$.

S-9: (a) The two curves intersect when x obeys $8x = x^2 + 15$ or $x^2 - 8x + 15 = (x-5)(x-3) = 0$. The points of intersection, in the first quadrant, are $(3, \sqrt{24})$ and $(5, \sqrt{40})$. The region R is the region between the blue and red curves, with $3 \leqslant x \leqslant 5$, in the figures below.



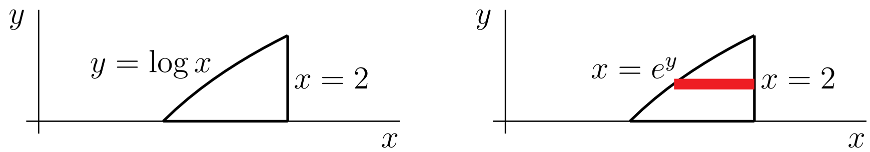


(b) The part of the solid with x coordinate between x and $x + \mathrm{d}x$ is a "washer" shaped region with inner radius $\sqrt{x^2+15}$, outer radius $\sqrt{8x}$ and thickness $\mathrm{d}x$. The surface area of the washer is $\pi(\sqrt{8x})^2 - \pi(\sqrt{x^2+15})^2 = \pi(8x - x^2 - 15)$ and its volume is

$\pi(8x - x^2 - 15)\,\mathrm{d}x$. The total volume is

$$\int_3^5 \pi(8x - x^2 - 15)\,\mathrm{d}x = \pi\Big[4x^2 - \frac{1}{3}x^3 - 15x\Big]_3^5 = \pi\Big[100 - \frac{125}{3} - 75 - 36 + 9 + 45\Big]$$
$$= \frac{4}{3}\pi \approx 4.19$$

S-10: (a) The region R is sketched in the figure on the left below.

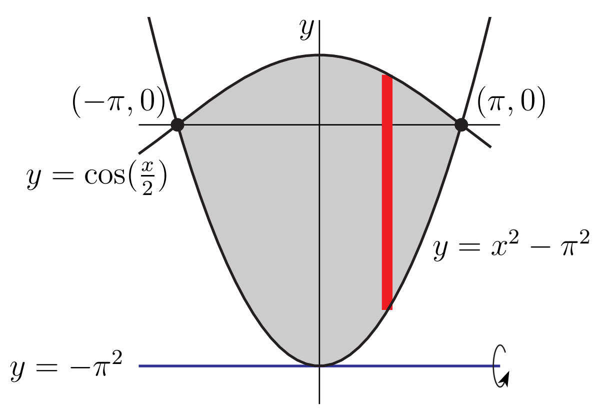


(b) We'll use horizontal washers as in Example 1.6.5 of the CLP 101 notes.

- We cut R into thin horizontal strips of width $\mathrm{d}y$ as in the figure on the right above.
- When we rotate R about the y–axis, i.e. about the line $x = 0$, each strip sweeps out a thin washer
  - whose inner radius is $r_{in} = e^y$ and outer radius is $r_{out} = 2$, and
  - whose thickness is $\mathrm{d}y$ and hence
  - whose volume $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi\big(4 - e^{2y}\big)\mathrm{d}y$.
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = \log 2$ (since at the top $x = 2$ and $x = e^y$), the total

$$\text{Volume} = \int_0^{\log 2} \pi\big(4 - e^{2y}\big)\,\mathrm{d}y = \pi\big[4y - e^{2y}/2\big]_0^{\log 2} = \pi\Big[4\log 2 - 2 + \frac{1}{2}\Big]$$
$$= \pi\Big[4\log 2 - \frac{3}{2}\Big] \approx 3.998$$

S-11: Here is a sketch of the curves $y = \cos(\frac{x}{2})$ and $y = x^2 - \pi^2$.



The curves meet at $x = \pm\pi$ where both $\cos(\frac{x}{2})$ and $x^2 - \pi^2$ take the value zero. We'll use vertical washers as specified in the question.

- We cut the specified region into thin vertical strips of width dx as in the figure above.
- When we rotate about the line $y = -\pi^2$, each strip sweeps out a thin washer
  - whose inner radius is $r_{in} = (x^2 - \pi^2) - (-\pi^2) = x^2$ and outer radius is $r_{out} = \cos(\frac{x}{2}) - (-\pi^2) = \cos(\frac{x}{2}) + \pi^2$, and
  - whose thickness is dx and hence
  - whose volume $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}x = \pi\big((\cos(\frac{x}{2}) + \pi^2)^2 - (x^2)^2\big)\mathrm{d}x$.
- As our leftmost strip is at $x = -\pi$ and our rightmost strip is at $x = \pi$,

the total volume is

$$\begin{aligned}
\pi \int_{-\pi}^{\pi} &\left(\cos^2(\tfrac{x}{2}) + 2\pi^2\cos(\tfrac{x}{2}) + \pi^4 - x^4\right)\,\mathrm{d}x \\
&= \pi \int_{-\pi}^{\pi} \left(\frac{1+\cos(x)}{2} + 2\pi^2\cos(\tfrac{x}{2}) + \pi^4 - x^4\right)\,\mathrm{d}x \\
&= 2\pi \int_{0}^{\pi} \left(\frac{1+\cos(x)}{2} + 2\pi^2\cos(\tfrac{x}{2}) + \pi^4 - x^4\right)\,\mathrm{d}x \\
&= 2\pi \left[\frac{1}{2}x + \frac{1}{2}\sin(x) + 4\pi^2\sin(\tfrac{x}{2}) + \pi^4 x - \frac{1}{5}x^5\right]_0^{\pi} \\
&= 2\pi\left[\frac{\pi}{2} + 0 + 4\pi^2 + \pi^5 - \frac{\pi^5}{5}\right] \\
&= \pi^2 + 8\pi^3 + \frac{8\pi^6}{5}
\end{aligned}$$

In the middle line, we used the fact that the integrand is an even function and the interval of integration $[-\pi, \pi]$ is symmetric, but one can also compute directly.

S-12: As in Example 1.6.6 of the CLP 101 notes, we slice V into thin horizontal "square pancakes".

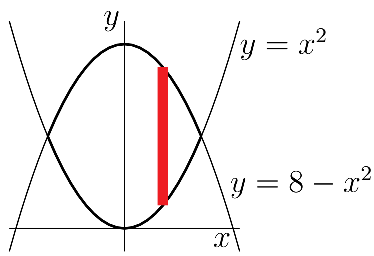
- We are told that the pancake at height x is a square of side $\frac{2}{1+x}$ and so
- has cross-sectional area $\left(\frac{2}{1+x}\right)^2$ and thickness dx and hence
- has volume $\left(\frac{2}{1+x}\right)^2\mathrm{d}x$.

Hence the volume of V is

$$\int_0^2 \left[\frac{2}{1+x}\right]^2 \mathrm{d}x = \int_1^3 \frac{4}{u^2}\,\mathrm{d}u = 4\frac{u^{-1}}{-1}\bigg|_1^3 = -4\left[\frac{1}{3} - 1\right] = \frac{8}{3}$$

We made the change of variables $u = 1 + x$, du = dx.

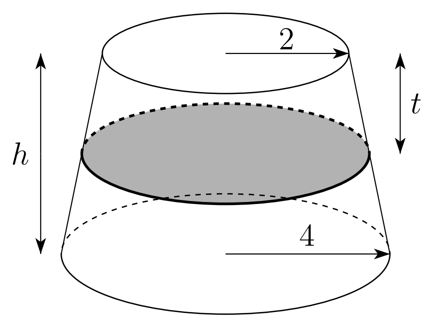
S-13: Here is a sketch of the base region.



Consider the thin vertical cross–section resting on the heavy red line in the figure above. It has thickness $\mathrm{d}x$. Its face is a square whose side runs from $y = x^2$ to $y = 8 - x^2$, a distance of $8 - 2x^2$. So the face has area $(8-2x^2)^2$ and the slice has volume $(8-2x^2)^2\,\mathrm{d}x$. The two curves cross when $x^2 = 8 - x^2$, i.e. when $x^2 = 4$ or $x = \pm 2$. So x runs from -2 to 2 and the total volume is

$$\begin{aligned}\int_{-2}^{2} (8-2x^2)^2\,\mathrm{d}x &= 2\int_0^2 4(4-x^2)^2\,\mathrm{d}x = 8\int_0^2 \left[16 - 8x^2 + x^4\right]\mathrm{d}x \\ &= 8\Big[16\times 2 - \frac{8}{3}2^3 + \frac{1}{5}2^5\Big] = \frac{256\times 8}{15} = 136.5\dot{3}\end{aligned}$$

S-14: Slice the frustrum into horizontal discs. When the disc is a distance t from the top of the frustrum it has radius $2 + 2t/h$. Note that as t runs from 0 (the top of the frustrum) to $t = h$ (the bottom of the frustrum) the radius $2 + 2t/h$ increases linearly from 2 to 4.



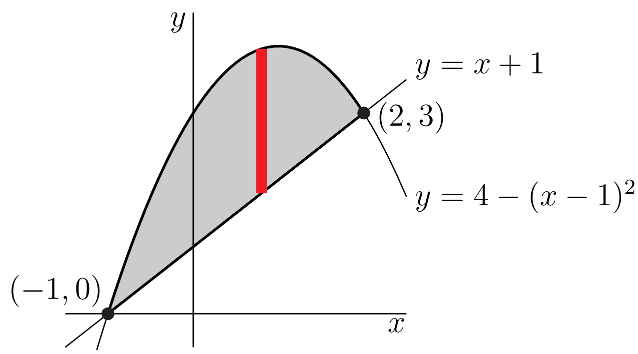
Thus the disk has volume $\pi\big(2+2t/h\big)^2\mathrm{d}t$. The total volume of the frustrum is

$$\pi\int_0^h \big(2+2t/h\big)^2\mathrm{d}t = 4\pi\int_0^h \big(1+t/h\big)^2\mathrm{d}t = 4\pi\frac{(1+t/h)^3}{3/h}\Bigg|_0^h = \frac{4}{3}\pi h\times 7 = \frac{28}{3}\pi h$$

S-15: (a) The curve $y = 4 - (x-1)^2$ is an “upside down parabola” and line $y = x+1$ has slope 1. They intersect at points (x, y) which satisfy both $y = x+1$ and $y = 4-(x-1)^2$. That is, when x obeys

$$\begin{aligned}x+1 = 4-(x-1)^2 &\iff x+1 = 4-x^2+2x-1 \iff x^2-x-2=0 \\ &\iff (x-2)(x+1) = 0 \qquad \iff x = -1, 2\end{aligned}$$

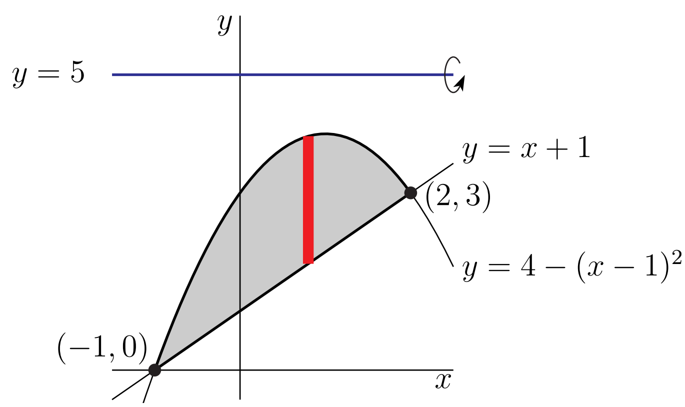
Thus the intersection points are $(-1, 0)$ and $(2, 3)$. Here is a sketch of R



The red strip in the sketch above runs from $y = x + 1$ to $y = 4 - (x-1)^2$ and so has area $[4 - (x-1)^2 - (x+1)]\,\mathrm{d}x = [2 + x - x^2]\,\mathrm{d}x$. All together R has

$$\begin{aligned}\text{Area} &= \int_{-1}^{2} \left[2 + x - x^2\right]\,\mathrm{d}x \\ &= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} \\ &= 6 + \frac{3}{2} - \frac{9}{3} = \frac{9}{2}\end{aligned}$$

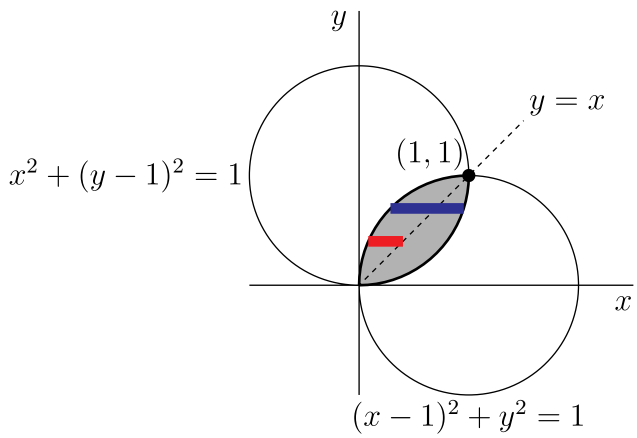
(b) We'll use vertical washers as in Example 1.6.3 of the CLP 101 notes.



- We cut R into thin vertical strips of width $\mathrm{d}x$ like the red strip in the figure above.
- When we rotate R about the horizontal line $y = 5$, each strip sweeps out a thin washer
  - whose inner radius is $r_{in} = 5 - [4 - (x-1)^2] = 1 + (x-1)^2$, and
  - whose outer radius is $r_{out} = 5 - [x+1] = 4 - x$ and
  - whose thickness is $\mathrm{d}x$ and hence
  - whose volume is $\pi\left[r_{out}^2 - r_{in}^2\right]\mathrm{d}x = \pi\left[\left(4-x\right)^2 - \left(1 + (x-1)^2\right)^2\right]\mathrm{d}x$
- As our leftmost strip is at $x = -1$ and our rightmost strip is at $x = 2$, the total

$$\text{Volume} = \pi \int_{-1}^{2} \left[\left(4-x\right)^2 - \left(1 + (x-1)^2\right)^2\right]\mathrm{d}x$$

S-16: (a) The curves $(x-1)^2+y^2=1$ and $x^2+(y-1)^2=1$ are circles of radius 1 centered on $(1,0)$ and $(0,1)$ respectively. Both circles pass through $(0,0)$ and $(1,1)$. They are sketched below.



The region $\mathcal{R}$ is symmetric about the line $y=x$, so the area of $\mathcal{R}$ is twice the area of the part of $\mathcal{R}$ to the left of the line $y=x$. The red strip in the sketch above runs from $x=1-\sqrt{1-y^2}$ to $x=y$

$$\begin{aligned}\text{Area} &= 2\int_0^1 \big[y-\big(1-\sqrt{1-y^2}\big)\big]\,\mathrm{d}y\\ &= 2\Big\{\Big[\frac{y^2}{2}-y\Big]_0^1+\int_0^1\sqrt{1-y^2}\,\mathrm{d}y\Big\}\\ &= \frac{\pi}{2}-1\end{aligned}$$

Here the integral $\int_0^1\sqrt{1-y^2}\,\mathrm{d}y$ was evaluated simply as the area of one quarter of a cicular disk of radius 1. It can also be evaluated by substituting $y=\sin\theta$.

(b) We'll use horizontal washers as in Example 1.6.5 of the CLP 101 notes.

- We cut $\mathcal{R}$ into thin horizontal strips of width $\mathrm{d}y$ like the blue strip in the figure above.
- When we rotate $\mathcal{R}$ about the y–axis, each strip sweeps out a thin washer
  - whose inner radius is $r_{in}=1-\sqrt{1-y^2}$, and
  - whose outer radius is $r_{out}=\sqrt{1-(y-1)^2}$ and
  - whose thickness is $\mathrm{d}y$ and hence
  - whose volume is $\pi\big[\big(\sqrt{1-(y-1)^2}\big)^2-\big(1-\sqrt{1-y^2}\,\big)^2\big]\,\mathrm{d}y$ $=2\pi\big[\sqrt{1-y^2}+y-1\big]\,\mathrm{d}y$

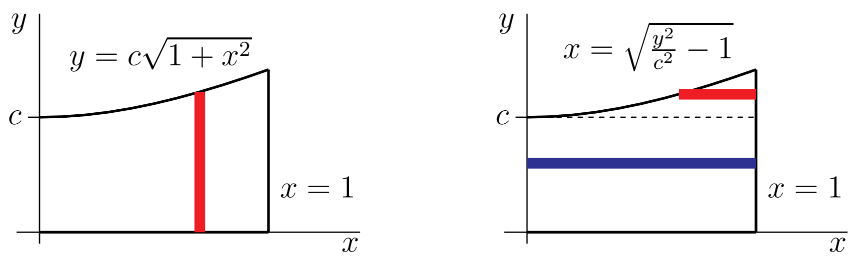
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = 1$, the total

$$\text{Volume} = 2\pi \int_0^1 \left[\sqrt{1-y^2} + y - 1\right] \mathrm{d}y = 2\pi\left[\frac{\pi}{4} + \frac{1}{2} - 1\right]$$
$$= \frac{\pi^2}{2} - \pi \approx 1.793$$

Here, we again used that $\int_0^1 \sqrt{1-y^2}\, dy$ is the area of a quarter circle of radius one.

S-17: (a) Let $\mathcal{V}_1$ be the solid obtained by revolving $\mathcal{R}$ about the x–axis. The portion of $\mathcal{V}_1$ with x–coordinate between x and $x + \mathrm{d}x$ is obtained by rotating the red vertical strip in the figure on the left below about the x–axis. That portion is a disk of radius $c\sqrt{1+x^2}$ and thickness $\mathrm{d}x$. The volume of this disk is $\pi(c\sqrt{1+x^2})^2\mathrm{d}x = \pi c^2(1+x^2)\,\mathrm{d}x$. So the total volume of $\mathcal{V}_1$ is

$$V_1 = \int_0^1 \pi c^2(1+x^2)\,\mathrm{d}x = \pi c^2\left[x + \frac{x^3}{3}\right]_0^1 = \frac{4}{3}\pi c^2$$



(b) We'll use horizontal washers as in Example 1.6.5 of the CLP 101 notes.

- We cut $\mathcal{R}$ into thin horizontal strips of width $\mathrm{d}y$ as in the figure on the right above.
- When we rotate $\mathcal{R}$ about the y–axis, i.e. about the line $x = 0$, each strip sweeps out a thin washer
  - whose outer radius is $r_{out} = 1$, and
  - whose inner radius is $r_{in} = \sqrt{\frac{y^2}{c^2} - 1}$ when $y \geqslant c\sqrt{1+0^2} = c$ (see the red strip in the figure on the right above), and whose inner radius is $r_{in} = 0$ when $y \leqslant c$ (see the blue strip in the figure on the right above) and
  - whose thickness is $\mathrm{d}y$ and hence
  - whose volume is $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi\big(2 - \frac{y^2}{c^2}\big)\mathrm{d}y$ when $y \geqslant c$ and whose volume is $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi\,\mathrm{d}y$ when $y \leqslant c$ and
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = \sqrt{2}\,c$ (since at

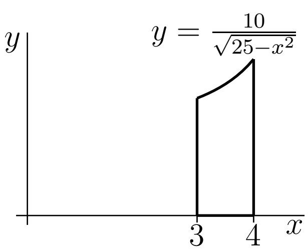
the top $x=1$ and $y=c\sqrt{1+x^2}$), the total

$$\begin{aligned} V_2 &= \int_c^{\sqrt{2}\,c} \pi\Big(2-\frac{y^2}{c^2}\Big)\,\mathrm{d}y + \int_0^c \pi\,\mathrm{d}y \\ &= \pi\Big[2y-\frac{y^3}{3c^2}\Big]_c^{\sqrt{2}\,c} + \pi c \\ &= \pi\,c\Big[\frac{4\sqrt{2}}{3}-\frac{5}{3}\Big] + \pi c \\ &= \frac{\pi\,c}{3}\big[4\sqrt{2}-2\big] = \frac{2\pi\,c}{3}\big[2^{3/2}-1\big] \end{aligned}$$

(c) We have $V_1 = V_2$ if and only if

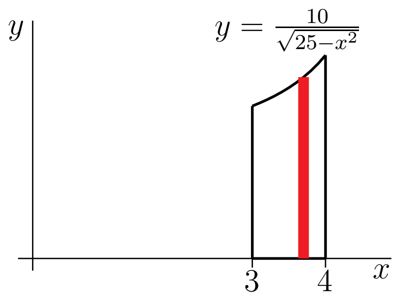
$$\frac{4}{3}\pi c^2 = \frac{2}{3}\pi c[2^{3/2}-1] \iff \boxed{c=0 \text{ or } c=\tfrac{1}{2}[2^{3/2}-1]}$$

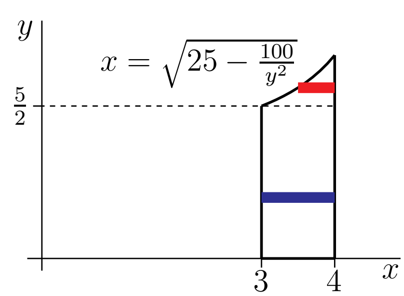
S-18: (a) The region R is



(b) Let $\mathcal{V}_1$ be the solid obtained by revolving R about the x–axis. The portion of $\mathcal{V}_1$ with x–coordinate between x and $x+\mathrm{d}x$ is obtained by rotating the red vertical strip in the figure on the left below about the x–axis. That portion is a disk of radius $\frac{10}{\sqrt{25-x^2}}$ and thickness $\mathrm{d}x$. The volume of this disk is $\pi\big(\frac{10}{\sqrt{25-x^2}}\big)^2\,\mathrm{d}x$. So the total volume of $\mathcal{V}_1$ is

$$\begin{aligned} \int_3^4 \pi\Big(\frac{10}{\sqrt{25-x^2}}\Big)^2\,\mathrm{d}x &= 100\pi\int_3^4 \frac{1}{25-x^2}\,\mathrm{d}x = 100\pi\int_3^4 \frac{1}{(5-x)(5+x)}\,\mathrm{d}x \\ &= 10\pi\int_3^4 \Big[\frac{1}{5-x}+\frac{1}{5+x}\Big]\,\mathrm{d}x = 10\pi\Big[-\log(5-x)+\log(5+x)\Big]_3^4 \\ &= 10\pi\Big[-\log 1+\log 9+\log 2-\log 8\Big] = 10\pi\log\frac{9}{4} = 20\pi\log\frac{3}{2} \end{aligned}$$



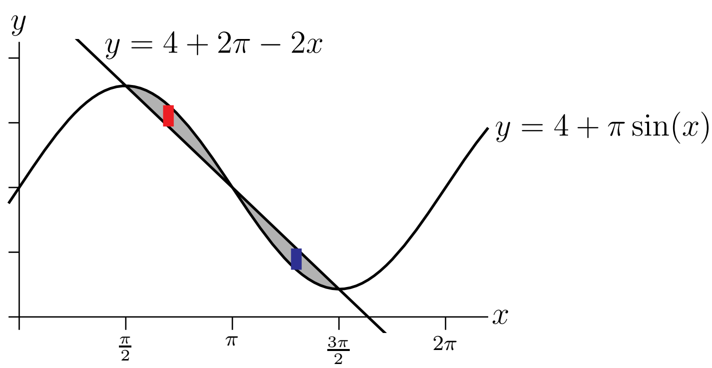


(c) We'll use horizontal washers as in Example 1.6.5 of the CLP 101 notes.

- We cut $\mathcal{R}$ into thin horizontal strips of width $\mathrm{d}y$ as in the figure on the right above.
- When we rotate $\mathcal{R}$ about the y–axis, each strip sweeps out a thin washer
  - whose outer radius is $r_{out} = 4$, and
  - whose inner radius is $r_{in} = \sqrt{25 - \frac{100}{y^2}}$ when $y \geqslant \frac{10}{\sqrt{25-3^2}} = \frac{10}{4} = \frac{5}{2}$ (see the red strip in the figure on the right above), and whose inner radius is $r_{in} = 3$ when $y \leqslant \frac{5}{2}$ (see the blue strip in the figure on the right above) and
  - whose thickness is $\mathrm{d}y$ and hence
  - whose volume is $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi\big(\frac{100}{y^2} - 9\big)\mathrm{d}y$ when $y \geqslant \frac{5}{2}$ and whose volume is $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = 7\pi\,\mathrm{d}y$ when $y \leqslant \frac{5}{2}$ and
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = \frac{10}{3}$ (since at the top $x = 4$ and $y = \frac{10}{\sqrt{25-x^2}} = \frac{10}{\sqrt{25-4^2}} = \frac{10}{3}$), the volume is

$$\begin{aligned}
&\int_{5/2}^{10/3} \pi\Big(\frac{100}{y^2} - 9\Big)\mathrm{d}y + \int_0^{5/2} 7\pi\,\mathrm{d}y \\
&= \pi\Big[-\frac{100}{y} - 9y\Big]_{5/2}^{10/3} + \frac{35}{2}\pi \\
&= \pi\Big[-30 + 40 - 30 + \frac{45}{2}\Big] + \frac{35}{2}\pi \\
&= 20\pi
\end{aligned}$$

S-19: We will compute the volume by rotating thin vertical strips as in the sketch



about the line $y = -1$ to generate thin washers. The line $y = 4 + 2\pi - 2x$ intersects the curve $y = 4 + \pi\sin x$ when

$$4 + 2\pi - 2x = 4 + \pi\sin x \iff \sin x = 2 - \frac{2}{\pi}x \iff x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

These solutions were guessed by looking at the sketch above, but then verified by substituting them back into the equation. From the sketch we see that

- when $\frac{\pi}{2} \leqslant x \leqslant \pi$, the top of the strip is at $y = 4 + \pi \sin x$ and the bottom of the strip is at $y = 4 + 2\pi - 2x$. So when the strip is rotated we get a thin washer with outer and inner radii $R(x) = 1 + 4 + \pi \sin x = 5 + \pi \sin x$ and $r(x) = 1 + 4 + 2\pi - 2x = 5 + 2\pi - 2x$, respectively.
- when $\pi \leqslant x \leqslant \frac{3\pi}{2}$, the top of the strip is at $y = 4 + 2\pi - 2x$ and the bottom of the strip is at $y = 4 + \pi \sin x$. So when the strip is rotated we get a thin washer with $R(x) = 1 + 4 + 2\pi - 2x = 5 + 2\pi - 2x$ and $r(x) = 1 + 4 + \pi \sin x = 5 + \pi \sin x$, respectively.

So the total

$$\begin{aligned}
\text{Volume} &= \int_{\pi/2}^{\pi} \pi\big[R(x)^2 - r(x)^2\big]\,\mathrm{d}x + \int_{\pi}^{3\pi/2} \pi\big[R(x)^2 - r(x)^2\big]\,\mathrm{d}x \\
&= \int_{\pi/2}^{\pi} \pi\big[(5 + \pi \sin x)^2 - (5 + 2\pi - 2x)^2\big]\,\mathrm{d}x \\
&\qquad + \int_{\pi}^{3\pi/2} \pi\big[(5 + 2\pi - 2x)^2 - (5 + \pi \sin x)^2\big]\,\mathrm{d}x
\end{aligned}$$

Solutions to Exercises 1.7 — Jump to TABLE OF CONTENTS

S-1: By integration by parts with $u = \log x$ and $\mathrm{d}v = x\,\mathrm{d}x$, so that $\mathrm{d}u = \frac{\mathrm{d}x}{x}$ and $v = \frac{x^2}{2}$,

$$\begin{aligned}
\int x \log x\,\mathrm{d}x &= \frac{x^2 \log x}{2} - \int \frac{x^2}{2}\,\frac{\mathrm{d}x}{x} = \frac{x^2 \log x}{2} - \frac{1}{2}\int x\,\mathrm{d}x \\
&= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C
\end{aligned}$$

S-2: By integration by parts with $u = \log x$ and $\mathrm{d}v = x^{-7}\,\mathrm{d}x$, so that $\mathrm{d}u = \frac{\mathrm{d}x}{x}$ and $v = -\frac{x^{-6}}{6}$,

$$\begin{aligned}
\int \frac{\log x}{x^7}\,\mathrm{d}x &= -\log x\,\frac{x^{-6}}{6} + \int \frac{x^{-6}}{6}\,\frac{\mathrm{d}x}{x} = -\frac{\log x}{6x^6} + \frac{1}{6}\int x^{-7}\,\mathrm{d}x \\
&= -\frac{\log x}{6x^6} - \frac{1}{36x^6} + C
\end{aligned}$$

S-3: We integrate by parts, using $u = x$, $\mathrm{d}v = \sin x\,\mathrm{d}x$ so that $v = -\cos x$ and $du = \mathrm{d}x$:

$$\int_0^{\pi} x \sin x\,\mathrm{d}x = -x\cos x\Big|_0^{\pi} - \int_0^{\pi} (-\cos x)\,\mathrm{d}x = \Big[-x\cos x + \sin x\Big]_0^{\pi} = -\pi(-1) = \pi$$

S-4: We integrate by parts, using $u = x$, $\mathrm{d}v = \cos x\,\mathrm{d}x$ so that $v = \sin x$ and $du = \mathrm{d}x$:

$$\int_0^{\frac{\pi}{2}} x\cos x\,\mathrm{d}x = x\sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,\mathrm{d}x = \Big[x\sin x + \cos x\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$$

S-5: We integrate by parts, using $u = \cos^{-1} y$, $\mathrm{d}v = \mathrm{d}y$ so that $v = y$ and $du = -\frac{\mathrm{d}y}{\sqrt{1-y^2}}$:

$$\begin{aligned}\int \cos^{-1} y\,\mathrm{d}y &= y\cos^{-1} y + \int \frac{y}{\sqrt{1-y^2}}\mathrm{d}y \\ &= y\cos^{-1} y - \sqrt{1-y^2} + C\end{aligned}$$

S-6: We integrate by parts, using $u = \arctan(2y)$, $\mathrm{d}v = 4y\,\mathrm{d}y$ so that $v = 2y^2$ and $du = \frac{2\,\mathrm{d}y}{1+(2y)^2}$:

$$\int 4y\arctan(2y)\,\mathrm{d}y = 2y^2\arctan(2y) - \int \frac{4y^2}{(2y)^2+1}\,\mathrm{d}y$$

We now notice that $\frac{4y^2}{4y^2+1} = \frac{4y^2+1}{4y^2+1} - \frac{1}{4y^2+1}$. We therefore have

$$\int \frac{4y^2}{4y^2+1}\,\mathrm{d}y = \int \left(1 - \frac{1}{4y^2+1}\right)\,\mathrm{d}y = y - \frac{1}{2}\arctan(2y) + C$$

The final answer is then

$$\int 4y\arctan(2y)\,\mathrm{d}y = 2y^2\arctan(2y) - y + \frac{1}{2}\arctan(2y) + C$$

S-7: (a) Integrate by parts with $u = \sin^{n-1} x$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$, so that $\mathrm{d}u = (n-1)\sin^{n-2} x\cos x$ and $v = -\cos x$.

$$\begin{aligned}\int \sin^n x\,\mathrm{d}x &= -\sin^{n-1} x\ \cos x + (n-1)\int \cos^2 x\ \sin^{n-2} x\,\mathrm{d}x \\ &= -\sin^{n-1} x\ \cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x\,\mathrm{d}x \\ &= -\sin^{n-1} x\ \cos x + (n-1)\int \sin^{n-2} x\,\mathrm{d}x - (n-1)\int \sin^n x\,\mathrm{d}x\end{aligned}$$

Moving the last term on the right hand side to the left hand side gives

$$n\int \sin^n x\,\mathrm{d}x = -\sin^{n-1} x\ \cos x + (n-1)\int \sin^{n-2} x\,\mathrm{d}x$$

Dividing across by n gives the desired reduction formula.

(b) By the reduction formula of part (a)

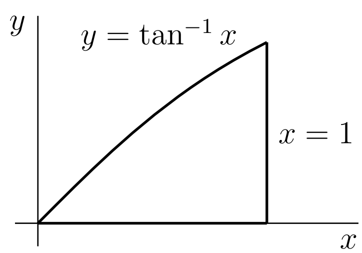
$$\int_0^{\pi/2} \sin^n(x)\,\mathrm{d}x = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}(x)\,\mathrm{d}x$$

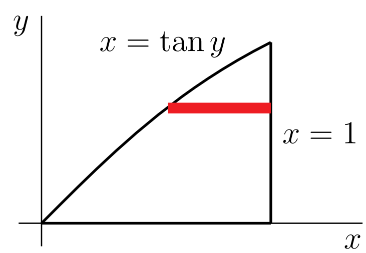
for all integers $n \geqslant 2$, since $\sin 0 = \cos\frac{\pi}{2} = 0$. Applying this reduction formula, with $n = 8, 6, 4, 2$,

$$\begin{aligned}\int_0^{\pi/2} \sin^8(x)\,\mathrm{d}x &= \frac{7}{8}\int_0^{\pi/2} \sin^6(x)\,\mathrm{d}x = \frac{7}{8}\frac{5}{6}\int_0^{\pi/2} \sin^4(x)\,\mathrm{d}x = \frac{7}{8}\frac{5}{6}\frac{3}{4}\int_0^{\pi/2} \sin^2(x)\,\mathrm{d}x \\ &= \frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\int_0^{\pi/2} \mathrm{d}x = \frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{35}{256}\pi \approx 0.4295\end{aligned}$$

S-8: (a) The sketch is the figure on the left below. By integration by parts with $u = \tan^{-1} x$, $\mathrm{d}v = \mathrm{d}x$, $v = x$ and $\mathrm{d}u = \frac{1}{1+x^2}\,\mathrm{d}x$,

$$\begin{aligned}A &= \int_0^1 \tan^{-1} x\,\mathrm{d}x = x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x = \tan^{-1} 1 - \tfrac{1}{2}\ln(1+x^2)\Big|_0^1 \\ &= \frac{\pi}{4} - \frac{\ln 2}{2}\end{aligned}$$





(b) We'll use horizontal washers as in Example 1.6.5 of the CLP 101 notes.

- We cut R into thin horizontal strips of width $\mathrm{d}y$ as in the figure on the right above.
- When we rotate R about the y–axis, each strip sweeps out a thin washer
  - whose inner radius is $r_{in} = \tan y$ and outer radius is $r_{out} = 1$, and
  - whose thickness is $\mathrm{d}y$ and hence
  - whose volume $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi(1 - \tan^2 y)\mathrm{d}y$.
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = \frac{\pi}{4}$ (since at the top $x = 1$ and $x = \tan y$), the total

$$\begin{aligned}\text{Volume} &= \int_0^{\pi/4} \pi(1 - \tan^2 y)\,\mathrm{d}y = \int_0^{\pi/4} \pi(2 - \sec^2 y)\,\mathrm{d}y = \pi\big[2y - \tan y\big]_0^{\pi/4} \\ &= \frac{\pi^2}{2} - \pi\end{aligned}$$

S-9: To get rid of the square root in the argument of f'', we make the change of variables $x = t^2$, $\mathrm{d}x = 2t\,\mathrm{d}t$.

$$\int_0^4 f''\big(\sqrt{x}\big)\,\mathrm{d}x = 2\int_0^2 t f''(t)\,\mathrm{d}t$$

Then, to convert f'' into f' ,we integration by parts with $u = t$, $\mathrm{d}v = f''(t)\,\mathrm{d}t$, $v = f'(t)$.

$$\begin{aligned}
\int_0^4 f''\big(\sqrt{x}\big)\,\mathrm{d}x &= 2\Big\{ t f'(t)\Big|_0^2 - \int_0^2 f'(t)\,\mathrm{d}t \Big\} \\
&= 2\Big[t f'(t) - f(t) \Big]_0^2 \\
&= 2\big[2f'(2) - f(2) + f(0)\big] = 2\big[2 \times 4 - 3 + 1\big] \\
&= 12
\end{aligned}$$

Solutions to Exercises 1.8 — Jump to TABLE OF CONTENTS

S-1: Make the substitution $u = \sin x$, so that $\mathrm{d}u = \cos x\,\mathrm{d}x$ and $\cos^2 x = 1 - \sin^2 x = 1 - u^2$:

$$\begin{aligned}
\int \cos^3 x\,\mathrm{d}x &= \int (1 - \sin^2 x)\cos x\,\mathrm{d}x = \int (1 - u^2)\,\mathrm{d}u \\
&= u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C
\end{aligned}$$

S-2: Make the substitution $u = \sin t$, so that $\mathrm{d}u = \cos t\,\mathrm{d}t$ and $\cos^2 t = 1 - \sin^2 t = 1 - u^2$:

$$\begin{aligned}
\int \sin^{36} t \cos^3 t\,\mathrm{d}t &= \int \sin^{36} t\,(1 - \sin^2 t)\cos t\,\mathrm{d}t = \int u^{36}(1 - u^2)\,\mathrm{d}u \\
&= \frac{u^{37}}{37} - \frac{u^{39}}{39} + C = \frac{\sin^{37} t}{37} - \frac{\sin^{39} t}{39} + C
\end{aligned}$$

S-3: *First solution:* Substituting $u = \cos x$, $\mathrm{d}u = -\sin x\,\mathrm{d}x$, $\sin^2 x = 1 - \cos^2 x = 1 - u^2$, gives

$$\begin{aligned}
\int \tan^3 x \sec^5 x\,\mathrm{d}x &= \int \frac{\sin^3 x}{\cos^8 x}\,\mathrm{d}x = \int \frac{(1 - \cos^2 x)\sin x}{\cos^8 x}\,\mathrm{d}x = -\int \frac{1 - u^2}{u^8}\,\mathrm{d}u \\
&= -\left[\frac{u^{-7}}{-7} - \frac{u^{-5}}{-5}\right] + C = \boxed{\frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C}
\end{aligned}$$

Second solution: Alternatively, substituting $u = \sec x$, $\mathrm{d}u = \sec x \tan x\,\mathrm{d}x$, $\tan^2 x = \sec^2 x - 1 = u^2 - 1$, gives

$$\begin{aligned}\int \tan^3 x \sec^5 x\,\mathrm{d}x &= \int \tan^2 x \sec^4 x\,(\tan x \sec x)\,\mathrm{d}x = \int (u^2-1)u^4\,\mathrm{d}u \\ &= \left[\frac{u^7}{7} - \frac{u^5}{5}\right] + C = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C\end{aligned}$$

S-4: Use the substitution $u = \tan x$, so that $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$:

$$\begin{aligned}\int \sec^4 x\ \tan^{46} x\,\mathrm{d}x &= \int (\tan^2 x + 1)\tan^{46} x\ \sec^2 x\,\mathrm{d}x = \int (u^2+1)u^{46}\,\mathrm{d}u \\ &= \frac{u^{49}}{49} + \frac{u^{47}}{47} + C = \frac{\tan^{49} x}{49} + \frac{\tan^{47} x}{47} + C\end{aligned}$$

S-5: Using the trig identity $\cos^2 x = \frac{1+\cos(2x)}{2}$, we have

$$\int \cos^2 x\mathrm{d}x = \frac{1}{2}\int_0^\pi [1+\cos(2x)]\,\mathrm{d}x = \frac{1}{2}\left[x + \frac{1}{2}\sin(2x)\right]_0^\pi = \frac{\pi}{2}$$

S-6: (a) Using the trig identity $\tan^2 x = \sec^2 x - 1$ and the substitution $y = \tan x$, $dy = \sec^2 x\ \mathrm{d}x$,

$$\begin{aligned}\int \tan^n x\ \mathrm{d}x = \int \tan^{n-2} x\ \tan^2 x\ \mathrm{d}x \qquad &= \int \tan^{n-2} x\ \sec^2 x\ \mathrm{d}x - \int \tan^{n-2} x\ \mathrm{d}x \\ = \int y^{n-2}\,dy - \int \tan^{n-2} x\ \mathrm{d}x &= \frac{y^{n-1}}{n-1} - \int \tan^{n-2} x\ \mathrm{d}x \\ = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\ \mathrm{d}x &\end{aligned}$$

(b) By the reduction formula of part (a)

$$\int_0^{\pi/4} \tan^n(x)\,\mathrm{d}x = \frac{1}{n-1} - \int_0^{\pi/4} \tan^{n-2}(x)\,\mathrm{d}x$$

for all integers $n \geqslant 2$, since $\tan 0 = 0$ and $\tan\frac{\pi}{4} = 1$. Applying this reduction formula, with $n = 6, 4, 2$,

$$\begin{aligned}\int_0^{\pi/4} \tan^6(x)\,\mathrm{d}x &= \frac{1}{5} - \int_0^{\pi/4} \tan^4(x)\,\mathrm{d}x = \frac{1}{5} - \frac{1}{3} + \int_0^{\pi/4} \tan^2(x)\,\mathrm{d}x = \frac{1}{5} - \frac{1}{3} + 1 - \int_0^{\pi/4} \mathrm{d}x \\ &= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4} = \frac{13}{15} - \frac{\pi}{4} \approx 0.0813\end{aligned}$$

Solutions to Exercises 1.9 — Jump to TABLE OF CONTENTS

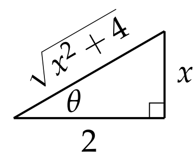
S-1: (a) The substitution $x = \frac{4}{3}\sec\theta$, combined with $\sec^2\theta - 1 = \tan^2\theta$ eliminates the square root $\sqrt{9x^2-16}$.

(b) The substitution $x = \frac{1}{2}\sin\theta$, combined with $1 - \sin^2\theta = \cos^2\theta$ eliminates the square root $\sqrt{1-4x^2}$.

(c) The substitution $x = 5\tan\theta$, combined with $1 + \tan^2\theta = \sec^2\theta$ eliminates the square root $\sqrt{25+x^2}$.

S-2: Let $x = 2\tan\theta$, so that $x^2 + 4 = 4\tan^2\theta + 4 = 4\sec^2\theta$ and $\mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta$. Then

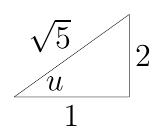
$$\begin{aligned}\int \frac{1}{(x^2+4)^{3/2}}\,\mathrm{d}x &= \int \frac{1}{(4\sec^2\theta)^{3/2}} \cdot 2\sec^2\theta\,\mathrm{d}\theta \\ &= \int \frac{2\sec^2\theta}{8\sec^3\theta}\,\mathrm{d}\theta \\ &= \frac{1}{4}\int \cos\theta\,\mathrm{d}\theta \\ &= \frac{1}{4}\sin\theta + C = \frac{1}{4}\frac{x}{\sqrt{x^2+4}} + C\end{aligned}$$



The fact that $\sin\theta = \frac{x}{\sqrt{x^2+4}}$ when $\tan\theta = \frac{x}{2}$ can be read off of the right angled triangle above. In that triangle, we have chosen the lengths of the right hand and bottom sides so that $\tan\theta = \frac{x}{2}$ and then we determined the length of the hypotheneuse by using Pythagorous.

S-3: Substitute $x = 2\tan u$, so that $\mathrm{d}x = 2\sec^2 u\,\mathrm{d}u$. Note that when $x = 4$ we have $4 = 2\tan u$, so that $\tan u = 2$.

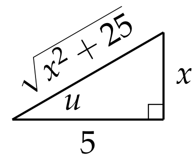
$$\begin{aligned}\int_0^4 \frac{1}{\left(4+x^2\right)^{3/2}}\,\mathrm{d}x &= \int_0^{\arctan 2} \frac{1}{\left(4+4\tan^2 u\right)^{3/2}}\, 2\sec^2 u\,\mathrm{d}u \\ &= \frac{1}{4}\int_0^{\arctan 2} \frac{\sec^2 u}{\sec^3 u}\,\mathrm{d}u \\ &= \frac{1}{4}\int_0^{\arctan 2} \cos u\,\mathrm{d}u \\ &= \frac{1}{4}\sin u\Big|_0^{\arctan 2} \\ &= \frac{1}{4}\big(\sin(\arctan 2) - 0\big) = \frac{1}{2\sqrt{5}}\end{aligned}$$



That the sin of arctan 2 is $\frac{2}{\sqrt{5}}$ has been read off of the triangle above. The lengths of the right hand side and bottom of the triangle were first chosen so that $\tan u = 2$. Then the hypotenuse was determined by using Pythagorous.

S-4: Substitute $x = 5\tan u$, so that $\mathrm{d}x = 5\sec^2 u\,\mathrm{d}u$.

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+25}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{25\tan^2 u + 25}}\,5\sec^2 u\,\mathrm{d}u \\ &= \int \frac{\sec^2 u}{\sec u}\,\mathrm{d}u = \int \sec u\,\mathrm{d}u \\ &= \log\left|\sec u + \tan u\right| + C \\ &= \log\left|\sqrt{1+\frac{x^2}{25}} + \frac{x}{5}\right| + C\end{aligned}$$



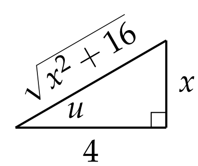
The fact that $\sec u = \sqrt{1+\frac{x^2}{25}}$ when $\tan u = \frac{x}{5}$ can be read off of the right angled triangle above. In that triangle, we have chosen the lengths of the right hand and bottom sides so that $\tan u = \frac{x}{5}$ and then we determined the length of the hypotheneuse by using Pythagorous.

S-5: Substitute $x = 4\tan u$, so that $\mathrm{d}x = 4\sec^2 u\,\mathrm{d}u$.

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+16}}\,\mathrm{d}x &= \int \frac{1}{16\tan^2 u\sqrt{16\tan^2 u+16}}\,4\sec^2 u\,\mathrm{d}u \\ &= \int \frac{\sec^2 u}{16\tan^2 u\sec u}\,\mathrm{d}u = \frac{1}{16}\int \frac{\sec u}{\tan^2 u}\,\mathrm{d}u \\ &= \frac{1}{16}\int \frac{\cos u}{\sin^2 u}\,\mathrm{d}u\end{aligned}$$

To finish off the integral, we'll substitute $v = \sin u$, $\mathrm{d}v = \cos u\,\mathrm{d}u$.

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+16}}\,\mathrm{d}x &= \frac{1}{16}\int \frac{\cos u}{\sin^2 u}\,\mathrm{d}u = \frac{1}{16}\int \frac{\mathrm{d}v}{v^2} = -\frac{1}{16v} + C \\ &= -\frac{1}{16\sin u} + C = -\frac{1}{16}\sqrt{1+\frac{16}{x^2}} + C\end{aligned}$$



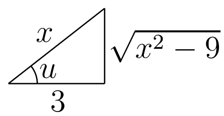
The fact that $\sin u = \frac{x}{\sqrt{x^2+16}}$ when $\tan u = \frac{x}{4}$ can be read off of the right angled triangle above. In that triangle, we have chosen the lengths of the right hand and bottom sides so that $\tan u = \frac{x}{4}$ and then we determined the length of the hypotheneuse by using Pythagorous.

S-6: Make the change of variables $x = 5\sin\theta$, $\mathrm{d}x = 5\cos\theta\,\mathrm{d}\theta$. Since $x = 0$ corresponds to $\theta = 0$ and $x = \frac{5}{2}$ correponds to $\sin\theta = \frac{1}{2}$ or $\theta = \frac{\pi}{6}$,

$$\int_0^{5/2} \frac{\mathrm{d}x}{\sqrt{25-x^2}} = \int_0^{\pi/6} \frac{5\cos\theta\,\mathrm{d}\theta}{\sqrt{25-25\sin^2\theta}} = \int_0^{\pi/6} \mathrm{d}\theta = \frac{\pi}{6}$$

S-7: Substituting $x = 3\sec u$, so that $\mathrm{d}x = 3\sec u \tan u \, \mathrm{d}u$ and $x^2 - 9 = 9\sec^2 u - 9 = 9\tan^2 u$, gives

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-9}} &= \int \frac{3\sec u \tan u \, \mathrm{d}u}{9\sec^2 u \sqrt{9\sec^2 u - 9}} \\
&= \int \frac{3\sec u \tan u \, \mathrm{d}u}{9\sec^2 u \sqrt{9\tan^2 u}} \\
&= \frac{1}{9}\int \frac{\mathrm{d}u}{\sec u} \\
&= \frac{1}{9}\int \cos u \, \mathrm{d}u = \frac{1}{9}\sin u + C.
\end{aligned}$$



The bottom and hypotenuse of the right-angled triangle above have been chosen so that $\sec u = \frac{x}{3}$. By Pythagorous the right hand side is $\sqrt{x^2-9}$. So $\sin u = \frac{\sqrt{x^2-9}}{x}$ and

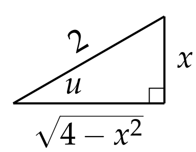
$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-9}} = \frac{\sqrt{x^2-9}}{9x} + C.$$

There are of course equivalent ways to write this answer—for example,

$$\frac{1}{9}\sqrt{1 - \left(\frac{3}{x}\right)^2} + C$$

S-8: Substitute $x = 2\sin u$, so that $\mathrm{d}x = 2\cos u \, \mathrm{d}u$.

$$\begin{aligned}
\int \sqrt{4-x^2} \, \mathrm{d}x &= \int \sqrt{4 - 4\sin^2 u} \, 2\cos u \, \mathrm{d}u \\
&= \int 4\cos^2 u \, \mathrm{d}u = 2\int \big[1 + \cos(2u)\big] \, \mathrm{d}u \\
&= 2u + \sin(2u) + C \\
&= 2u + 2\sin u \cos u + C \\
&= 2\arcsin\tfrac{x}{2} + x\sqrt{1 - \tfrac{x^2}{4}} + C
\end{aligned}$$



The fact that $\cos u = \sqrt{1 - \frac{x^2}{4}}$ when $\sin u = \frac{x}{2}$ can be read off of the right angled triangle above. In that triangle, we have chosen the lengths of the right hand side and of the hypotheneuse so that $\sin u = \frac{x}{2}$ and then we determined the length of the bottom side by Pythagorous.

S-9: This integrand looks very different from those above. But it is only slightly disguised. If we complete the square

$$\int \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}} = \int \frac{\mathrm{d}x}{\sqrt{4-(x+1)^2}}$$

and then make the substitution $y = x+1$, $\mathrm{d}y = \mathrm{d}x$

$$\int \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}} = \int \frac{\mathrm{d}x}{\sqrt{4-(x+1)^2}} = \int \frac{\mathrm{d}y}{\sqrt{4-y^2}}$$

we get a typical trig substitution integral. So we substitute $y = 2\sin\theta$, $\mathrm{d}y = 2\cos\theta\,\mathrm{d}\theta$ to get

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}} &= \int \frac{\mathrm{d}y}{\sqrt{4-y^2}} = \int \frac{2\cos\theta\,\mathrm{d}\theta}{\sqrt{4-4\sin^2\theta}} = \int \mathrm{d}\theta = \theta + C = \arcsin\frac{y}{2} + C \\ &= \arcsin\frac{x+1}{2} + C\end{aligned}$$

An experienced integrator would probably substitute $x+1 = 2\sin\theta$ directly, without going through y.

S-10: We'll use the trig identity $\cos 2\theta = 2\cos^2\theta - 1$. It implies that

$$\begin{aligned}\cos^2\theta = \frac{\cos 2\theta + 1}{2} \implies \cos^4\theta &= \frac{1}{4}\big[\cos^2 2\theta + 2\cos 2\theta + 1\big] = \frac{1}{4}\Big[\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\Big] \\ &= \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}\end{aligned}$$

So

$$\begin{aligned}\int_0^{\pi/4} \cos^4\theta\,\mathrm{d}\theta &= \int_0^{\pi/4} \Big[\frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}\Big]\,\mathrm{d}\theta \\ &= \Big[\frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3}{8}\theta\Big]_0^{\pi/4} \\ &= \frac{1}{4} + \frac{3}{8}\,\frac{\pi}{4} \\ &= \frac{8+3\pi}{32}\end{aligned}$$

as required.

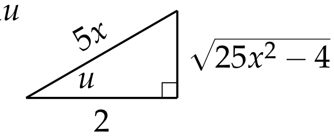
(b) We'll use the trig substitution $x = \tan\theta$, $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. Note that when $\theta = \pm\frac{\pi}{4}$, we have $x = \pm 1$. Also note that dividing the trig identity $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ gives the trig identity $\tan^2\theta + 1 = \sec^2\theta$. So

$$\begin{aligned}\int_{-1}^{1} \frac{\mathrm{d}x}{(x^2+1)^3} &= 2\int_0^1 \frac{\mathrm{d}x}{(x^2+1)^3} \\ &= 2\int_0^{\pi/4} \frac{\sec^2\theta\,\mathrm{d}\theta}{(\tan^2\theta+1)^3} \\ &= 2\int_0^{\pi/4} \frac{\sec^2\theta\,\mathrm{d}\theta}{(\sec^2\theta)^3} \\ &= 2\int_0^{\pi/4} \cos^4\theta\,\mathrm{d}\theta \\ &= \frac{8+3\pi}{16}\end{aligned}$$

by part (a).

S-11: Substitute $x = \frac{2}{5}\sec u$, so that $\mathrm{d}x = \frac{2}{5}\sec u\ \tan u\,\mathrm{d}u$ and $25x^2 - 4 = 4(\sec^2 u - 1) = 4\tan^2 u$.

$$\begin{aligned}
\int \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x &= \int \frac{2\tan u}{\frac{2}{5}\sec u}\,\frac{2}{5}\sec u\tan u\,\mathrm{d}u \\
&= 2\int \tan^2 u\,\mathrm{d}u = 2\int \big[\sec^2 u - 1\big]\,\mathrm{d}u \\
&= 2\tan u - 2u + C \\
&= \sqrt{25x^2-4} - 2\,\mathrm{arcsec}\,\tfrac{5x}{2} + C
\end{aligned}$$



The fact that $\tan u = \frac{1}{2}\sqrt{25x^2-4}$ when $\sec u = \frac{5x}{2}$ can be read off of the right angled triangle above. In that triangle, we have chosen the lengths of the bottom side and of the hypotheneuse so that $\sec u = \frac{5x}{2}$, i.e. $\cos u = \frac{2}{5x}$, and then we determined the length of the right hand side by Pythagorous.

Solutions to Exercises 1.10 — Jump to TABLE OF CONTENTS

S-1: The partial fraction expansion has the form

$$\frac{3x^3 - 2x^2 + 11}{x^2(x-1)(x^2+3)} = \frac{A}{x-1} + \text{various terms}$$

When we multiply through by the original denominator, this becomes

$$3x^3 - 2x^2 + 11 = x^2(x^2+3)A + (x-1)(\text{other terms}).$$

Evaluating both sides at $x = 1$ yields $3 \cdot 1^3 - 2 \cdot 1^2 + 11 = 1^2(1^2+3)A + 0$, or $A = 3$.

S-2:

$$\begin{aligned}
\frac{x^3+3}{(x^2-1)^2(x^2+1)} &= \frac{x^3+3}{(x-1)^2(x+1)^2(x^2+1)} \\
&= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1}
\end{aligned}$$

S-3: This is a section on partial fractions. So of course we are going to use partial fractions. We start by expressing the integrand, i.e. the fraction $\frac{1}{x+x^2} = \frac{1}{x(1+x)}$, as a linear combination of the simpler fractions $\frac{1}{x}$ and $\frac{1}{x+1}$ (which we already know how to integrate). We will have

$$\frac{1}{x+x^2} = \frac{1}{x(1+x)} = \frac{a}{x} + \frac{b}{x+1} = \frac{a(x+1)+bx}{x(1+x)}$$

The fraction on the left hand side is the same as the fraction on the right hand side if and only if the numerator on the left hand side, which is $1 = 0x + 1$, is equal to the numerator on the right hand side, which is $a(x+1) + bx = (a+b)x + a$. This in turn is the case if and only of $a = 1$ (i.e. the constant terms are the same in the two numerators) and $a + b = 0$ (i.e. the coefficients of x are the same in the two numerators). So $a = 1$ and $b = -1$. Now we can easily do the integral

$$\int_1^2 \frac{\mathrm{d}x}{x+x^2} = \int_1^2 \frac{\mathrm{d}x}{x(x+1)} = \int_1^2 \left[\frac{1}{x} - \frac{1}{x+1}\right] \mathrm{d}x = \Big[\log x - \log(x+1)\Big]_1^2 = \log 2 - \log\frac{3}{2}$$
$$= \log\frac{4}{3}$$

S-4: We'll first do a partial fractions expansion. The sneaky way is to temporarily rename x^2 to y. Then $x^4 + x^2 = y^2 + y$ and

$$\frac{1}{x^4+x^2} = \frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$$

Now we restore y to x^2 giving

$$\int \frac{1}{x^4+x^2}\,\mathrm{d}x = \int \left[\frac{1}{x^2} - \frac{1}{x^2+1}\right] \mathrm{d}x = -\frac{1}{x} - \arctan x + C$$

S-5: The integrand is of the form $N(x)/D(x)$ with $D(x)$ already factored and $N(x)$ of lower degree. We immediately look for a partial fractions decomposition:

$$\frac{12x+4}{(x-3)(x^2+1)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1}.$$

Multiplying through by the denominator yields

$$12x + 4 = A(x^2+1) + (Bx+C)(x-3) \tag{$*$}$$

Setting $x = 3$ we find:

$$36 + 4 = A(9+1) + 0 \implies 40 = 10A \implies A = 4$$

Substituting $A = 4$ in $(*)$ gives

$$\begin{aligned} 12x + 4 = 4(x^2+1) + (Bx+C)(x-3) &\implies -4x^2 + 12x = (x-3)(Bx+C) \\ &\implies (-4x)(x-3) = (Bx+C)(x-3) \\ &\implies B = -4,\ C = 0 \end{aligned}$$

So we have found that $A = 4$, $B = -4$, and $C = 0$. Therefore

$$\begin{aligned} \int \frac{12x+4}{(x-3)(x^2+1)}\,dx &= \int \left(\frac{4}{x-3} - \frac{4x}{x^2+1}\right) dx \\ &= 4\log|x-3| - 2\log(x^2+1) + C \end{aligned}$$

Here the second integral was found just by guessing an antiderivative. Alternatively, one could use the substitution $u = x^2 + 1$, $\mathrm{d}u = 2x\,\mathrm{d}x$.

S-6: The integrand is of the form $N(x)/D(x)$ with $D(x)$ already factored and $N(x)$ of lower degree. We immediately look for a partial fractions decomposition:

$$\frac{3x^2-4}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

Multiplying through by the denominator gives

$$3x^2 - 4 = A(x^2+4) + (Bx+C)(x-2) \tag{$*$}$$

Setting $x = 2$ we find:

$$12 - 4 = A(4+4) + 0 \implies 8 = 8A \implies A = 1$$

Substituting $A = 1$ in $(*)$ gives

$$\begin{aligned} 3x^2 - 4 = (x^2+4) + (x-2)(Bx+C) &\implies 2x^2 - 8 = (x-2)(Bx+C) \\ &\implies (x-2)(2x+4) = (x-2)(Bx+C) \\ &\implies B = 2,\ C = 4 \end{aligned}$$

Thus, we have:

$$\frac{3x^2-4}{(x-2)(x^2+4)} = \frac{1}{x-2} + \frac{2x+4}{x^2+4} = \frac{1}{x-2} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}$$

The first two of these are directly integrable:

$$F(x) = \log|x-2| + \log|x^2+4| + \int \frac{4}{x^2+4}\,\mathrm{d}x$$

(The second integral was found just by guessing an antiderivative. Alternatively, one could use the substitution $u = x^2 + 4$, $\mathrm{d}u = 2x\,\mathrm{d}x$.) For the final integral, we substitute: $x = 2y$, $\mathrm{d}x = 2\mathrm{d}y$, and see that:

$$\int \frac{4}{x^2+4}\,\mathrm{d}x = 2\int \frac{1}{y^2+1}\,dy = 2\arctan y + D = 2\arctan(x/2) + D$$

for any constant D. All together we have:

$$F(x) = \log|x-2| + \log|x^2+4| + 2\arctan(x/2) + D$$

S-7: This sure looks like a partial fractions problem. So let's go through our protocol.

- The degree of the numerator $x - 13$ is one, which is strictly smaller than the dergee of the denominator $x^2 - x - 6$, which is two. So we do not long divide to pull out a polynomial.
- Next we factor the denominator.

$$x^2 - x - 6 = (x-3)(x+2)$$

- Next we find the partial fractions expansion of the integrand. It is of the form

$$\frac{x-13}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

To find A and B, using the sneaky method, we cross multiply by the denominator.

$$x - 13 = A(x+2) + B(x-3)$$

Now we can find A by evaluating at $x = 3$

$$3 - 13 = A(3+2) + B(3-3) \implies A = -2$$

and find B by evaluating at $x = -2$.

$$-2 - 13 = A(-2+2) + B(-2-3) \implies B = 3$$

(Hmmm. A and B are nice round numbers. Sure looks like a rigged exam or problem set problem.) So our partial fraction expansion is

$$\frac{x-13}{(x-3)(x+2)} = \frac{-2}{x-3} + \frac{3}{x+2}$$

As a check, we recombine the right hand side and make sure that it matches the left hand side

$$\frac{-2}{x-3} + \frac{3}{x+2} = \frac{-2(x+2)+3(x-3)}{(x-3)(x+2)} = \frac{x-13}{(x-3)(x+2)}$$

- Finally, we do the integral

$$\int \frac{x-13}{x^2-x-6}\mathrm{d}x = \int \left[\frac{-2}{x-3} + \frac{3}{x+2}\right]\mathrm{d}x = -2\log|x-3| + 3\log|x+2| + C$$

S-8: Again, this sure looks like a partial fractions problem. So let's go through our protocol.

- The degree of the numerator $5x+1$ is one, which is strictly smaller than the dergee of the denominator x^2+5x+6, which is two. So we do not long divide to pull out a polynomial.
- Next we factor the denominator.

$$x^2 + 5x + 6 = (x+2)(x+3)$$

- Next we find the partial fractions expansion of the integrand. It is of the form

$$\frac{5x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

To find A and B, using the sneaky method, we cross multiply by the denominator.

$$5x+1 = A(x+3)+B(x+2)$$

Now we can find A by evaluating at $x=-2$

$$-10+1 = A(-2+3)+B(-2+2) \implies A=-9$$

and find B by evaluating at $x=-3$.

$$-15+1 = A(-3+3)+B(-3+2) \implies B=14$$

So our partial fraction expansion is

$$\frac{5x+1}{(x+2)(x+3)} = \frac{-9}{x+2}+\frac{14}{x+3}$$

As a check, we recombine the right hand side and make sure that it matches the left hand side

$$\frac{-9}{x+2}+\frac{14}{x+3} = \frac{-9(x+3)+14(x+2)}{(x+2)(x+3)} = \frac{5x+1}{(x+2)(x+3)}$$

- Finally, we do the integral

$$\int \frac{5x+1}{x^2+5x+6}\mathrm{d}x = \int\left[\frac{-9}{x+2}+\frac{14}{x+3}\right]\mathrm{d}x = -9\log|x+2|+14\log|x+3|+C$$

Solutions to Exercises 1.11 — Jump to TABLE OF CONTENTS

S-1: True. Because $f(x)$ is positive and concave up, the graph of $f(x)$ is always below the top of the trapezoids used in the trapezoidal rule.

S-2: By (1.11.2) in the CLP 101 notes, the midpoint rule approximation to $\int_a^b f(x)\,\mathrm{d}x$ with $n=3$ is

$$\int_a^b f(x)\,\mathrm{d}x \approx \big[f(\bar{x}_1)+f(\bar{x}_2)+f(\bar{x}_3)\big]\Delta x$$

where $\Delta x = \frac{b-a}{3}$ and

$$\begin{array}{llll} x_0=a & x_1=a+\Delta x & x_2=a+2\Delta x & x_3=b \\ & \bar{x}_1=\frac{x_0+x_1}{2} & \bar{x}_2=\frac{x_1+x_2}{2} & \bar{x}_3=\frac{x_2+x_3}{2} \end{array}$$

For this problem, $a=0$, $b=\pi$ and $f(x)=\sin x$, so that $\Delta x=\frac{\pi}{3}$ and

$$\begin{array}{llll} x_0=0 & x_1=\frac{\pi}{3} & x_2=\frac{2\pi}{3} & x_3=\pi \\ & \bar{x}_1=\frac{\pi}{6} & \bar{x}_2=\frac{\pi}{2} & \bar{x}_3=\frac{5\pi}{6} \end{array}$$

and

$$\int_0^\pi \sin x\,\mathrm{d}x \approx \big[\sin\tfrac{\pi}{6}+\sin\tfrac{\pi}{2}+\sin\tfrac{5\pi}{6}\big]\tfrac{\pi}{3} = \big[\tfrac{1}{2}+1+\tfrac{1}{2}\big]\tfrac{\pi}{3} = \tfrac{2\pi}{3}$$

S-3: Let $f(x)$ be the diameter a distance x from the left end of the log. The cross sectional area a distance x from the left end of the log is then $\pi\big(\frac{f(x)}{2}\big)^2 = \frac{\pi}{4}f(x)^2$. The volume is

$$\begin{aligned} V = \int_0^6 \frac{\pi}{4} f(x)^2\,\mathrm{d}x &\approx \frac{\pi}{4}\frac{1}{3}\Big[f(0)^2 + 4f(1)^2 + 2f(2)^2 + 4f(3)^2 + 2f(4)^2 + 4f(5)^2 + f(6)^2\Big] \\ &= \frac{\pi}{12}\Big[1.2^2 + 4(1)^2 + 2(0.8)^2 + 4(0.8)^2 + 2(1)^2 + 4(1)^2 + 1.2^2\Big] \\ &= 4.377\ \mathrm{m}^3 \end{aligned}$$

where we have approximated the integral using Simpson's Rule with $\Delta x = 1$.

S-4: Let $f(x)$ denote the diameter at height x. As in Example 1.6.6 of the CLP 101 notes, we slice V into thin horizontal "pancakes", which in this case are circular.

- We are told that the pancake at height x is a circular disk of diameter $f(x)$ and so
- has cross-sectional area $\pi\big(\frac{f(x)}{2}\big)^2$ and thickness $\mathrm{d}x$ and hence
- has volume $\pi\big(\frac{f(x)}{2}\big)^2\mathrm{d}x$.

Hence the volume of V is

$$\begin{aligned} \int_0^2 \pi\Big[\tfrac{f(x)}{2}\Big]^2\,\mathrm{d}x &\approx \frac{\pi}{4}10\Big[\tfrac{1}{2}f(0)^2 + f(10)^2 + f(20)^2 + f(30)^2 + \tfrac{1}{2}f(40)^2\Big] \\ &= \frac{\pi}{4}10\Big[\tfrac{1}{2}24^2 + 16^2 + 10^2 + 6^2 + \tfrac{1}{2}4^2\Big] \\ &= 688 \times 2.5\pi = 1720\pi = 5403.5 \end{aligned}$$

where we have approximated the integral using the trapezoidal rule with $\Delta x = 10$.

S-5: Call the circumference at height x, $c(x)$. The corresponding radius is $\frac{c(x)}{2\pi}$ and the corresponding cross–sectional area is $\pi\big(\frac{c(x)}{2\pi}\big)^2 = \frac{c(x)^2}{4\pi}$. Hence the total volume is

$$\begin{aligned} \int_0^8 \frac{c(x)^2}{4\pi}\,\mathrm{d}x &\approx \frac{1}{4\pi}\frac{2}{3}\Big[c(0)^2 + 4c(2)^2 + 2c(4)^2 + 4c(6)^2 + c(8)^2\Big] \\ &= \frac{1}{4\pi}\frac{2}{3}\Big[1.2^2 + 4(1.1)^2 + 2(1.3)^2 + 4(0.9)^2 + 0.2^2\Big] = 0.6865 \end{aligned}$$

S-6: (a) The Trapezoidal Rule gives

$$\begin{aligned} V &= \int_0^{60} A(h)\,\mathrm{d}h \approx 10\Big[\tfrac{1}{2}A(0) + A(10) + A(20) + A(30) + A(40) + A(50) + \tfrac{1}{2}A(60)\Big] \\ &= 363{,}500 \end{aligned}$$

(b) Simpson's Rule gives

$$\begin{aligned} V &= \int_0^{60} A(h)\,\mathrm{d}h \approx \frac{20}{6}\Big[A(0) + 4A(10) + 2A(20) + 4A(30) + 2A(40) + 4A(50) + A(60)\Big] \\ &= 367{,}000 \end{aligned}$$

S-7: Call the curve in the graph $y = f(x)$. It looks like

$$f(2) = 3 \qquad f(3) = 8 \qquad f(4) = 7 \qquad f(5) = 6 \qquad f(6) = 4$$

(a) The Trapezoidal Rule gives

$$T_4 = \tfrac{1}{2}\{3 + 2 \times 8 + 2 \times 7 + 2 \times 6 + 4\} \times 1 = \frac{49}{2}$$

(b) Simpson's Rule gives

$$S_4 = \frac{1}{3}\{3 + 4 \times 8 + 2 \times 7 + 4 \times 6 + 4\} \times 1 = \frac{77}{3}$$

S-8: Let $f(x) = \sin(x^2)$. Then $f'(x) = 2x\cos(x^2)$ and

$$f''(x) = 2\cos(x^2) - 4x^2\sin(x^2).$$

Since $|x^2| \leqslant 1$ when $|x| \leqslant 1$, and $|\sin\theta| \leqslant 1$ and $|\cos\theta| \leqslant 1$ for all θ, we have

$$\left|2\cos(x^2) - 4x^2\sin(x^2)\right| \leqslant 2|\cos(x^2)| + 4x^2|\sin(x^2)| \leqslant 2 \times 1 + 4 \times 1 \times 1 = 2 + 4 = 6$$

We can therefore choose $K = 6$, and it follows that the error is at most

$$\frac{K[b-a]^3}{24n^2} \leqslant \frac{6 \cdot [1-(-1)]^3}{24 \cdot 1000^2} = \frac{2}{10^6} = 2 \cdot 10^{-6}$$

S-9: Setting $f(x) = 2x^4$ and $b - a = 1 - (-2) = 3$, we compute $f''(x) = 24x^2$. The largest value of $24x^2$ on the interval $[-2, 1]$ occurs at $x = -2$, so we can take $M = 24 \cdot (-2)^2 = 96$. Thus the total error for the midpoint rule with $n = 60$ points is bounded by

$$\frac{M(b-a)^3}{24n^2} = \frac{96 \times 3^3}{24 \times 60 \times 60} = \frac{3}{100}$$

S-10: (a) Since $a = 0$, $b = 2$ and $n = 6$, we have $\Delta x = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3}$, and so $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 1$, $x_4 = \frac{4}{3}$, $x_5 = \frac{5}{3}$, and $x_6 = 2$. Since Simpson's Rule with $n = 6$ in general is

$$\frac{\Delta x}{3}\big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\big],$$

the desired approximation is

$$\frac{1/3}{3}\left((-3)^5 + 4\left(\frac{1}{3} - 3\right)^5 + 2\left(\frac{2}{3} - 3\right)^5 + 4(-2)^5 + 2\left(\frac{4}{3} - 3\right)^5 + 4\left(\frac{5}{3} - 3\right)^5 + (-1)^5\right)$$

(b) Here $f(x) = (x-3)^5$, which has derivatives

$$f'(x) = 5(x-3)^4 \qquad f''(x) = 20(x-3)^3$$
$$f^{(3)}(x) = 60(x-3)^2 \qquad f^{(4)}(x) = 120(x-3).$$

For $0 \leqslant x \leqslant 2$, $(x-3)$ runs from -3 to -1, so the maximum absolute values are found at $x = 0$, giving $K = 20 \cdot |0-3|^3 = 540$ and $L = 120 \cdot |0-3| = 360$. Consequently, for the Midpoint Rule with $n = 100$,

$$|E_M| \leqslant \frac{K(b-a)^3}{24n^2} = \frac{540 \times 2^3}{24 \times 10^4} = \frac{180}{10^4};$$

whereas for Simpson's Rule with $n = 10$,

$$|E_S| \leqslant \frac{360 \times 2^5}{180 \times 10^4} = \frac{64}{10^4}.$$

Since $64 < 180$, Simpson's Rule results in a smaller error bound.

S-11: In this case, $a = 1$, $b = 4$ and we may take $K = 2$. So we need n to obey

$$\frac{2(4-1)^3}{12n^2} \leqslant 0.001 \iff n^2 \geqslant \frac{2(3)^3}{12}1000 = \frac{27000}{6} = \frac{9000}{2} = 4500$$

One obvious allowed n is 100. Any $n \geqslant 68$ works.

S-12: In general the error in approximating $\int_a^b f(x)\,\mathrm{d}x$ using Simpson's rule with n steps is bounded by $\frac{K(b-a)}{180}(\Delta x)^4$ where $\Delta x = \frac{b-a}{n}$ and $K \geqslant |f^{(4)}(x)|$ for all $a \leqslant x \leqslant b$. In this case, $a = 1$, $b = 5$, $n = 4$ and $f(x) = \frac{1}{x}$. So

$$f'(x) = -\frac{1}{x^2} \qquad f''(x) = \frac{2}{x^3} \qquad f^{(3)}(x) = -\frac{6}{x^4} \qquad f^{(4)}(x) = \frac{24}{x^5}$$

and

$$\left|f^{(4)}(x)\right| \leqslant 24 \text{ for all } x \geqslant 1$$

So we may take $K = 24$ and $\Delta x = \frac{5-1}{4} = 1$ and

$$\text{Error } \leqslant \frac{24(5-1)}{180}(1)^4 = \frac{24}{45} = \frac{8}{15}$$

S-13: In general the error in approximating $\int_a^b f(x)\,\mathrm{d}x$ using Simpson's rule with n steps is bounded by $\frac{K(b-a)}{180}(\Delta x)^4$ where $\Delta x = \frac{b-a}{n}$ and $K \geqslant |f^{(4)}(x)|$ for all $a \leqslant x \leqslant b$. In this case, $a = 0$, $b = 1$, $n = 6$ and $f(x) = e^{-2x} + 3x^3$. So

$$f'(x) = -2e^{-2x} + 9x^2 \qquad f''(x) = 4e^{-2x} + 18x \qquad f^{(3)}(x) = -8e^{-2x} + 18 \qquad f^{(4)}(x) = 16e^{-2x}$$

and

$$\left|f^{(4)}(x)\right| \leqslant 16 \text{ for all } x \geqslant 0$$

So we may take $K = 16$ and $\Delta x = \frac{1-0}{6} = \frac{1}{6}$ and

$$\text{Error } \leqslant \frac{16(1-0)}{180}(1/6)^4 = \frac{16}{180 \times 6^4} = \frac{1}{180 \times 3^4} = \frac{1}{14580}$$

S-14: Set $a = 1$, $b = 2$, $n = 4$, $f(x) = \frac{1}{x}$ and $h = \frac{b-a}{n} = \frac{1}{4}$. Then

(a)

$$\begin{aligned} T_4 &= \frac{h}{2}\big[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\big] \\ &= \frac{h}{2}\big[f(1) + 2f(5/4) + 2f(6/4) + 2f(7/4) + f(2)\big] \\ &= \frac{1}{8}\Big[1 + 2 \times \frac{4}{5} + 2 \times \frac{4}{6} + 2 \times \frac{4}{7} + \frac{1}{2}\Big] \end{aligned}$$

(b)

$$\begin{aligned} S_4 &= \frac{h}{3}\big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\big] \\ &= \frac{h}{3}\big[f(1) + 4f(5/4) + 2f(6/4) + 4f(7/4) + f(2)\big] \\ &= \frac{1}{12}\Big[1 + 4 \times \frac{4}{5} + 2 \times \frac{4}{6} + 4 \times \frac{4}{7} + \frac{1}{2}\Big] \end{aligned}$$

(c) In this case, $a = 1$, $b = 2$, $n = 4$ and $f(x) = \frac{1}{x}$. So

$$f'(x) = -\frac{1}{x^2} \qquad f''(x) = \frac{2}{x^3} \qquad f^{(3)}(x) = -\frac{6}{x^4} \qquad f^{(4)}(x) = \frac{24}{x^5}$$

and

$$\left|f^{(4)}(x)\right| \leqslant 24 \text{ for all } x \geqslant 1$$

So we may take $K = 24$ and

$$\text{Error } \leqslant \frac{K(b-a)^5}{180 \times n^4} \leqslant \frac{24(2-1)^5}{180 \times 4^4} = \frac{24}{180 \times 4^4} = \frac{3}{5760}$$

S-15: Set $a = 0$, $b = 8$, $n = 4$, $f(x) = s(x)$ and $h = \frac{b-a}{n} = 2$. Then

(a)

$$\begin{aligned} T_4 &= \frac{h}{2}\big[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\big] \\ &= 1.00664 + 2\times 1.00543 + 2\times 1.00435 + 2\times 1.00331 + 1.00233 \\ &= 8.03515 \\ S_4 &= \frac{h}{3}\big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\big] \\ &= \frac{2}{3}\big[1.00664 + 4\times 1.00543 + 2\times 1.00435 + 4\times 1.00331 + 1.00233\big] \\ &= 8.03509 \end{aligned}$$

(b)

$$\begin{aligned} \left|\int_a^b f(x)\,dx - T_n\right| &\leqslant \frac{K_2(b-a)^3}{12n^2} \leqslant \frac{2}{1000}\frac{8^3}{12(4)^2} = 0.00533 \\ \left|\int_a^b f(x)\,dx - S_n\right| &\leqslant \frac{K_4(b-a)^5}{180n^4} \leqslant \frac{4}{1000}\frac{8^5}{180(4)^4} = 0.00284 \end{aligned}$$

S-16: Denote by $f(x)$ the width of the pool a distance x from the left hand end. Thus $f(0) = 0$, $f(2) = 10$, $f(4) = 12$, $f(6) = 10$, $f(8) = 8$, $f(10) = 6$, $f(12) = 8$, $f(14) = 10$ and $f(16) = 0$. The volume of the part of the pool with x–coordinate running from x to $x + \mathrm{d}x$ is $\frac{1}{2}\pi\big(\frac{f(x)}{2}\big)^2\,\mathrm{d}x$. So the total volume is

$$\begin{aligned} V &= \frac{\pi}{8}\int_0^{16} f(x)^2\,\mathrm{d}x \\ &\approx \frac{\pi}{8}\frac{\Delta x}{3}\Big[f(0)^2 + 4f(2)^2 + 2f(4)^2 + 4f(6)^2 + 2f(8)^2 + 4f(10)^2 + 2f(12)^2 + 4f(14)^2 + f(16)^2\Big] \\ &= \frac{\pi}{8}\frac{2}{3}\Big[0 + 4(10)^2 + 2(12)^2 + 4(10)^2 + 2(8)^2 + 4(6)^2 + 2(8)^2 + 4(10)^2 + 0\Big] \\ &\approx 494 \text{ ft}^3 \end{aligned}$$

S-17: (a) The Trapezoidal Rule gives

$$\begin{aligned} M &= 2\pi 10^{-6}\int_0^1 rg(r)\,\mathrm{d}r \approx 2\pi 10^{-6}\,\frac{1}{4}\Big[\frac{1}{2}0g(0) + \frac{1}{4}g\big(\tfrac{1}{4}\big) + \frac{1}{2}g\big(\tfrac{1}{2}\big) + \frac{3}{4}g\big(\tfrac{3}{4}\big) + \frac{1}{2}g(1)\Big] \\ &= 0.025635 \end{aligned}$$

(b) In this case, the integrand $f(r) = 2\pi 10^{-6} rg(r)$ obeys

$$f''(r) = 2\pi 10^{-6}\frac{\mathrm{d}}{\mathrm{d}r}\big[g(r) + rg'(r)\big] = 2\pi 10^{-6}\big[2g'(r) + rg''(r)\big]$$

and hence, for $0 \leqslant r \leqslant 1$,

$$\left|f''(r)\right| \leqslant 2\pi 10^{-6}\left[2 \times 200 + 1 \times 150\right] = 1.1\pi 10^{-3}$$

So we may take $K = 1.1\pi 10^{-3}$ and, as $a = 0$, $b = 1$, and $n = 4$,

$$\text{error} \leqslant \frac{1.1\pi 10^{-3}(1-0)^3}{12(4)^2} \leqslant 1.8 \times 10^{-5}$$

S-18: (a) Let $f(x) = \frac{1}{x}$, $a = 1$, $b = 2$ and $\Delta x = \frac{b-a}{6} = \frac{1}{6}$. By Simpson's rule

$$\begin{aligned}\int_1^2 \frac{1}{x}\,\mathrm{d}x &\approx \frac{\Delta x}{3}\left[f(1) + 4f\left(\frac{7}{6}\right) + 2f\left(\frac{8}{6}\right) + 4f\left(\frac{9}{6}\right) + 2f\left(\frac{10}{6}\right) + 4f\left(\frac{11}{6}\right) + f(2)\right] \\ &= \frac{1}{18}\left[1 + \frac{24}{7} + \frac{12}{8} + \frac{24}{9} + \frac{12}{10} + \frac{24}{11} + \frac{1}{2}\right] = 0.6931698\end{aligned}$$

(b) The integrand is $f(x) = \frac{1}{x}$. The first four derivatives of $f(x)$ are $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f^{(3)}(x) = -\frac{6}{x^4}$, $f^{(4)}(x) = \frac{24}{x^5}$. On the interval $1 \leqslant x \leqslant 2$, the fourth derivative is never bigger in magnitude than $K = 24$. So

$$|E_n| \leqslant \frac{K(b-a)^5}{180n^4} = \frac{24(2-1)^5}{180n^4} = \frac{4}{30n^4}$$

This is no more than $0.00001 = 10^{-5}$ if $n^4 \geqslant \frac{4}{30}10^5$ or $n \geqslant \sqrt[4]{\frac{4}{30}10^5} = 10.75$ or $\boxed{n \geqslant 12}$ (since n must be even for Simpson's Rule).

S-19: (a) From the figure, we see that the magnitude of $f''''(x)$ never exceeds 310 for $0 \leqslant x \leqslant 2$. So the error is bounded by

$$\frac{310(2-0)^5}{180 \times 8^4} = 0.01345$$

(b) We need to choose n so that

$$\frac{310(2-0)^5}{180 \times n^4} \leqslant 10^{-4} \iff n^4 \geqslant \frac{310 \times 2^5}{180}10^4 \iff n \geqslant 10\sqrt[4]{\frac{310 \times 32}{180}} = 27.2$$

For Simpson's rule, n must be even so choose an even integer obeying $n = 28$.

S-20: Let $g(x) = \int_0^x \sin(\sqrt{t})\,\mathrm{d}t$. Since $g'(x) = \sin(\sqrt{x})$ and $f(x) = g(x^2)$,

$$f'(x) = 2xg'(x^2) = 2x\sin x \qquad f''(x) = 2\sin x + 2x\cos x$$

Since $|\sin x|, |\cos x| \leqslant 1$, we have $|f''(x)| \leqslant 2 + 2|x|$ and, for $0 \leqslant t \leqslant 1$, $|f''(t)| \leqslant 4$. When the trapezoidal rule with n subintervals is applied, the resulting error E_n obeys

$$E_n \leqslant \frac{4(1-0)^3}{12n^2} \leqslant 0.000005 \iff n^2 \geqslant \frac{4}{12 \times 0.000005} \iff n \geqslant 259$$

S-21: The Trapezoidal Rule gives

$$M = 2\pi 10^{-6}\int_0^1 rg(r)\,\mathrm{d}r \approx 2\pi 10^{-6}\,\frac{1}{4}\Big[\frac{1}{2}0g(0)+\frac{1}{4}g\Big(\frac{1}{4}\Big)+\frac{1}{2}g\Big(\frac{1}{2}\Big)+\frac{3}{4}g\Big(\frac{3}{4}\Big)+\frac{1}{2}g(1)\Big]$$
$$= 0.025635$$

(b) In this case, the integrand $f(r) = 2\pi 10^{-6} rg(r)$ obeys

$$f''(r) = 2\pi 10^{-6}\frac{\mathrm{d}}{\mathrm{d}r}\big[g(r)+rg'(r)\big] = 2\pi 10^{-6}\big[2g'(r)+rg''(r)\big]$$

and hence, for $0 \leqslant r \leqslant 1$,

$$\big|f''(r)\big| \leqslant 2\pi 10^{-6}\big[2\times 200 + 1\times 150\big] = 1.1\pi 10^{-3}$$

So

$$\text{error} \leqslant \frac{1.1\pi 10^{-3}(1-0)^3}{12(4)^2} \leqslant 1.8\times 10^{-5}$$

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S-1: False. For example if $f(x) = e^{-x}$ and $g(x) = 1$ then $\int_1^\infty f(x)\,\mathrm{d}x$ converges but $\int_1^\infty g(x)\,\mathrm{d}x$ diverges.

S-2: Notice that

$$\int_1^t \frac{1}{x^{5q}}\,\mathrm{d}x = \begin{cases} \frac{1}{1-5q}(t^{1-5q}-1) \text{ with } 1-5q>0, & \text{if } q<\frac{1}{5},\\ \log t, & \text{if } q = \frac{1}{5},\\ \frac{1}{5q-1}(1-\frac{1}{t^{5q-1}}) \text{ with } 5q-1>0, & \text{if } q>\frac{1}{5}.\end{cases}$$

Therefore

$$\int_1^\infty \frac{1}{x^{5q}}\,\mathrm{d}x = \lim_{t\to\infty}\left(\int_1^t \frac{1}{x^{5q}}\,\mathrm{d}x\right) = \begin{cases} \frac{1}{1-5q}(\lim_{t\to\infty} t^{1-5q}-1) = \infty, & \text{if } q<\frac{1}{5},\\ \lim_{t\to\infty}\log t = \infty, & \text{if } q=\frac{1}{5},\\ \frac{1}{5q-1}(1-\lim_{t\to\infty}\frac{1}{t^{5q-1}}) = \frac{1}{5q-1}, & \text{if } q>\frac{1}{5};\end{cases}$$

The first two cases are divergent, and so the largest such value is $q = \frac{1}{5}$. (Alternatively, we might recognize this as a "p-integral" with $p = 5q$, and recall that the p-integral diverges precisely when $p \leqslant 1$.)

S-3: The denominator is zero when $x = 1$, so the integrand has a singularity at $x = 1$. So

$$\int_0^1 \frac{x^4}{x^5-1}\,\mathrm{d}x = \lim_{t\to 1^-}\int_0^t \frac{x^4}{x^5-1}\,\mathrm{d}x$$

To evaluate this integral we use the substitution $u = x^5$, $\mathrm{d}u = 5x^4\mathrm{d}x$. When $x = 0$ we have $u = 0$ and when $x = t$, we have $u = t^5$, so

$$\int_{x=0}^{x=t} \frac{x^4}{x^5-1}\,\mathrm{d}x = \int_{u=0}^{u=t^5} \frac{1}{5(u-1)}\,\mathrm{d}u = \frac{1}{5}\log|u-1|\Big|_0^{t^5} = \frac{1}{5}\log|t^5-1|$$

This diverges as $t \to 1^-$, so the integral diverges.

S-4: The integrand has a singularity at $x = -1$. So

$$\int_{-2}^{2} \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x = \lim_{t\to-1^-}\int_{-2}^{t} \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x + \lim_{t\to-1^+}\int_{t}^{2} \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x$$

Since

$$\int_{-2}^{t} \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x = -\frac{3}{(x+1)^{1/3}}\Big|_{-2}^{t} = -\frac{3}{(t+1)^{1/3}} + \frac{3}{(-1)^{1/3}}$$

diverges as $t \to -1^-$, the integral diverges. (A similar argument shows that the second integral diverges, which is also enough to conclude that the original integral diverges.)

S-5: For all $x \geqslant 1$, $\sqrt{4x^2-x} \leqslant \sqrt{4x^2} = 2x$ so

$$\int_1^\infty \frac{1}{\sqrt{4x^2-x}}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{4x^2-x}}\,\mathrm{d}x \geqslant \lim_{t\to\infty}\int_1^t \frac{1}{2x}\,\mathrm{d}x = \lim_{t\to\infty}\frac{1}{2}\ln x\Big|_1^t = \lim_{t\to\infty}\frac{1}{2}\ln t = \infty$$

So the integral *does not converge*.

S-6: The integrand is positive everywhere. So either the integral converges to some finite number or it is infinite. Since

$$\frac{1}{x^2+\sqrt{x}} \leqslant \frac{1}{\sqrt{x}} \quad \text{and the integral } \int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} \text{ converges by Example 1.12.9}$$
$$\frac{1}{x^2+\sqrt{x}} \leqslant \frac{1}{x^2} \quad \text{and the integral } \int_1^\infty \frac{\mathrm{d}x}{x^2} \text{ converges by Example 1.12.8}$$

the integral *converges* by the comparison test. (The examples are in the CLP 101 notes.)

S-7: You might think that, because the integrand is odd, the integral converges to 0. But you would be wrong. There are two "sources of impropriety", namely $x \approx +\infty$ and $x \approx -\infty$. So we split the integral in two

$$\int_{-\infty}^{+\infty} \frac{x}{x^2+1}\mathrm{d}x = \int_{-\infty}^{0} \frac{x}{x^2+1}\mathrm{d}x + \int_{0}^{+\infty} \frac{x}{x^2+1}\mathrm{d}x$$

and treat the two halves separately

$$\int_0^{+\infty} \frac{x}{x^2+1}\mathrm{d}x = \lim_{R\to\infty}\int_0^R \frac{x}{x^2+1}\mathrm{d}x = \lim_{R\to\infty}\frac{1}{2}\log(x^2+1)\Big|_0^R = \lim_{R\to\infty}\frac{1}{2}\log(R^2+1) = +\infty$$
$$\int_{-\infty}^{0} \frac{x}{x^2+1}\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^0 \frac{x}{x^2+1}\mathrm{d}x = \lim_{R\to\infty}\frac{1}{2}\log(x^2+1)\Big|_{-R}^0 = \lim_{R\to\infty}-\frac{1}{2}\log(R^2+1) = -\infty$$

Both halves diverge, so the whole integral diverges. Don't make the mistake of thinking that $\infty - \infty = 0$. That can get you into *big* trouble. ∞ is not a normal number. For example $2\infty = \infty$. So if ∞ were a normal number we would have both $\infty - \infty = 0$ and $\infty - \infty = 2\infty - \infty = \infty$.

S-8: Since

$$\begin{aligned}
\frac{|\sin x|}{x^{3/2}+x^{1/2}} &\leqslant \frac{1}{x^{1/2}} \quad \text{and the integral } \int_0^1 \frac{\mathrm{d}x}{x^{1/2}} \text{ converges by Example 1.12.9} \\
\frac{|\sin x|}{x^{3/2}+x^{1/2}} &\leqslant \frac{1}{x^{3/2}} \quad \text{and the integral } \int_1^\infty \frac{\mathrm{d}x}{x^{3/2}} \text{ converges by Example 1.12.8}
\end{aligned}$$

the integral *converges* by the comparison test. (The examples are in the CLP 101 notes.)

S-9: The integrand is positive everywhere. So either the integral converges to some finite number or it is infinite. There are two potential "sources of impropriety" — a possible singularity at $x = 0$ and the fact that the domain of integration extends to ∞. So we split up the integral.

$$\int_0^\infty \frac{x+1}{x^{1/3}(x^2+x+1)}\,\mathrm{d}x = \int_0^1 \frac{x+1}{x^{1/3}(x^2+x+1)}\,\mathrm{d}x + \int_1^\infty \frac{x+1}{x^{1/3}(x^2+x+1)}\,\mathrm{d}x$$

When $x \approx 0$, the integrand

$$\frac{x+1}{x^{1/3}(x^2+x+1)} \approx \frac{1}{x^{1/3}(1)} = \frac{1}{x^{1/3}}$$

When x is very large

$$\frac{x+1}{x^{1/3}(x^2+x+1)} \approx \frac{x}{x^{1/3}(x^2)} = \frac{1}{x^{4/3}}$$

In fact

$$\begin{aligned}
\int_0^1 \frac{x+1}{x^{1/3}(x^2+x+1)}\,\mathrm{d}x &\leqslant \int_0^1 \frac{2}{x^{1/3}}\,\mathrm{d}x = 2\frac{x^{2/3}}{2/3}\bigg|_0^1 = \frac{2}{2/3} = 3 \\
\int_1^\infty \frac{x+1}{x^{1/3}[x^2+x+1]}\,\mathrm{d}x &= \lim_{R\to\infty}\int_1^R \frac{x+1}{x^{1/3}[x^2+x+1]}\,\mathrm{d}x \leqslant \lim_{R\to\infty}\int_1^R \frac{x+1}{x^{1/3}[x(x+1)]}\,\mathrm{d}x \\
&= \lim_{R\to\infty}\int_1^R \frac{1}{x^{4/3}}\,\mathrm{d}x = \lim_{R\to\infty}\frac{x^{-1/3}}{-1/3}\bigg|_1^R = \frac{1}{1/3} = 3 \\
\implies \int_0^\infty \frac{x+1}{x^{1/3}(x^2+x+1)}\,\mathrm{d}x &\leqslant 6
\end{aligned}$$

So the integral converges.

S-10: The integrand is positive everywhere. So either the integral converges to some finite number or it is infinite. There are two potential "sources of impropriety" — a possible singularity at $x = 0$ and the fact that the domain of integration extends to ∞. So we split up the integral.

$$\int_0^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x = \int_0^1 \frac{\sin^4 x}{x^2}\,\mathrm{d}x + \int_1^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x$$

We'll treat the first integral first. By l'Hôpital's rule (or recall Example **??** in the CLP100 notes)

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1$$

Consequently

$$\lim_{x\to 0}\frac{\sin^4 x}{x^2} = \Big(\lim_{x\to 0}\sin^2 x\Big)\Big(\lim_{x\to 0}\frac{\sin x}{x}\Big)\Big(\lim_{x\to 0}\frac{\sin x}{x}\Big) = 0\times 1\times 1 = 0$$

and the first integral is not even improper.

Now for the second integral. Since $|\sin x| \leqslant 1$

$$\int_1^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x \leqslant \int_1^\infty \frac{\mathrm{d}x}{x^2} = \frac{x^{-1}}{-1}\bigg|_1^\infty = 1$$

and second integral converges by the comparison test. So the original integral *converges* too.

S-11: Let's first find a t such that $\int_t^\infty \frac{e^{-x}}{1+x}\,\mathrm{d}x \leqslant \frac{1}{2}10^{-4}$. For all $x \geqslant 0$, $0 < \frac{e^{-x}}{1+x} \leqslant e^{-x}$, so

$$\int_t^\infty \frac{e^{-x}}{1+x}\,\mathrm{d}x \leqslant \int_t^\infty e^{-x}\,\mathrm{d}x = e^{-t} \leqslant \frac{1}{2}10^{-4} \text{ if } t \geqslant -\log\Big(\frac{1}{2}10^{-4}\Big) = 9.90$$

Choose, for example, $t = 10$.

$$f(x) = \frac{e^{-x}}{1+x} \implies f'(x) = -\frac{e^{-x}}{1+x} - \frac{e^{-x}}{(1+x)^2} \implies f''(x) = \frac{e^{-x}}{1+x} + 2\frac{e^{-x}}{(1+x)^2} + 2\frac{e^{-x}}{(1+x)^3}$$

Since $f''(x)$ is positive and decreases as x increases

$$|f''(x)| \leqslant f''(0) = 5 \implies |E_n| \leqslant \frac{5(10-0)^3}{24n^2} = \frac{5000}{24n^2} = \frac{625}{3n^2}$$

and $|E_n| \leqslant \frac{1}{2}10^{-4}$ if

$$\frac{625}{3n^2} \leqslant \frac{1}{2}10^{-4} \iff n^2 \geqslant \frac{1250\times 10^4}{3} \iff n \geqslant \sqrt{\frac{1.25\times 10^7}{3}} = 2041.2$$

So $t = 10$ and $n = 2042$ will do the job. There are many other correct answers.

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S-1: Substitute $u = x^2 - 3$, $\mathrm{d}u = 2x\,\mathrm{d}x$. This gives

$$\int \frac{x}{x^2-3}\mathrm{d}x = \int \frac{\mathrm{d}u/2}{u} = \frac{1}{2}\log|u| + C = \frac{1}{2}\log\left|x^2-3\right| + C$$

S-2: (a) Substituting $y = 9 + x^2$, $\mathrm{d}y = 2x\mathrm{d}x$, $x\mathrm{d}x = \frac{\mathrm{d}y}{2}$, $y(0) = 9$, $y(4) = 25$

$$\begin{aligned}\int_0^4 \frac{x}{\sqrt{9+x^2}}\,\mathrm{d}x &= \int_9^{25} \frac{1}{\sqrt{y}}\,\frac{\mathrm{d}y}{2} = \frac{1}{2}\frac{\sqrt{y}}{1/2}\Big|_9^{25} = 5-3 \\ &= 2\end{aligned}$$

(b) Substituting $y = \sin x$, $\mathrm{d}y = \cos x\,\mathrm{d}x$, $y(0) = 0$, $y(\pi/2) = 1$, $\cos^2 x = 1 - y^2$

$$\begin{aligned}\int_0^{\pi/2} \cos^3 x\ \sin^2 x\,\mathrm{d}x &= \int_0^{\pi/2} \cos^2 x\ \sin^2 x\ \cos x\,\mathrm{d}x = \int_0^1 (1-y^2)y^2\,\mathrm{d}y = \int_0^1 (y^2 - y^4)\,\mathrm{d}y \\ &= \left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = \frac{1}{3} - \frac{1}{5} \\ &= \frac{2}{15}\end{aligned}$$

(c) Integrate by parts with $u(x) = \log x$ and $\mathrm{d}v = x^3\,\mathrm{d}x$ so that $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and $v = x^4/4$. Then

$$\begin{aligned}\int_1^e x^3 \log x\,\mathrm{d}x &= \frac{x^4}{4}\log x\Big|_1^e - \int_1^e \frac{x^4}{4}\frac{1}{x}\,\mathrm{d}x = \frac{e^4}{4} - \int_1^e \frac{x^3}{4}\,\mathrm{d}x = \frac{e^4}{4} - \frac{x^4}{16}\Big|_1^e \\ &= \frac{3e^4}{16} + \frac{1}{16}\end{aligned}$$

S-3: (a) Integrate by parts with $u = x$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$ so that $\mathrm{d}u = \mathrm{d}x$ and $v = -\cos x$. Then

$$\int x\sin x\,\mathrm{d}x = -x\cos x - \int(-\cos x)\,\mathrm{d}x = -x\cos x + \sin x + C$$

so that

$$\int_0^{\pi/2} x\sin x\,\mathrm{d}x = \Big[-x\cos x + \sin x\Big]_0^{\pi/2} = 1$$

(b) Make the substitution $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$.

$$\begin{aligned}\int_0^{\pi/2} \cos^5 x\,\mathrm{d}x &= \int_0^{\pi/2} \left(1 - \sin^2 x\right)^2 \cos x\,\mathrm{d}x = \int_0^1 \left(1-u^2\right)^2\mathrm{d}u = \int_0^1 \left(1 - 2u^2 + u^4\right)\mathrm{d}u \\ &= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}\end{aligned}$$

S-4: (a) By integration by parts, with $u = x$ and $\mathrm{d}v = e^x\,\mathrm{d}x$, so that $\mathrm{d}u = \mathrm{d}x$ and $v = e^x$,

$$\int_0^2 xe^x\,\mathrm{d}x = xe^x\Big|_0^2 - \int_0^2 e^x\,\mathrm{d}x = 2e^2 - e^x\Big|_0^2 = e^2 + 1$$

(b) Substitute $x = \tan y$, $\mathrm{d}x = \sec^2 y\,\mathrm{d}y$. When $x = 0$, $\tan y = 0$ so $y = 0$. When $x = 1$, $\tan y = 1$ so $y = \frac{\pi}{4}$. Also $\sqrt{1+x^2} = \sqrt{1+\tan^2 y} = \sqrt{\sec^2 y} = \sec y$, since $\sec y \geqslant 0$ for all $0 \leqslant y \leqslant \frac{\pi}{4}$.

$$\int_0^1 \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \int_0^{\pi/4} \frac{\sec^2 y\,\mathrm{d}y}{\sec y} = \int_0^{\pi/4} \sec y\,\mathrm{d}y = \log|\sec y + \tan y|\Big|_0^{\pi/4}$$

Since $\sec\frac{\pi}{4} = \sqrt{2},\ \tan\frac{\pi}{4} = 1, \sec 0 = 1,\ \tan 0 = 0$, the top evaluation $\log|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}| = \log(\sqrt{2}+1)$, the bottom evaluation $\log|\sec 0 + \tan 0| = \log 1 = 0$ and

$$\int_0^1 \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \log(\sqrt{2}+1)$$

(c) We use partial fractions.

$$\begin{aligned}
\frac{4x}{(x^2-1)(x^2+1)} &= \frac{4x}{(x-1)(x+1)(x^2+1)} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+1} \\
&= \frac{a(x+1)(x^2+1) + b(x-1)(x^2+1) + (cx+d)(x-1)(x+1)}{(x-1)(x+1)(x^2+1)}
\end{aligned}$$

provided $a(x+1)(x^2+1) + b(x-1)(x^2+1) + (cx+d)(x-1)(x+1) = 4x$. Setting $x = 1$ gives $4a = 4$ or $a = 1$. Setting $x = -1$ gives $-4b = -4$ or $b = 1$. Substituting in $a = b = 1$ gives

$$\begin{aligned}
&(x+1)(x^2+1) + (x-1)(x^2+1) + (cx+d)(x-1)(x+1) = 4x \\
&\iff 2x(x^2+1) + (cx+d)(x-1)(x+1) = 4x \\
&\iff (cx+d)(x^2-1) = -2x^3 + 2x = -2x(x^2-1) \\
&\iff (cx+d) = -2x \iff c = -2,\ d = 0
\end{aligned}$$

So

$$\begin{aligned}
\int_3^5 \frac{4x}{(x^2-1)(x^2+1)}\,\mathrm{d}x &= \int_3^5 \left[\frac{1}{x-1} + \frac{1}{x+1} - \frac{2x}{x^2+1}\right]\mathrm{d}x \\
&= \Big[\log|x-1| + \log|x+1| - \log(x^2+1)\Big]_3^5 \\
&= \log 4 + \log 6 - \log 26 - \log 2 - \log 4 + \log 10 \\
&= \log\frac{6\times 10}{26\times 2} = \log\frac{15}{13} \approx 0.1431
\end{aligned}$$

S-5: (a) Make the substitution $y = \sin x$, $\mathrm{d}y = \cos x\,\mathrm{d}x$ and use the trig identity $\cos^2 x = 1 - \sin^2 x = 1 - y^2$.

$$\begin{aligned}\int_0^{\pi/2} \cos^5(x)\,\mathrm{d}x &= \int_0^{\pi/2} \cos^4(x)\ \cos(x)\,\mathrm{d}x = \int_0^1 (1-y^2)^2\,\mathrm{d}y = \int_0^1 \left[y^4 - 2y^2 + 1\right]\mathrm{d}y \\ &= \left[\frac{y^5}{5} - 2\frac{y^3}{3} + y\right]_0^1 = \frac{1}{5} - \frac{2}{3} + 1 = \frac{8}{15} \approx 0.53333\end{aligned}$$

(b) $\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x$ is the area of the portion of the disk $x^2 + y^2 \leqslant 9$ that lies in the first quadrant. It is $\frac{1}{4}\pi 3^3 = \boxed{\frac{9}{4}\pi}$. Alternatively, you could also evaluate this integral using the substitution $x = 3\sin y$, $\mathrm{d}x = 3\cos y\,\mathrm{d}y$.

$$\begin{aligned}\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x &= \int_0^{\pi/2} \sqrt{9 - 9\sin^2 y}\ (3\cos y)\,\mathrm{d}y = 9\int_0^{\pi/2} \cos^2 y\,\mathrm{d}y \\ &= \frac{9}{2}\int_0^{\pi/2} [1 + \cos(2y)]\,\mathrm{d}y = \frac{9}{2}\left[y + \frac{\sin(2y)}{2}\right]_0^{\pi/2} \\ &= \frac{9}{4}\pi\end{aligned}$$

(c) Integrate by parts, using $u = \log(1+x^2)$ and $\mathrm{d}v = \mathrm{d}x$, so that $\mathrm{d}u = \frac{2x}{1+x^2}$, $v = x$.

$$\begin{aligned}\int_0^1 \log(1+x^2)\,\mathrm{d}x &= x\log(1+x^2)\Big|_0^1 - \int_0^1 x\frac{2x}{1+x^2}\,\mathrm{d}x = \log 2 - 2\int_0^1 \frac{x^2}{1+x^2}\,\mathrm{d}x \\ &= \log 2 - 2\int_0^1 \left[1 - \frac{1}{1+x^2}\right]\mathrm{d}x = \log 2 - 2\big[x - \arctan x\big]_0^1 \\ &= \log 2 - 2 + \frac{\pi}{2} \approx 0.264\end{aligned}$$

(d) Use partial fractions.

$$\begin{aligned}\frac{x}{(x-1)^2(x-2)} = \frac{a}{(x-1)^2} + \frac{b}{x-1} + \frac{c}{x-2} &= \frac{a(x-2) + b(x-1)(x-2) + c(x-1)^2}{(x-1)^2(x-2)} \\ &\iff a(x-2) + b(x-1)(x-2) + c(x-1)^2 = x\end{aligned}$$

Setting $x = 1$ gives $-a = 1$. Setting $x = 2$ gives $c = 2$. Substituting in $a = -1$ and $c = 2$ gives

$$\begin{aligned}&b(x-1)(x-2) = x + (x-2) - 2(x-1)^2 = -2x^2 + 6x - 4 = -2(x-1)(x-2) \\ &\implies b = -2\end{aligned}$$

Hence

$$\begin{aligned}
\int_3^\infty \frac{x}{(x-1)^2(x-2)}\,\mathrm{d}x &= \lim_{M\to\infty}\int_3^M \Big[-\frac{1}{(x-1)^2}-\frac{2}{x-1}+\frac{2}{x-2}\Big]\,\mathrm{d}x \\
&= \lim_{M\to\infty}\Big[\frac{1}{x-1}-2\log|x-1|+2\log|x-2|\Big]_3^M \\
&= \lim_{M\to\infty}\Big[\frac{1}{x-1}+2\log\Big|\frac{x-2}{x-1}\Big|\Big]_3^M \\
&= \lim_{M\to\infty}\Big[\frac{1}{M-1}+2\log\Big|\frac{M-2}{M-1}\Big|\Big]-\Big[\frac{1}{3-1}+2\log\Big|\frac{3-2}{3-1}\Big|\Big] \\
&= 2\log 2-\frac{1}{2}\approx 0.886
\end{aligned}$$

since

$$\lim_{M\to\infty}\log\frac{M-2}{M-1}=\lim_{M\to\infty}\log\frac{1-2/M}{1-1/M}=\log 1=0$$

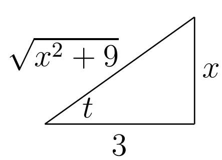
S-6: (a) Make the substitution $u=\sin(2x)$, $\mathrm{d}u=2\cos(2x)\,\mathrm{d}x$.

$$\begin{aligned}
\int_0^{\frac{\pi}{4}}\sin^2(2x)\cos^3(2x)\,\mathrm{d}x &= \int_0^{\frac{\pi}{4}}\sin^2(2x)\big[1-\sin^2(2x)\big]\cos(2x)\,\mathrm{d}x=\frac{1}{2}\int_0^1 u^2\big[1-u^2\big]\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^1\big(u^2-u^4\big)\,\mathrm{d}u=\frac{1}{2}\Big[\frac{1}{3}u^3-\frac{1}{5}u^5\Big]_0^1=\frac{1}{15}
\end{aligned}$$

(b) Make the substitution $x=3\tan t$, $\mathrm{d}x=3\sec^2 t\,\mathrm{d}t$ and use the trig identity $9+9\tan^2 t=9\sec^2 t$.

$$\begin{aligned}
\int\big(9+x^2\big)^{-\frac{3}{2}}\,\mathrm{d}x &= \int\big(9+9\tan^2 t\big)^{-\frac{3}{2}}\,3\sec^2 t\,\mathrm{d}t=\int\big(3\sec t\big)^{-3}\,3\sec^2 t\,\mathrm{d}t \\
&= \frac{1}{9}\int\cos t\,\mathrm{d}t=\frac{1}{9}\sin t+C=\frac{1}{9}\frac{x}{\sqrt{x^2+9}}+C
\end{aligned}$$

To convert back to x, in the last step, we used the triangle below, which is rigged to have $\tan t=\frac{x}{3}$.



(c) We use partial fractions.

$$\frac{1}{(x-1)(x^2+1)}=\frac{a}{x-1}+\frac{bx+c}{x^2+1}=\frac{a(x^2+1)+(bx+c)(x-1)}{(x-1)(x^2+1)}$$

provided $a(x^2+1)+(bx+c)(x-1)=1$ for all x. Setting $x=1$ gives $2a=1$ or $a=\frac{1}{2}$. Substituting in $a=\frac{1}{2}$ gives

$$\begin{aligned}
&\frac{1}{2}(x^2+1)+(bx+c)(x-1)=1\\
&\iff (bx+c)(x-1)=\frac{1}{2}(1-x^2)=-\frac{1}{2}(x-1)(x+1)\\
&\iff (bx+c)=-\frac{1}{2}(x+1) \iff b=c=-\frac{1}{2}
\end{aligned}$$

So

$$\begin{aligned}
\int \frac{\mathrm{d}x}{(x-1)(x^2+1)} &= \int \Big[\frac{1}{2}\frac{1}{x-1}-\frac{1}{2}\frac{x+1}{x^2+1}\Big]\,\mathrm{d}x = \int \Big[\frac{1}{2}\frac{1}{x-1}-\frac{1}{4}\frac{2x}{x^2+1}-\frac{1}{2}\frac{1}{x^2+1}\Big]\,\mathrm{d}x\\
&= \frac{1}{2}\log|x-1|-\frac{1}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x+C
\end{aligned}$$

(d) Integrate by parts with $u=\tan^{-1}x$ and $\mathrm{d}v=x\,\mathrm{d}x$ so that $\mathrm{d}u=\frac{1}{1+x^2}\mathrm{d}x$ and $v=\frac{1}{2}x^2$. Then

$$\begin{aligned}
\int x\tan^{-1}x\,\mathrm{d}x &= \frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}\,\mathrm{d}x\\
&= \frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int\frac{1+x^2}{1+x^2}\,\mathrm{d}x+\frac{1}{2}\int\frac{1}{1+x^2}\,\mathrm{d}x\\
&= \frac{1}{2}\big[x^2\tan^{-1}x-x+\tan^{-1}x\big]+C
\end{aligned}$$

S-7: (a) Substituting $y=\sin(2x)$, $\mathrm{d}y=2\cos(2x)\,\mathrm{d}x$, $y(x=0)=0$ and $y\big(x=\frac{\pi}{4}\big)=1$,

$$\int_0^{\pi/4}\sin^5(2x)\,\cos(2x)\,\mathrm{d}x=\int_0^1 y^5\,\frac{\mathrm{d}y}{2}=\frac{1}{2}\frac{1}{6}y^6\Big|_0^1=\frac{1}{12}$$

(b) Substituting $x=2\sin y$, $\mathrm{d}x=2\cos y\,\mathrm{d}y$,

$$\begin{aligned}
\int\sqrt{4-x^2}\,\mathrm{d}x &= \int\sqrt{4-4\sin^2 y}\,2\cos y\,\mathrm{d}y=4\int\cos^2 y\,\mathrm{d}y=2\int\big[1+\cos(2y)\big]\,\mathrm{d}y\\
&= 2y+\sin(2y)+C=2y+2\sin y\cos y+C\\
&= 2\sin^{-1}\frac{x}{2}+x\sqrt{1-\frac{x^2}{4}}+C
\end{aligned}$$

since $\sin y=\frac{x}{2}$ and $\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\frac{x^2}{4}}$.

(c) Integrate by parts, using $u=\log(1+x^2)$, $\mathrm{d}v=\mathrm{d}x$, $v=x$ and $du=\frac{2x}{1+x^2}\,\mathrm{d}x$

$$\begin{aligned}
\int\log(1+x^2)\,\mathrm{d}x &= x\log(1+x^2)-\int\frac{2x^2}{1+x^2}\mathrm{d}x=x\log(1+x^2)-\int\Big[2-\frac{2}{1+x^2}\Big]\mathrm{d}x\\
&= x\log(1+x^2)-2x+2\tan^{-1}x+C
\end{aligned}$$

(d) The partial fractions expansion

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax(x-1)+B(x-1)+Cx^2}{x^2(x-1)}$$

is true provided

$$Ax(x-1)+B(x-1)+Cx^2 = x+1$$

for all x. Setting $x=1$, gives the requirement $C=2$. Setting $x=0$, gives the requirement $B=-1$. As well, the net coefficient of x^2 on the left hand side, namely $A+C$, must be the same as the net coefficient of x^2 on the right hand side, namely 0. So $A+C=0$ and $A=-2$. Checking,

$$-2x(x-1)-(x-1)+2x^2 = -2x^2+2x-x+1+2x^2 = x+1$$

as desired. Thus

$$\int \frac{x+1}{x^2(x-1)}\,\mathrm{d}x = \int \Big[-\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1}\Big]\,\mathrm{d}x = -2\log|x| + \frac{1}{x} + 2\log|x-1| + C$$

S-8: (a) Define

$$I_1 = \int_0^\infty e^{-x}\sin(2x)\,\mathrm{d}x \qquad I_2 = \int_0^\infty e^{-x}\cos(2x)\,\mathrm{d}x$$

Integrating by parts, with $u=\sin(2x)$ or $\cos(2x)$ and $\mathrm{d}v = e^{-x}\,\mathrm{d}x$. That is, $v=-e^{-x}$.

$$\begin{aligned}
I_1 = \int_0^\infty e^{-x}\sin(2x)\,\mathrm{d}x &= \lim_{R\to\infty}\int_0^R e^{-x}\sin(2x)\,\mathrm{d}x \\
&= \lim_{R\to\infty}\Big[-e^{-x}\sin(2x)\Big|_0^R + 2\int_0^R e^{-x}\cos(2x)\,\mathrm{d}x\Big] = 2I_2 \\
I_2 = \int_0^\infty e^{-x}\cos(2x)\,\mathrm{d}x &= \lim_{R\to\infty}\int_0^R e^{-x}\cos(2x)\,\mathrm{d}x \\
&= \lim_{R\to\infty}\Big[-e^{-x}\cos(2x)\Big|_0^\infty - 2\int_0^\infty e^{-x}\sin(2x)\,\mathrm{d}x\Big] = 1-2I_1
\end{aligned}$$

Substituting $I_2 = \frac{1}{2}I_1$ into $I_2 = 1-2I_1$ gives $\frac{5}{2}I_1 = 1$ or $\int_0^\infty e^{-x}\sin(2x)\,\mathrm{d}x = \frac{2}{5}$.

(b) Subsitute $x=\sqrt{2}\tan y$, $\mathrm{d}x = \sqrt{2}\sec^2 y\,\mathrm{d}y$.

$$\int_0^{\sqrt{2}} \frac{1}{(2+x^2)^{3/2}}\,\mathrm{d}x = \sqrt{2}\int_0^{\pi/4}\frac{\sec^2 y}{(2+2\tan^2 y)^{3/2}}\,\mathrm{d}y = \frac{1}{2}\int_0^{\pi/4}\cos y\,\mathrm{d}y = \frac{1}{2}\sin y\Big|_0^{\pi/4} = \frac{1}{2\sqrt{2}}$$

(c) Integrate by parts, using $u=\log(1+x^2)$ and $\mathrm{d}v = x\,\mathrm{d}x$, so that $\mathrm{d}u = \frac{2x}{1+x^2}$, $v=\frac{x^2}{2}$.

$$\begin{aligned}
\int_0^1 x\log(1+x^2)\,\mathrm{d}x &= \frac{1}{2}x^2\log(1+x^2)\Big|_0^1 - \int_0^1 \frac{x^3}{1+x^2}\,\mathrm{d}y = \frac{1}{2}\log 2 - \int_0^1\Big[x - \frac{x}{1+x^2}\Big]\,\mathrm{d}x \\
&= \frac{1}{2}\log 2 - \Big[\frac{x^2}{2} - \frac{1}{2}\log(1+x^2)\Big]_0^1 = \log 2 - \frac{1}{2} \approx 0.193
\end{aligned}$$

(c) (Second solution) First substitute $y = 1 + x^2$, $\mathrm{d}y = 2x\,\mathrm{d}x$.

$$\int_0^1 x\log(1+x^2)\,\mathrm{d}x = \frac{1}{2}\int_1^2 \log y\,\mathrm{d}y$$

Then integrate by parts, using $u = \log y$ and $\mathrm{d}v = \mathrm{d}y$, so that $\mathrm{d}u = \frac{1}{y}$, $v = y$.

$$\int_0^1 x\log(1+x^2)\,\mathrm{d}x = \frac{1}{2}\int_1^2 \log y\,\mathrm{d}y = \frac{1}{2}y\log y\Big|_1^2 - \frac{1}{2}\int_1^2 y\frac{1}{y}\,\mathrm{d}y = \log 2 - \frac{1}{2} \approx 0.193$$

(d) Use partial fractions.

$$\frac{1}{(x-1)^2(x-2)} = \frac{a}{(x-1)^2} + \frac{b}{x-1} + \frac{c}{x-2} = \frac{a(x-2) + b(x-1)(x-2) + c(x-1)^2}{(x-1)^2(x-2)}$$
$$\iff a(x-2) + b(x-1)(x-2) + c(x-1)^2 = 1$$

Setting $x = 1$ gives $-a = 1$. Setting $x = 2$ gives $c = 1$. Substituting in $a = -1$ and $c = 1$ gives

$$b(x-1)(x-2) = 1 + (x-2) - (x-1)^2 = -x^2 + 3x - 2 = -(x-1)(x-2) \implies b = -1$$

Hence

$$\begin{aligned}
\int_3^\infty \frac{x}{(x-1)^2(x-2)}\,\mathrm{d}x &= \lim_{M\to\infty}\int_3^M \Big[-\frac{1}{(x-1)^2} - \frac{1}{x-1} + \frac{1}{x-2}\Big]\,\mathrm{d}x \\
&= \lim_{M\to\infty}\Big[\frac{1}{x-1} - \log(x-1) + \log(x-2)\Big]_3^M \\
&= \lim_{M\to\infty}\Big[\frac{1}{M-1} + \log\frac{M-2}{M-1}\Big] - \Big[\frac{1}{3-1} + \log\frac{3-2}{3-1}\Big] \\
&= \log 2 - \frac{1}{2} \approx 0.193
\end{aligned}$$

since

$$\lim_{M\to\infty}\log\frac{M-2}{M-1} = \lim_{M\to\infty}\log\frac{1-2/M}{1-1/M} = \log 1 = 0$$

S-9: (a) Integrate by parts with $u = \log x$ and $\mathrm{d}v = x\,\mathrm{d}x$, so that $\mathrm{d}u = \frac{\mathrm{d}x}{x}$ and $v = \frac{1}{2}x^2$.

$$\int x\,\log x\,\mathrm{d}x = \frac{1}{2}x^2\log x - \frac{1}{2}\int x^2\frac{1}{x}\mathrm{d}x = \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + C$$

(b)

$$\begin{aligned}
\int \frac{(x-1)\mathrm{d}x}{x^2+4x+5} &= \int \frac{x-1}{(x+2)^2+1}\mathrm{d}x = \int \frac{x+2}{(x+2)^2+1}\mathrm{d}x - 3\int \frac{1}{(x+2)^2+1}\mathrm{d}x \\
&= \frac{1}{2}\log[(x+2)^2+1] - 3\arctan(x+2) + C
\end{aligned}$$

(c) We use partial fractions.

$$\frac{1}{x^2-4x+3}=\frac{1}{(x-3)(x-1)}=\frac{a}{x-3}+\frac{b}{x-1}=\frac{a(x-1)+b(x-3)}{(x-3)(x-1)}$$

provided $a(x-1)+b(x-3)=1$. Setting $x=3$ gives $a=\frac{1}{2}$. Setting $x=1$ gives $b=-\frac{1}{2}$. So

$$\int\frac{\mathrm{d}x}{x^2-4x+3}=\int\Big[\frac{1}{2}\frac{1}{x-3}-\frac{1}{2}\frac{1}{x-1}\Big]\,\mathrm{d}x=\frac{1}{2}\log|x-3|-\frac{1}{2}\log|x-1|+C$$

(d) Substitute $y=x^3$, $\mathrm{d}y=3x^2\,\mathrm{d}x$.

$$\int\frac{x^2\,\mathrm{d}x}{1+x^6}=\frac{1}{3}\int\frac{\mathrm{d}y}{1+y^2}=\frac{1}{3}\arctan y+C=\frac{1}{3}\arctan x^3+C$$

S-10: (a) Integrate by parts with $u=\tan^{-1}x$, $\mathrm{d}v=\mathrm{d}x$, $\mathrm{d}u=\frac{\mathrm{d}x}{1+x^2}$ and $v=x$. This gives

$$\int_0^1\tan^{-1}x\,\mathrm{d}x=x\tan^{-1}x\Big|_0^1-\int_0^1\frac{x}{1+x^2}\mathrm{d}x=\tan^{-1}1-\Big[\frac{1}{2}\log(1+x^2)\Big]_0^1=\frac{\pi}{4}-\frac{1}{2}\log 2$$

(b) Note that the derivative of the denominator is $2x-2$, which differs from the numerator only by 1. So

$$\begin{aligned}\int\frac{2x-1}{x^2-2x+5}\,\mathrm{d}x&=\int\frac{2x-2}{x^2-2x+5}\,\mathrm{d}x+\int\frac{1}{x^2-2x+5}\,\mathrm{d}x\\&=\int\frac{2x-2}{x^2-2x+5}\,\mathrm{d}x+\int\frac{1}{(x-1)^2+4}\,\mathrm{d}x\\&=\log|x^2-2x+5|+\frac{1}{2}\tan^{-1}\frac{x-1}{2}+C\end{aligned}$$

S-11: (a) Integrating by parts with $u=\log x$, $\mathrm{d}v=x\,\mathrm{d}x$, $\mathrm{d}u=\frac{\mathrm{d}x}{x}$ and $v=\frac{x^2}{2}$

$$\int x\,\log x\,\mathrm{d}x=uv-\int v\,\mathrm{d}u=\frac{x^2}{2}\log x-\int\frac{x}{2}\,\mathrm{d}x=\frac{x^2}{2}\log x-\frac{x^2}{4}+C$$

(b) Substituting $u=x^3+1$, $\mathrm{d}u=3x^2\,\mathrm{d}x$

$$\int\frac{x^2}{(x^3+1)^{101}}\mathrm{d}x=\int\frac{1}{u^{101}}\frac{\mathrm{d}u}{3}=\frac{u^{-100}}{-100}\frac{1}{3}+C=-\frac{1}{300(x^3+1)^{100}}+C$$

(c) Substituting $\mathrm{d}u=\cos x\,\mathrm{d}x$, $u=\sin x$, $\cos^2x=1-\sin^2x=1-u^2$,

$$\begin{aligned}\int\cos^3x\,\sin^4x\,\mathrm{d}x&=\int\cos^2x\,\sin^4x\,\cos x\,\mathrm{d}x=\int(1-u^2)u^4\,\mathrm{d}u=\frac{u^5}{5}-\frac{u^7}{7}+C\\&=\frac{\sin^5x}{5}-\frac{\sin^7x}{7}+C\end{aligned}$$

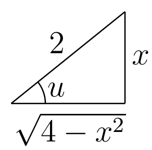
(d) Substituting $x = 2\sin u$, $\mathrm{d}x = 2\cos u\,\mathrm{d}u$,

$$\int \sqrt{4-x^2}\,\mathrm{d}x = \int \sqrt{4-4\sin^2 u}\,2\cos u\,\mathrm{d}u = 4\int \cos^2 u\,\mathrm{d}u = 4\int \frac{1+\cos(2u)}{2}\mathrm{d}u$$
$$= 2\Big[u + \frac{1}{2}\sin(2u)\Big] + C = 2\Big[u + \sin u\cos u\Big] + C$$

We still need to express our answer as a function of x. Recall that $\sin u = \frac{x}{2}$. So $u = \arcsin\frac{x}{2}$. Our answer also contains $\cos u$, so we also need to express it as a function of x. We can do this either by using the trig identity

$$\cos u = \sqrt{1-\sin^2 u} = \sqrt{1-\frac{x^2}{4}}$$

or by reading $\cos u$ off of the triangle below, which has u as an angle and whose sides have been chosen so that $\sin u = \frac{x}{2}$.



So

$$\int \sqrt{4-x^2}\,\mathrm{d}x = 2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + C$$

S-12: (a) If the integrand had x's instead of e^x's it would be a rational function, ripe for the application of partial fractions. So let's start by making the substitution $u = e^x$, $\mathrm{d}u = e^x\,\mathrm{d}x$:

$$\int \frac{e^x}{(e^x+1)(e^x-3)}\mathrm{d}x = \int \frac{\mathrm{d}u}{(u+1)(u-3)}$$

Now, we follow the partial fractions protocol, starting with expressing

$$\frac{1}{(u+1)(u-3)} = \frac{A}{u+1} + \frac{B}{u-3}$$

To find A and B, the sneaky way, we cross multiply by the denominator

$$1 = A(u-3) + B(u+1)$$

and find A and B by evaluating at $u = -1$ and $u = 3$, respectively.

$$1 = A(-1-3) + B(-1+1) \iff A = -\frac{1}{4}$$
$$1 = A(3-3) + B(3+1) \iff B = \frac{1}{4}$$

Finally, we can do the integral:

$$\int \frac{e^x}{(e^x+1)(e^x-3)}\mathrm{d}x = \int \frac{\mathrm{d}u}{(u+1)(u-3)} = \int \left[\frac{-1/4}{u+1} + \frac{1/4}{u-3}\right]\mathrm{d}u$$
$$= -\frac{1}{4}\log|u+1| + \frac{1}{4}\log|u-3| + C$$
$$= -\frac{1}{4}\log|e^x+1| + \frac{1}{4}\log|e^x-3| + C$$

(b) The argument of the square root is

$$12 + 4x - x^2 = 12 - (x-2)^2 + 4 = 16 - (x-2)^2$$

Hmmm. The numerator is $x^2 - 4x + 4 = (x-2)^2$. So let's make the integral look somewhat simpler by substituting $u = x - 2$, $\mathrm{d}u = \mathrm{d}x$. When $x = 2$ we have $u = 0$, and when $x = 4$ we have $u = 2$ so

$$\int_{x=2}^{x=4} \frac{x^2-4x+4}{\sqrt{12+4x-x^2}}\mathrm{d}x = \int_{u=0}^{u=2} \frac{u^2}{\sqrt{16-u^2}}\mathrm{d}u$$

This is perfect for the trig substitution $u = 4\sin\theta$, $\mathrm{d}u = 4\cos(\theta)\,\mathrm{d}\theta$. When $u = 0$ we have $4\sin\theta = 0$ and hence $\theta = 0$. When $u = 2$ we have $4\sin\theta = 2$ and hence $\theta = \frac{\pi}{6}$. So

$$\int_{u=0}^{u=2} \frac{u^2}{\sqrt{16-u^2}}\mathrm{d}u = \int_{\theta=0}^{\theta=\pi/6} \frac{16\sin^2\theta}{\sqrt{16-16\sin^2\theta}} 4\cos\theta\mathrm{d}\theta$$
$$= 16\int_0^{\pi/6} \sin^2\theta\;\mathrm{d}\theta$$
$$= 8\int_0^{\pi/6} \big[1-\cos(2\theta)\big]\;\mathrm{d}\theta$$
$$= 8\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi/6} = 8\left[\frac{\pi}{6} - \frac{1}{2}\frac{\sqrt{3}}{2}\right]$$
$$= \frac{4\pi}{3} - 2\sqrt{3}$$

S-13: (a) Substituting $y = \cos x$, $\mathrm{d}y = -\sin x\,\mathrm{d}x$, $\sin^2 x = 1 - \cos^2 x = 1 - y^2$

$$\int \frac{\sin^3 x}{\cos^3 x}\,\mathrm{d}x = \int \frac{\sin^2 x}{\cos^3 x}\,\sin x\,\mathrm{d}x = \int \frac{1-y^2}{y^3}\,(-\mathrm{d}y) = -\int \left(y^{-3} - y^{-1}\right)\,\mathrm{d}y$$
$$= -\frac{y^{-2}}{-2} + \log|y| + C = \frac{1}{2}\sec^2 x + \log|\cos x| + C$$

(b) Substituting $x^5 = 4y$, $5x^4\,\mathrm{d}x = 4\,\mathrm{d}y$, and using that $x = 2 \implies 2^5 = 4y \implies y = 8$,

$$\int_{-2}^{2} \frac{x^4}{x^{10}+16}\,\mathrm{d}x = 2\int_0^2 \frac{x^4}{x^{10}+16}\,\mathrm{d}x = 2\frac{4}{5}\int_0^8 \frac{1}{16y^2+16}\,\mathrm{d}y = \frac{1}{10}\int_0^8 \frac{1}{y^2+1}\,\mathrm{d}y$$
$$= \frac{1}{10}\tan^{-1} 8 \approx 0.1446$$

(c) Integrate by parts, using $u = \log(1+x^2)$, $\mathrm{d}v = \mathrm{d}x$, $v = x$ and $\mathrm{d}u = \frac{2x}{1+x^2}\,\mathrm{d}x$

$$\begin{aligned}\int_0^1 \log(1+x^2)\,\mathrm{d}x &= x\log(1+x^2)\Big|_0^1 - \int_0^1 \frac{2x^2}{1+x^2}\,\mathrm{d}x \\ &= \log 2 - \int_0^1 \Big[2 - \frac{2}{1+x^2}\Big]\,\mathrm{d}x \\ &= \log 2 - \Big[2x - 2\tan^{-1} x\Big]_0^1 \\ &= \log 2 - 2 + \frac{\pi}{2} \approx 0.2639\end{aligned}$$

S-14: (a) Split the specified integral into

$$\int_0^3 (x+1)\sqrt{9-x^2}\,\mathrm{d}x = \int_0^3 \sqrt{9-x^2}\,\mathrm{d}x + \int_0^3 x\sqrt{9-x^2}\,\mathrm{d}x$$

The first piece represents the area above the x–axis and below the curve $y = \sqrt{9-x^2}$, i.e. $x^2 + y^2 = 9$, with $0 \leqslant x \leqslant 3$. That's the area of one quadrant of a disk of radius 3. So

$$\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x = \frac{1}{4}\pi 3^2 = \frac{9}{4}\pi$$

For the second part we substitute $u = 9 - x^2$, $\mathrm{d}u = -2x\,\mathrm{d}x$, $u(x=0) = 9$ and $u(x=3) = 0$. So

$$\int_0^3 x\sqrt{9-x^2}\,\mathrm{d}x = \int_9^0 \sqrt{u}\,\frac{\mathrm{d}u}{-2} = -\frac{1}{2}\Big[\frac{u^{3/2}}{3/2}\Big]_9^0 = -\frac{1}{2}\Big[-\frac{27}{3/2}\Big] = 9$$

All together

$$\int_0^3 (x+1)\sqrt{9-x^2}\,\mathrm{d}x = \frac{9}{4}\pi + 9$$

(b) The integrand is of the form $N(x)/D(x)$ with $D(x)$ already factored and $N(x)$ of lower degree. We immediately look for a partial fractions decomposition:

$$\frac{4x+8}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}.$$

Multiplying through by the denominator yields

$$4x + 8 = A(x^2+4) + (Bx+C)(x-2) \tag{$*$}$$

Setting $x = 2$ we find:

$$8 + 8 = A(4+4) + 0 \implies 16 = 8A \implies A = 2$$

Substituting $A = 2$ in $(*)$ gives

$$\begin{aligned}
4x + 8 &= A(x^2+4) + (Bx+C)(x-2) \\
&\implies -2x^2 + 4x = (x-2)(Bx+C) \\
&\implies (-2x)(x-2) = (Bx+C)(x-2) \\
&\implies B = -2,\ C = 0
\end{aligned}$$

So we have found that $A = 2$, $B = -2$, and $C = 0$. Therefore

$$\begin{aligned}
\int \frac{4x+8}{(x-2)(x^2+4)}\, \mathrm{d}x &= \int \left(\frac{2}{x-2} - \frac{2x}{x^2+4} \right) dx \\
&= 2\log|x-2| - \log(x^2+4) + C
\end{aligned}$$

Here the second integral was found just by guessing an antiderivative. Alternatively, one could use the substitution $u = x^2+4$, $\mathrm{d}u = 2x\,\mathrm{d}x$.

(c) The given integral is improper and so is

$$\int_{-\infty}^{+\infty} \frac{1}{e^x + e^{-x}}\, \mathrm{d}x = \lim_{R,R'\to\infty} \int_{-R}^{R'} \frac{1}{e^x+e^{-x}}\, \mathrm{d}x = \lim_{R,R'\to\infty} \int_{-R}^{R'} \frac{e^x\, \mathrm{d}x}{e^{2x}+1}$$

We substitute $u = e^x$, $\mathrm{d}u = e^x\,\mathrm{d}x$, giving

$$\int_{-\infty}^{+\infty} \frac{1}{e^x+e^{-x}}\, \mathrm{d}x = \lim_{R,R'\to\infty} \int_{e^{-R}}^{e^{R'}} \frac{\mathrm{d}u}{u^2+1} = \lim_{R,R'\to\infty} \big[\arctan u\big]_{e^{-R}}^{e^{R'}} = \frac{\pi}{2}$$

S-15: (a) Substituting $u = \log x$, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$, $\mathrm{d}x = x\,\mathrm{d}u = e^u\,\mathrm{d}u$

$$\int \sin(\log x)\ \mathrm{d}x = \int \sin(u)\ e^u\ \mathrm{d}u$$

We have already seen, in Example 1.7.11 of the CLP 101 notes, that

$$\int \sin(u)\ e^u\ \mathrm{d}u = \frac{1}{2}e^u\big(\sin u - \cos u\big) + C$$

So

$$\int \sin(\log x)\ \mathrm{d}x = \frac{1}{2}x\big[\sin(\log x) - \cos(\log x)\big] + C$$

(b) The integrand is of the form $N(x)/D(x)$ with $N(x)$ of lower degree than $D(x)$. So we factor $D(x) = (x-2)(x-3)$ and look for a partial fractions decomposition:

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Multiplying through by the denominator yields

$$1 = A(x-3) + B(x-2)$$

Setting $x = 2$ we find:

$$1 = A(2-3) + 0 \implies A = -1$$

Setting $x = 3$ we find:

$$1 = 0 + B(3-2) \implies B = 1$$

So we have found that $A = -1$ and $B = 1$. Therefore

$$\begin{aligned}\int \frac{1}{(x-2)(x-3)}\,\mathrm{d}x &= \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right) dx \\ &= \log|x-3| - \log|x-2| + C\end{aligned}$$

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S-1: By Hooke's Law, the force exerted by the spring at displacement x m from its natural length is $F = kx$, where k is the spring constant. Measuring distance in meters and force in newtons, the total work is

$$\int_0^{0.1\text{ m}} kx\,\mathrm{d}x = \frac{1}{2}kx^2\bigg|_0^{0.1\text{ m}} = \frac{1}{2}\cdot 50\cdot(0.1)^2\text{ J} = \frac{1}{4}\text{ J}.$$

S-2: By definition, the work done in moving the object from $x = 1$ meters to $x = 16$ meters by the force $F(x)$ is

$$W = \int_1^{16} F(x)\,\mathrm{d}x = \int_1^{16} \frac{a}{\sqrt{x}}\,\mathrm{d}x = 2a\sqrt{x}\,\bigg|_{x=1}^{x=16} = 6a$$

To have $W = 18$, we need $a = 3$.

As a side remark, $F(x) = \frac{a}{\sqrt{x}}$ has to have units Newtons. As x, a distance, has units meters, a has to have the bizarre units Newtons$\times\sqrt{\text{meters}}$.

S-3: First note that Newtons and Joules are MKS units, so we should measure distances in meters rather than centimeters. Next recall that a (linear) spring with spring constant k exerts a force $F(x) = kx$ when the spring is stretched xm beyond its natural length. So in this case $0.05k = 10$, or $k = 200$. The work done is

$$\int_0^{0.5} F(x)\,\mathrm{d}x = \int_0^{0.5} 200x\,\mathrm{d}x = \Big[100x^2\Big]_0^{0.5} = 25$$

S-4: Note that the cable has mass density $\frac{8}{5}$ kg/m. When the bucket is at height y, the cable that remains to be lifted has length $(5-y)$ m and mass $\frac{8}{5}(5-y) = 8\big(1-\frac{y}{5}\big)$ kg. So, at height y, the cable is subject to a downward gravitational force of $8\big(1-\frac{y}{5}\big)\cdot 9.8$; to raise the cable we need to apply a compensating upward force of $8\big(1-\frac{y}{5}\big)\cdot 9.8$. So the work required is

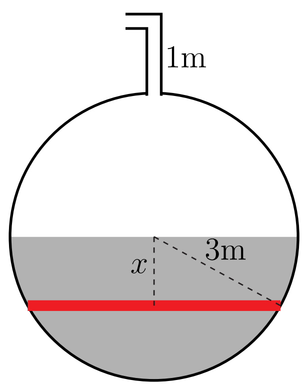
$$\int_0^5 8\Big(1-\frac{y}{5}\Big)\cdot 9.8\,\mathrm{d}y = 8\Big(y-\frac{y^2}{10}\Big)\cdot 9.8\Big|_0^5 = 8\cdot 2.5\cdot 9.8 = 196\text{ J}.$$

Alternatively, the cable has linear density 8 kg/5 m = 1.6 kg/m, and so the work required to lift a small piece of the cable (of length Δy) from height y m to height 5 m is $1.6\Delta y\cdot 9.8(5-y)$. The total work required is therefore

$$\int_0^5 1.6\cdot 9.8(5-y)\,\mathrm{d}y = 1.6\cdot 9.8\cdot 12.5 = 196\text{ J}$$

as before.

S-5: Imagine slicing the water into horizontal pancakes as in the sketch



Denote by x the distance of a pancake below the surface of the water. So x runs from 0 to 3. Each pancake

- has radius $\sqrt{3^2-x^2}$ (by Pythagorous) and hence
- has cross–sectional area $\pi(9-x^2)$ and hence
- has volume $\pi(9-x^2)\,\mathrm{d}x$ and hence
- has mass $1000\pi(9-x^2)\,\mathrm{d}x$ and hence
- is subject to a gravitational force of $9.8\times 1000\pi(9-x^2)\,\mathrm{d}x$ and hence
- requires work $9800\pi(9-x^2)(x+4)\,\mathrm{d}x$ to raise it to the spout. (It has to be raised xm to bring it to the height of the centre of the sphere, then 3m more to bring it to the top of the sphere, and then 1m more to bring it to the spout.)

So the total work is

$$\begin{aligned}\int_0^3 9800\pi(9-x^2)(x+4)\,\mathrm{d}x &= \int_0^3 9800\pi\big(36+9x-4x^2-x^3\big)\,\mathrm{d}x\\ &= 9800\pi\Big[36x+\frac{9}{2}x^2-\frac{4}{3}x^3-\frac{1}{4}x^4\Big]_0^3\\ &= 9800\frac{369}{4}\pi = 904{,}050\pi \text{ joules}\end{aligned}$$

S-6: The plate at height z has

- has side length $3-z$ and hence
- has area $(3-z)^2$ and hence
- has volume $(3-z)^2\,\mathrm{d}z$ and hence
- has mass $8000(3-z)^2\,\mathrm{d}z$ and hence
- is subject to a gravitational force of $9.8\times 8000(3-z)^2\,\mathrm{d}z$ and hence
- requires work $9.8\times 8000(2+z)(3-z)^2\,\mathrm{d}z$ to raise it from 2m below ground level to zm above ground level.

So the total work is

$$\int_0^3 9.8\times 8000(2+z)(3-z)^2\,\mathrm{d}z \text{ joules}$$

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S-1: By definition, the average value is

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\big(\sin(5x)+1\big)\,\mathrm{d}x$$

We now observe that $\sin(5x)$ is an odd function, and hence its integral over the symmetric interval $[-\frac{\pi}{2},\frac{\pi}{2}]$ equals zero. So the average value of $f(x)$ on this interval is 1.

Alternatively, the average equals, by the fundamental theorem of calculus,

$$\frac{1}{\pi}\left[\frac{-\cos(5x)}{5}+x\right]_{-\pi/2}^{\pi/2} = \frac{1}{\pi}\left\{\left[\frac{-\cos(5\pi/2)}{5}+\frac{\pi}{2}\right]-\left[\frac{-\cos(-5\pi/2)}{5}+\frac{-\pi}{2}\right]\right\} = \frac{\pi}{\pi} = 1$$

S-2: By definition, the average is

$$\begin{aligned}\frac{1}{e-1}\int_1^e x^2\log x\,\mathrm{d}x &= \frac{1}{e-1}\left[\frac{x^3}{3}\log x-\frac{x^3}{9}\right]_{x=1}^{x=e} = \frac{1}{e-1}\left[\frac{e^3}{3}-\frac{e^3}{9}+\frac{1}{9}\right]\\ &= \frac{1}{e-1}\left[\frac{2}{9}e^3+\frac{1}{9}\right]\end{aligned}$$

The indefinite integral $\int x^2 \log x \, dx = \frac{x^3}{3} \log x - \frac{x^3}{9} + C$ was guessed and then verified by checking that $\frac{d}{dx}\left[\frac{x^3}{3} \log x - \frac{x^3}{9}\right] = x^2 \log x$. The same indefinite integral can be found by using integration by parts with $u = \log x$, $dv = x^2 \, dx$, $v = \frac{x^3}{3}$.

S-3: By definition, the average value in question equals

$$\frac{1}{\pi/2 - 0} \int_0^{\pi/2} (3\cos^3 x + 2\cos^2 x) \, dx = \frac{2}{\pi}\left(\int_0^{\pi/2} 3\cos^3 x \, dx + \int_0^{\pi/2} 2\cos^2 x \, dx \right)$$

For the first integral we use the substitution $u = \sin x$, $du = \cos x \, dx$, $\cos^2 x = 1 - \sin^2 x = 1 - u^2$. Note that the endpoints $x = 0$ and $x = \frac{\pi}{2}$ become $u = 0$ and $u = 1$, respectively.

$$\begin{aligned}
\int_0^{\pi/2} 3\cos^3 x \, dx &= \int_0^{\pi/2} 3\cos^2 x \cos x \, dx \\
&= \int_0^1 3(1 - u^2) \, du \\
&= (3u - u^3)\Big|_0^1 = 2.
\end{aligned}$$

For the second integral we use the trigonometric identity $\cos^2 x \, dx = \frac{1+\cos(2x)}{2}$.

$$\begin{aligned}
2\int_0^{\pi/2} \cos^2 x \, dx &= \int_0^{\pi/2} \big(1 + \cos(2x)\big) \, dx \\
&= \left[x + \frac{1}{2}\sin(2x)\right]_0^{\pi/2} = \frac{\pi}{2}
\end{aligned}$$

Therefore the average value in question is

$$\frac{2}{\pi}\left(\int_0^{\pi/2} 3\cos^3 x \, dx + \int_0^{\pi/2} 2\cos^2 x \, dx \right) = \frac{2}{\pi}\left(2 + \frac{\pi}{2}\right) = \frac{4}{\pi} + 1.$$

S-4: By definition, the average value in question equals

$$\text{Ave} = \frac{1}{\pi/k - 0} \int_0^{\pi/k} \sin(kx) \, dx$$

To evaluate the integral, we use the substitution $u = kx$, $du = k \, dx$. Note that the endpoints $x = 0$ and $x = \pi/k$ become $u = 0$ and $u = \pi$, respectively. So

$$\text{Ave} = \frac{k}{\pi} \int_0^{\pi} \sin(u) \, \frac{du}{k} = \frac{1}{\pi}\Big[-\cos(u)\Big]_0^{\pi} = \frac{2}{\pi}$$

S-5: By definition, the average temperature is

$$\begin{aligned}\frac{1}{3}\int_0^3 T(x)\,\mathrm{d}x &= \frac{1}{3}\int_0^3 \frac{80}{16-x^2}\,\mathrm{d}x = \frac{1}{3}\int_0^3 \frac{80}{(4-x)(4+x)}\,\mathrm{d}x = \frac{1}{3}\int_0^3 \Big[\frac{10}{4-x}+\frac{10}{4+x}\Big]\,\mathrm{d}x\\ &= \frac{1}{3}\int_0^3 \Big[-\frac{10}{x-4}+\frac{10}{4+x}\Big]\,\mathrm{d}x = \frac{10}{3}\Big[-\log|x-4|+\log|x+4|\Big]_0^3\\ &= \frac{10}{3}\log\Big|\frac{x+4}{x-4}\Big|\Big|_0^3 = \frac{10}{3}[\log 7-\log 1]\\ &= \frac{10}{3}\log 7\end{aligned}$$

S-6: By definition, the average value is

$$\frac{1}{e-1}\int_1^e \frac{\log x}{x}\,\mathrm{d}x = \frac{1}{e-1}\int_0^1 u\,\mathrm{d}u = \frac{1}{e-1}\frac{u^2}{2}\Big|_0^1 = \frac{1}{2(e-1)}$$

where we made the change of variables $u=\log x$, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$.

S-7: By definition, the average value is

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2 x\,\mathrm{d}x = \frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi}\big[\cos(2x)+1\big]\,\mathrm{d}x = \frac{1}{2\pi}\frac{1}{2}\Big[\frac{\sin(2x)}{2}+x\Big]_0^{2\pi} = \frac{1}{2\pi}\frac{1}{2}2\pi = \frac{1}{2}$$

S-8: (a) a) Let $v(t)$ be the speed of the car at time t. Then, by the trapezoidal rule with $a=0$, $b=2$, $\Delta t = 1/3$, the distance traveled is

$$\begin{aligned}\int_0^2 v(t)\,dt &\approx \Delta t\Big[\tfrac{1}{2}v(0)+v(1/3)+v(2/3)+v(3/3)+v(4/3)+v(5/3)+\tfrac{1}{2}v(2)\Big]\\ &= \frac{1}{3}\Big[\tfrac{1}{2}50+70+80+55+60+80+\tfrac{1}{2}40\Big] = 130\text{ km}\end{aligned}$$

(b) The average speed is $\frac{1}{2}\int_0^2 v(t)\,dt \approx 65$ km/hr.

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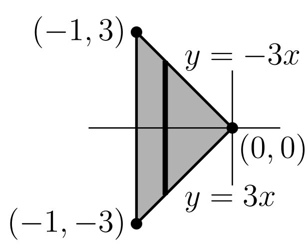
S-1: We use vertical strips, as in the sketch below. (To use horizontal strips we would have to split the domain of integration in two: $-3\leqslant y\leqslant 0$ and $0\leqslant y\leqslant 3$.)



The equations of the top and bottom of the triangle are

$$y = T(x) = -3x \qquad \text{and} \qquad y = B(x) = 3x.$$

The area of the triangle is $A = \frac{1}{2}(6)(1) = 3$. Using vertical slices,

$$\bar{x} = \frac{1}{A}\int_{-1}^{0} x\big[T(x) - B(x)\big]\,\mathrm{d}x = \frac{1}{3}\int_{-1}^{0} x\big[(-3x) - (3x)\big]\,\mathrm{d}x = -\frac{1}{3}\int_{-1}^{0} 6x^2\,\mathrm{d}x$$

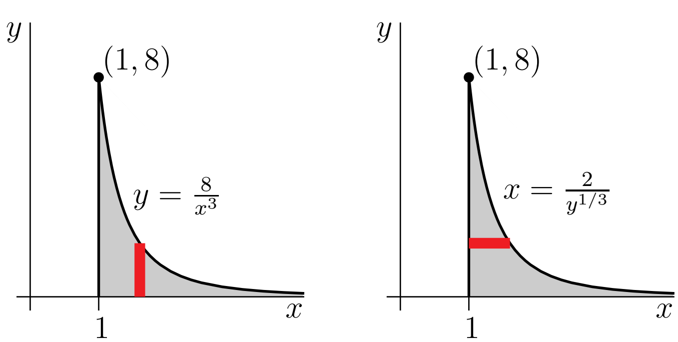
S-2: The equation of the top of the region is $y = T(x) = 1$, the equation of the bottom of the region is $y = B(x) = -e^x$ and x rus from $a = 0$ to $b = 1$. So the y-coordinate of the centre of mass is

$$\begin{aligned}\bar{y} &= \frac{1}{2A}\int_0^1 \big[T(x)^2 - B(x)^2\big]\,\mathrm{d}x = \frac{1}{2e}\int_0^1 \left(1 - e^{2x}\right)\mathrm{d}x = \frac{1}{2e}\left[x - \frac{1}{2}e^{2x}\right]_0^1 \\ &= \frac{1}{2e}\left[1 - \frac{e^2}{2} - 0 + \frac{1}{2}\right] = \frac{3}{4e} - \frac{e}{4}\end{aligned}$$

S-3: The area of the region is

$$A = \int_1^\infty \frac{8}{x^3}\,\mathrm{d}x = \lim_{t\to\infty}\left(\int_1^t \frac{8}{x^3}\,\mathrm{d}x\right) = \lim_{t\to\infty}\left[-\frac{4}{x^2}\right]_1^t = \lim_{t\to\infty}\left[-\frac{4}{t^2} + \frac{4}{1^2}\right] = 0 + 4$$

We'll now compute $\bar{y}$ twice, once with vertical strips, as in the figure in the left below, and once with horizontal strips as in the figure on the right below.



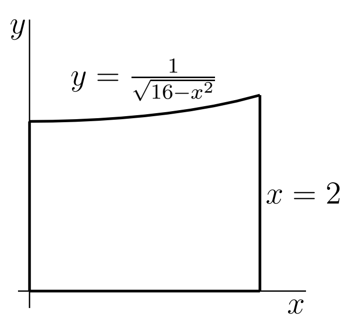
Vertical strips: The equation of the top of the region is $y = T(x) = \frac{8}{x^3}$ and the equation of the bottom of the region is $y = B(x) = 0$. So, using vertical strips, as in the figure on the left above, the y-coordinate of the centre of mass is

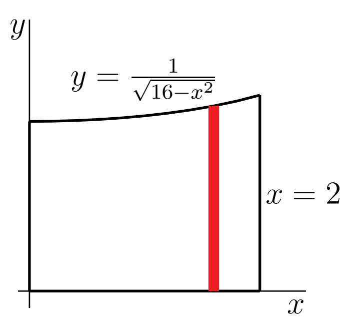
$$\begin{aligned}\bar{y} &= \frac{1}{2A}\int_1^\infty \left[T(x)^2 - B(x)^2\right] \mathrm{d}x \\ &= \frac{1}{8}\int_1^\infty \left(\frac{8}{x^3}\right)^2 \mathrm{d}x \\ &= \lim_{t\to\infty}\left(\int_1^t \frac{8}{x^6}\,\mathrm{d}x\right) \\ &= \lim_{t\to\infty}\left[-\frac{8}{5x^5}\right]_1^t \\ &= \lim_{t\to\infty}\left[-\frac{8}{5t^5}+\frac{8}{5\times 1^5}\right] = \frac{8}{5}\end{aligned}$$

Vertical strips: Since $y = \frac{8}{x^3}$ is equivalent to $x = \sqrt[3]{\frac{8}{y}}$, the equation of the right hand side of the region is $x = R(y) = \frac{2}{y^{1/3}}$ and the equation of the left hand side of the region is $x = L(y) = 1$. The x– and y–coordinates of the point at the top of the region obeys both $x = 1$ and $y = \frac{8}{x^3} = 8$. Thus y runs from 0 to 8. So, using horizontal strips, as in the figure on the right above, the y-coordinate of the centre of mass is

$$\begin{aligned}\bar{y} &= \frac{1}{A}\int_0^8 y\left[R(y)-L(y)\right]\mathrm{d}y \\ &= \frac{1}{4}\int_0^8 y\left[2y^{-1/3}-1\right]\mathrm{d}y \\ &= \frac{1}{4}\int_0^8 \left[2y^{2/3}-y\right]\mathrm{d}y \\ &= \frac{1}{4}\left[\frac{6}{5}y^{5/3}-\frac{y^2}{2}\right]_0^8 \\ &= \frac{1}{4}\left[\frac{6\times 32}{5}-\frac{8\times 8}{2}\right] = 8\left[\frac{6}{5}-1\right] = \frac{8}{5}\end{aligned}$$

S-4: (a) The sketch is the figure on the left below.





(b) The part of the region with x coordinate between x and $x + \mathrm{d}x$ is a strip of width $\mathrm{d}x$ running from $y = 0$ to $y = \frac{1}{\sqrt{16-x^2}}$. It is illustrated in red in the figure on the right above.

So the area of the region is

$$A = \int_0^2 \frac{1}{\sqrt{16-x^2}}\,\mathrm{d}x = \int_0^{\sin^{-1}1/2} \frac{1}{4\cos t} 4\cos t\,\mathrm{d}t = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

where we made the substitution $x = 4\sin t,\ \mathrm{d}x = 4\cos t\mathrm{d}t, \sqrt{16-x^2} = 4\cos t$. On the (red) strip with x coordinate between x and $x+\mathrm{d}x$, the average value of y is $\frac{1/2}{\sqrt{16-x^2}}$. The y–coordinate of the centroid is the weighted average of $\frac{1/2}{\sqrt{16-x^2}}$ with the strip counted as having weight $\frac{1}{\sqrt{16-x^2}}\,\mathrm{d}x$.

$$\begin{aligned}
\bar{y} &= \frac{1}{A}\int_0^2 \frac{1/2}{\sqrt{16-x^2}}\frac{1}{\sqrt{16-x^2}}\,\mathrm{d}x = \frac{1}{2A}\int_0^2 \frac{1}{16-x^2}\,\mathrm{d}x = \frac{1}{2A}\int_0^2 \frac{1}{(4-x)(4+x)}\,\mathrm{d}x \\
&= \frac{1}{2A}\int_0^2 \Big[\frac{1/8}{4+x} + \frac{1/8}{4-x}\Big]\,\mathrm{d}x = \frac{1}{16A}\int_0^2 \Big[\frac{1}{x+4} - \frac{1}{x-4}\Big]\,\mathrm{d}x \\
&= \frac{1}{16A}\Big[\log|x+4| - \log|x-4|\Big]_0^2 = \frac{6}{16\pi}\big[\log 6 - \log 2 - \log 4 + \log 4\big] \\
&= \frac{3\log 3}{8\pi}
\end{aligned}$$

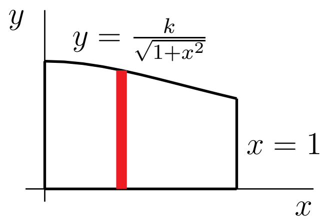
S-5: The top of the region is $y = T(x) = \cos(x)$ and the bottom of the region is $y = B(x) = \sin(x)$. So the area of the region is

$$\begin{aligned}
A &= \int_0^{\pi/4} \big(T(x) - B(x)\big)\,\mathrm{d}x = \int_0^{\pi/4} \big(\cos(x) - \sin(x)\big)\,\mathrm{d}x = \Big[\sin(x) + \cos(x)\Big]_0^{\pi/4} \\
&= \sqrt{2} - 1
\end{aligned}$$

and region has centroid $(\bar{x}, \bar{y})$ with

$$\begin{aligned}
\bar{x} &= \frac{1}{A}\int_0^{\pi/4} x\big(T(x) - B(x)\big)\,\mathrm{d}x = \frac{1}{A}\int_0^{\pi/4} x\big(\cos(x) - \sin(x)\big)\,\mathrm{d}x \\
&= \frac{1}{A}\Big[x\sin(x) + \cos x + x\cos(x) - \sin x\Big]_0^{\pi/4} = \frac{\frac{\pi}{4}\sqrt{2} - 1}{\sqrt{2} - 1} \\
\bar{y} &= \frac{1}{2A}\int_0^{\pi/4} \big(T(x)^2 - B(x)^2\big)\,\mathrm{d}x = \frac{1}{2A}\int_0^{\pi/4} \big(\cos^2(x) - \sin^2(x)\big)\,\mathrm{d}x \\
&= \frac{1}{2A}\int_0^{\pi/4} \cos(2x)\,\mathrm{d}x = \frac{1}{2A}\Big[\frac{1}{2}\sin(2x)\Big]_0^{\pi/4} = \frac{1}{4(\sqrt{2} - 1)}
\end{aligned}$$

S-6: (a) Imagine that the plane region is a metal plate of density one unit per unit area. Then the part of the plate with x–coordinate between x and $x + \mathrm{d}x$ has width $\mathrm{d}x$ and



height $\frac{k}{\sqrt{1+x^2}}$. So it has area, and hence weight, $\frac{k}{\sqrt{1+x^2}}\mathrm{d}x$. The x–coordinate of the centroid is the weighted average of x or

$$\bar{x} = \frac{1}{A}\int_0^1 x\frac{k}{\sqrt{1+x^2}}\,\mathrm{d}x = \frac{1}{A}\int_1^2 \frac{k}{\sqrt{u}}\frac{du}{2} = \frac{k}{2A}\Big[\frac{\sqrt{u}}{1/2}\Big]_1^2 = \frac{k}{A}\big[\sqrt{2}-1\big]$$

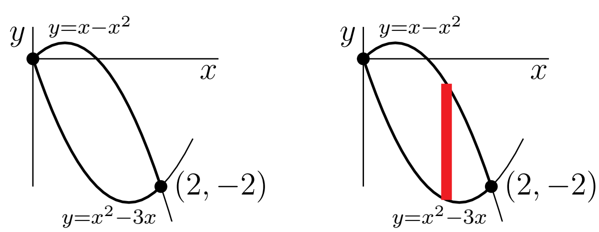
We made the substitution $u = 1 + x^2$, $du = 2x\mathrm{d}x$. The average value of y on the part of the plate with x–coordinate between x and $x + \mathrm{d}x$ is $\frac{k}{2\sqrt{1+x^2}}$. The y–coordinate of the centroid is the weighted average of $\frac{k}{2\sqrt{1+x^2}}$ or

$$\bar{y} = \frac{1}{A}\int_0^1 \frac{k}{2\sqrt{1+x^2}}\frac{k}{\sqrt{1+x^2}}\,\mathrm{d}x = \frac{k^2}{2A}\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \frac{k^2}{2A}\Big[\arctan 1 - \arctan 0\Big] = \frac{k^2}{2A}\frac{\pi}{4} = \frac{k^2\pi}{8A}$$

(b) We have $\bar{x} = \bar{y}$ if and only if

$$\frac{k}{A}\big[\sqrt{2}-1\big] = \frac{k^2\pi}{8A} \implies k = \frac{8}{\pi}\big[\sqrt{2}-1\big]$$

S-7: (a) The sketch is the figure on the left below.



(b) The curves cross when $x^2 - 3x = x - x^2 \implies 2x^2 = 4x \implies x = 0,\ x = 2$. The corresponding values of y are $y = 0$ and $y = 2 - 2^2 = -2$. Using vertical strips, as in the figure on the right above, the area is

$$\int_0^2 \big[(x-x^2) - (x^2-3x)\big]\,\mathrm{d}x = \int_0^2 \big[4x - 2x^2\big]\,\mathrm{d}x = \big[2x^2 - \frac{2}{3}x^3\big]_0^2 = 8 - \frac{16}{3} = \frac{8}{3}$$

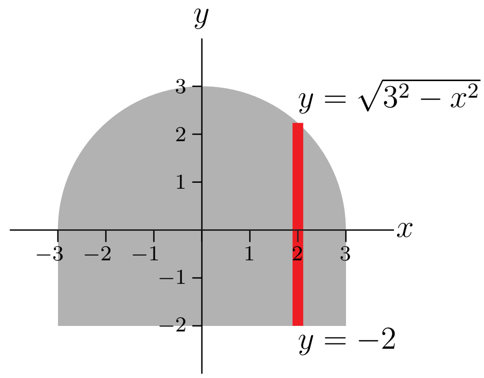
(c) The x–coordinate of the centroid of R, i.e. the weighted average of x over R, is

$$\bar{x} = \frac{3}{8}\int_0^2 x\big[(x-x^2) - (x^2-3x)\big]\,\mathrm{d}x = \frac{3}{8}\int_0^2 \big[4x^2 - 2x^3\big]\,\mathrm{d}x = \frac{3}{8}\Big[\frac{4}{3}x^3 - \frac{1}{2}x^4\Big]_0^2 = \frac{3}{8}\Big[\frac{32}{3} - 8\Big]$$
$$= 1$$

S-8: By definition the x–coordinate of the centroid is

$$\bar{x} = \frac{\int_0^1 x \frac{1}{1+x^2}\,\mathrm{d}x}{\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x} = \frac{\frac{1}{2}\log(1+x^2)\big|_0^1}{\tan^{-1} x\big|_0^1} = \frac{\frac{1}{2}\log 2}{\pi/4} = \frac{2}{\pi}\log 2 \approx 0.44127$$

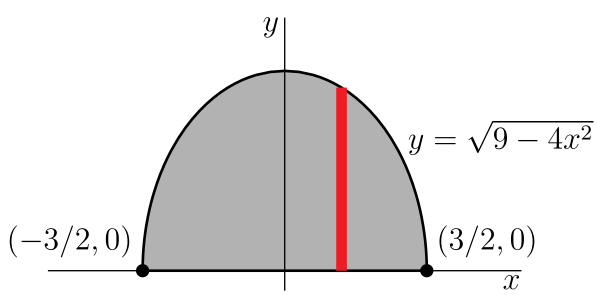
S-9: By symmetry, the centroid lies on the y–axis, so $\bar{x} = 0$. We'll use vertical strips as in the sketch



to compute the y–coordinate of the centroid. The strip with x–coordinate x has y running from -2 to $\sqrt{3^2 - x^2}$. So the average value of y on the strip is $\frac{1}{2}[\sqrt{3^2 - x^2} + (-2)]$. The stripe has "weight" (area) $[\sqrt{3^2 - x^2} - (-2)]\,\mathrm{d}x$. Thus, as the area of the region is $\frac{1}{2}\pi 3^2 + 2 \times 6 = 12 + 9\pi/2$, the y–coordinate of the centroid is

$$\begin{aligned}
\bar{y} &= \frac{1}{12+9\pi/2}\int_{-3}^{3} \frac{1}{2}[\sqrt{3^2 - x^2} - 2]\,[\sqrt{3^2 - x^2} + 2]\,\mathrm{d}x \\
&= \frac{1}{24+9\pi}\int_{-3}^{3} [(3^2 - x^2) - 4]\,\mathrm{d}x \;=\; \frac{1}{24+9\pi}\int_{-3}^{3} [5 - x^2]\,\mathrm{d}x \\
&= \frac{2}{24+9\pi}\int_{0}^{3} [5 - x^2]\,\mathrm{d}x \qquad\quad = \frac{2}{24+9\pi}\Big[5x - \frac{x^3}{3}\Big]_0^3 \\
&= \frac{2}{24+9\pi}[15-9] \qquad\qquad\qquad = \frac{12}{24+9\pi}
\end{aligned}$$

S-10: (a) Notice that when $x = 0$, $y = 3$ and as x^2 increases, y decreases until y hits zero at $x^2 = \frac{9}{4}$, i.e. at $x = \pm\frac{3}{2}$. For $x^2 > \frac{9}{4}$, y is not even defined. So, on D, x runs from $-\frac{3}{2}$ to $+\frac{3}{2}$ and, for each x, y runs from0 to $\sqrt{9 - 4x^2}$. Here is a sketch of D.



As an aside, we can rewrite $y = \sqrt{9-4x^2}$ as $4x^2 + y^2 = 9$, $y \geqslant 0$, which is the top half of the ellipse which passes through $(\pm a, 0)$ and $(0, \pm b)$ with $a = \frac{3}{2}$ and $b = 3$. The area of the full ellipse is $\pi ab = \frac{9}{2}\pi$. The area of D is half of that, which is $\frac{9}{4}\pi$. But we are told to use an integral, so we will do so.

The area is

$$\text{Area} = \int_{-3/2}^{3/2} \sqrt{9-4x^2}\,\mathrm{d}x$$

We can evaluate this integral by substituting $x = \frac{3}{2}\sin\theta$, $\mathrm{d}x = \frac{3}{2}\cos\theta\,\mathrm{d}\theta$ and using

$$x = \pm\frac{3}{2} \iff \sin\theta = \pm 1$$

So $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$ and

$$\begin{aligned}
\text{Area} &= \int_{-\pi/2}^{\pi/2} \sqrt{9-4\left(\tfrac{3}{2}\sin\theta\right)^2}\;\frac{3}{2}\cos\theta\,\mathrm{d}\theta &&= \int_{-\pi/2}^{\pi/2} \sqrt{9-9\sin^2\theta}\;\frac{3}{2}\cos\theta\,\mathrm{d}\theta \\
&= \frac{9}{2}\int_{-\pi/2}^{\pi/2} \cos^2\theta\,\mathrm{d}\theta &&= \frac{9}{2}\int_{-\pi/2}^{\pi/2} \frac{\cos(2\theta)+1}{2}\mathrm{d}\theta \\
&= \frac{9}{4}\left[\frac{\sin(2\theta)}{2} + \theta\right]_{-\pi/2}^{\pi/2} &&= \frac{9}{4}\pi
\end{aligned}$$

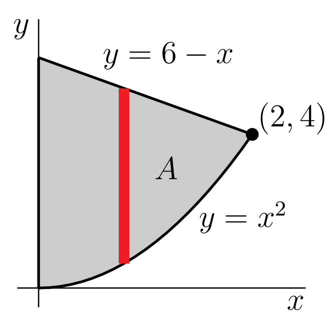
(b) The region D is symmetric about the y axis. So the centre of mass lies on the y axis. That is, $\bar{x} = 0$. Since D has area $A = \frac{9}{4}\pi$, top equation $y = T(x) = \sqrt{9-4x^2}$ and bottom equation $y = B(x) = 0$, with x running from $a = -\frac{3}{2}$ to $b = \frac{3}{2}$,

$$\begin{aligned}
\bar{y} &= \frac{1}{2A}\int_a^b \left[T(x)^2 - B(x)^2\right]\,\mathrm{d}x &&= \frac{2}{9\pi}\int_{-3/2}^{3/2}\left[9-4x^2\right]\,\mathrm{d}x &&= \frac{4}{9\pi}\int_0^{3/2}\left[9-4x^2\right]\,\mathrm{d}x \\
&= \frac{4}{9\pi}\left[9x - \frac{4}{3}x^3\right]_0^{3/2} &&= \frac{4}{9\pi}\left[9\frac{3}{2} - \frac{4}{3}\frac{3^3}{2^3}\right] &&= \frac{4}{9\pi}\left[9\frac{3}{2} - 9\frac{1}{2}\right] \\
&= \frac{4}{\pi}
\end{aligned}$$

S-11: (a) The two curves cross at points (x, y) that satisfy both $y = x^2$ and $y = 6 - x$, and hence

$$x^2 = 6 - x \iff x^2 + x - 6 = 0 \iff (x+3)(x-2)$$

So we see that the two curves intersect at $x = 2$ (as well as $x = -3$, which is to the left of the y-axis and therefore irrelevant). Here is a sketch of A.



The top of A has equation $y = T(x) = 6 - x$, the bottom has equation $y = B(x) = x^2$ and x runs from 0 to 2. So, using vertical strips,

$$\begin{aligned}
\bar{x} &= \frac{1}{\text{area}} \int_0^2 x\big[T(x) - B(x)\big]\, \mathrm{d}x \\
&= \frac{1}{22/3} \int_0^2 x\big[(6-x) - x^2\big]\, \mathrm{d}x = \frac{3}{22} \int_0^2 (6x - x^2 - x^3)\, \mathrm{d}x \\
&= \frac{3}{22} \left[3x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 \\
&= \frac{3}{22} \left[12 - \frac{8}{3} - 4\right] = \frac{3}{22} \frac{16}{3} = \frac{8}{11}
\end{aligned}$$

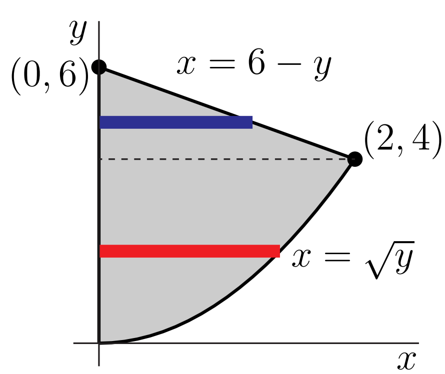
and

$$\begin{aligned}
\bar{y} &= \frac{1}{2(\text{area})} \int_0^2 \big[T(x)^2 - B(x)^2\big]\, \mathrm{d}x \\
&= \frac{1}{2} \frac{1}{22/3} \int_0^2 \big((6-x)^2 - x^4\,\big)\, \mathrm{d}x = \frac{3}{44} \left[-\frac{(6-x)^3}{3} - \frac{x^5}{5}\right]_0^2 \\
&= \frac{3}{44} \left(-\frac{64 - 216}{3} - \frac{32}{5}\right) = \frac{3}{44} \frac{664}{15} = \frac{166}{55}
\end{aligned}$$

The integral was evaluated by guessing an antiderivative for the integrand. It could also be evaluated as

$$\begin{aligned}
\frac{3}{44} \int_0^2 \big(36 - 12x + x^2 - x^4\,\big)\, \mathrm{d}x &= \frac{3}{44} \left[36x - 6x^2 + \frac{x^3}{3} - \frac{x^5}{5}\right]_0^2 \\
&= \frac{3}{44} \left(72 - 24 + \frac{8}{3} - \frac{32}{5}\right) = \frac{3}{44} \frac{664}{15} = \frac{166}{55}
\end{aligned}$$

(b) The question specifies the use of horizontal slices (as in Example 1.6.5 CLP 101 notes). We start by converting both equations $y = 6 - x$ and $y = x^2$ into equations of the form $x = f(y)$. To do so we solve for x in both equations, yielding $x = \sqrt{y}$ and $x = 6 - y$.



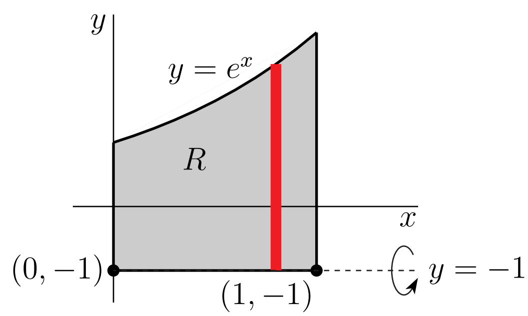
- We use thin horizontal strips of width $\mathrm{d}y$ as in the figure above.
- When we rotate about the y–axis, each strip sweeps out a thin disk

- whose radius is $r = 6 - y$ when $4 \leqslant y \leqslant 6$ (see the blue strip in the figure above), and whose radius is $r = \sqrt{y}$ when $0 \leqslant y \leqslant 4$ (see the red strip in the figure above) and
- whose thickness is $\mathrm{d}y$ and hence
- whose volume is $\pi r^2\,\mathrm{d}y = \pi(6-y)^2\,\mathrm{d}y$ when $4 \leqslant y \leqslant 6$ and whose volume is $\pi r^2\,\mathrm{d}y = \pi y\,\mathrm{d}y$ when $0 \leqslant y \leqslant 4$.

• As our bottommost strip is at $y = 0$ and our topmost strip is at $y = 6$, the total volume is

$$\pi \int_0^4 y\,\mathrm{d}y + \pi \int_4^6 (6-y)^2\,\mathrm{d}y$$

S-12: (a) Here is a sketch of the specified region, which we shall call R.



The top of R has equation $y = T(x) = e^x$, the bottom has equation $y = B(x) = -1$ and x runs from 0 to 1. So, using vertical strips, we see that R has

$$\text{area} = \int_0^1 \big[T(x) - B(x)\big]\,\mathrm{d}x = \int_0^1 \big[e^x - (-1)\big]\,\mathrm{d}x = \int_0^1 \big[e^x + 1\big]\,\mathrm{d}x = \big[e^x + x\big]_0^1 = e$$

and

$$\begin{aligned}\bar{y} &= \frac{1}{2(\text{area})}\int_0^1 \big[T(x)^2 - B(x)^2\big]\,\mathrm{d}x \\ &= \frac{1}{2e}\int_0^1 \big[e^{2x} - 1\,\big]\,\mathrm{d}x = \frac{1}{2e}\left[\frac{e^{2x}}{2} - x\right]_0^1 \\ &= \frac{1}{2e}\left(\frac{e^2}{2} - 1 - \frac{1}{2}\right) = \frac{e}{4} - \frac{3}{4e}\end{aligned}$$

(b) To compute the volume when R is rotated about $y = -1$

• We use thin vertical strips of width $\mathrm{d}x$ as in the figure above.

• When we rotate about the line $y = -1$, each strip sweeps out a thin disk

- whose radius is $r = T(x) - B(x) = e^x + 1$ and
- whose thickness is $\mathrm{d}x$ and hence

- whose volume is $\pi r^2\,\mathrm{d}x = \pi(e^x+1)^2\,\mathrm{d}x$.

- As our leftmost strip is at $x = 0$ and our rightmost strip is at $z = 1$, the total volume is

$$\begin{aligned}
\pi\int_0^1 (e^x+1)^2\,\mathrm{d}x = \pi\int_0^1 (e^{2x}+2e^x+1)\,\mathrm{d}x &= \pi\left[\frac{e^{2x}}{2}+2e^x+x\right]_0^1 \\
&= \pi\left[\left(\frac{e^2}{2}+2e+1\right)-\left(\frac{1}{2}+2+0\right)\right] \\
&= \pi\left(\frac{e^2}{2}+2e-\frac{3}{2}\right)
\end{aligned}$$

Solutions to Exercises 2.4 — Jump to TABLE OF CONTENTS

S-1: Rearranging, we have:

$$e^y\,\mathrm{d}y = 2x\,\mathrm{d}x.$$

Integrating both sides:

$$e^y = x^2 + C.$$

Since $y = \log 2$ when $x = 0$, we have

$$\begin{aligned}
e^{\log 2} &= 0^2 + C \\
2 &= C,
\end{aligned}$$

and therefore

$$\begin{aligned}
e^y &= x^2 + 2 \\
y &= \log(x^2+2)
\end{aligned}$$

S-2: Using separation of variables

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{x^2+1} \iff \frac{\mathrm{d}y}{y} = \frac{x}{x^2+1}\mathrm{d}x \iff \log|y| = \frac{1}{2}\log(1+x^2)+C$$

To satisfy $y(0) = 3$, we need $\log 3 = \frac{1}{2}\log(1+0)+C$ or $C = \log 3$. Thus

$$\log|y| = \frac{1}{2}\log(1+x^2)+\log 3 = \log\sqrt{1+x^2}+\log 3 = \log 3\sqrt{1+x^2} \implies y(x) = 3\sqrt{1+x^2}$$

The other sign of y would violate $y(0) = 3$ and so is unacceptable.

S-3: The given differential equation is separable and we solve it accordingly.

$$y' = e^{\frac{y}{3}} \cos t \iff e^{-y/3} \mathrm{d}y = \cos t \, \mathrm{d}t \iff -3e^{-y/3} = \sin t + C \iff e^{y/3} = \frac{-3}{C + \sin t}$$
$$\iff y(t) = 3 \log \frac{-3}{C + \sin t}$$

for any constant C. The solution only exists for $C + \sin t < 0$.

S-4: The given differential equation is separable and we solve it accordingly.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xe^{x^2 - \log(y^2)} = \frac{xe^{x^2}}{y^2} \iff y^2 \mathrm{d}y = xe^{x^2} \, \mathrm{d}x \iff \frac{y^3}{3} = \frac{1}{2}e^{x^2} + C' \iff y = \sqrt[3]{\frac{3}{2}e^{x^2} + C}$$

for any constant C.

S-5: The given differential equation is separable and we solve it accordingly.

$$y' = xe^y \iff \frac{\mathrm{d}y}{e^y} = x \, \mathrm{d}x \iff -e^{-y} = \frac{1}{2}x^2 - C \iff y = -\log\left(C - \frac{x^2}{2}\right)$$

for any constant C. The solution only exists for $C - \frac{x^2}{2} > 0$, i.e. for $C > 0$ and $|x| < \sqrt{2C}$.

S-6: The given differential equation is separable and we solve it accordingly. Cross–multiplying, we rewrite the equation as

$$y^2 \frac{\mathrm{d}y}{\mathrm{d}x} = e^x - 2x$$
$$y^2 \, \mathrm{d}y = (e^x - 2x) \, \mathrm{d}x.$$

Integrating both sides, we find

$$\frac{1}{3}y^3 = e^x - x^2 + C$$

Setting $x = 0$ and $y = 3$, we find $\frac{1}{3}3^3 = e^0 - 0^2 + C$ and hence $C = 8$; therefore the solution is

$$\frac{1}{3}y^3 = e^x - x^2 + 8$$
$$y = (3e^x - 3x^2 + 24)^{1/3}.$$

S-7: This is a separable differential equation that we solve in the usual way.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -xy^3 \implies \int \frac{\mathrm{d}y}{y^3} = -\int x \, \mathrm{d}x \implies \frac{y^{-2}}{-2} = -\frac{x^2}{2} + C \implies y^{-2} = x^2 - 2C.$$

To have $y = -\frac{1}{4}$ when $x = 0$, we must choose C to obey

$$\left(-\frac{1}{4}\right)^{-2} = 0 - 2C \implies -2C = 16 \implies y^{-2} = x^2 - 2C = x^2 + 16.$$

So $y = f(x) = -\frac{1}{\sqrt{x^2+16}}$. We need to take the negative square root to have $f(0) = -\frac{1}{4}$.

S-8: This is a separable differential equation that we solve in the usual way. Cross-multiplying and integrating,

$$\begin{aligned} y\,\mathrm{d}y &= (15x^2 + 4x + 3)\,\mathrm{d}x \\ \int y\,\mathrm{d}y &= \int (15x^2 + 4x + 3)\,\mathrm{d}x \\ \frac{y^2}{2} &= 5x^3 + 2x^2 + 3x + C. \end{aligned}$$

Plugging in $x = 1$ and $y = 4$ gives $\frac{4^2}{2} = 5 + 2 + 3 + C$, and so $C = -2$. Therefore

$$\begin{aligned} \frac{y^2}{2} &= 5x^3 + 2x^2 + 3x - 2 \\ y &= \sqrt{10x^3 + 4x^2 + 6x - 4}. \end{aligned}$$

We must choose the positive square root since $y(1)$ is positive.

S-9: The given differential equation is separable and we solve it accordingly.

$$y' = x^3 y \iff \frac{\mathrm{d}y}{y} = x^3\,\mathrm{d}x \iff \log|y| = \frac{x^4}{4} + C$$

We are told that $y = 1$ when $x = 0$. So $\log 1 = \frac{0^4}{4} + C$ which gives $C = 0$ so that $\log|y| = \frac{x^4}{4}$ or $|y(x)| = e^{x^4/4}$. Since $y(0) = 1 > 0$, we should drop the absolute values. (Since $|y(x)| = e^{x^4/4} \geqslant 1$, the magnitude of $y(x)$ is always at least one. As $y(x)$ must be continuous, it cannot change sign.) Thus $y(x) = e^{x^4/4}$.

S-10: This is a separable differential equation, even if it doesn't quite look like it. First move the y from the left hand side to the right hand side.

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^2 &\iff x\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - y = y(y-1) \\ &\iff \frac{\mathrm{d}y}{y(y-1)} = \left[\frac{1}{y-1} - \frac{1}{y}\right]\mathrm{d}y = \frac{\mathrm{d}x}{x} \\ &\iff \log|y-1| - \log|y| = \log|x| + C \\ &\iff \log\frac{|y-1|}{|y|} = \log|x| + C \end{aligned}$$

To determine C we set $x = 1$ and $y = -1$.

$$\log \frac{|-2|}{|-1|} = \log|1| + C \Rightarrow C = \log 2$$

So the solution is

$$\log \frac{|y-1|}{|y|} = \log|x| + \log 2 = \log 2|x| \iff \frac{|y-1|}{|y|} = 2|x|$$

For x near 1 and y near -1, both x and $\frac{y-1}{y}$ are positive and we may drop the absolute value signs.

$$\frac{y-1}{y} = 2x \iff y - 1 = 2xy \iff y = \frac{1}{1-2x}$$

As a check, we compute

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x\frac{2}{(1-2x)^2} + \frac{1}{1-2x} = \frac{2x + (1-2x)}{(1-2x)^2} = \frac{1}{(1-2x)^2} = y^2$$

and

$$y(1) = \frac{1}{1 - 2 \times 1} = -1$$

S-11: We are told that $y = f(x)$ obeys the separable differential equation $y' = xy$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = xy &\iff \frac{\mathrm{d}y}{y} = x\,\mathrm{d}x \\
&\iff \int \frac{\mathrm{d}y}{y} = \int x\,\mathrm{d}x \\
&\iff \log|y| = \frac{x^2}{2} + C
\end{aligned}$$

To determine C we set $x = 0$ and $y = e$.

$$\log e = \frac{0^2}{2} + C \Rightarrow C = 1$$

So the solution is

$$\log|y| = \frac{x^2}{2} + 1$$

We are told that $y = f(x) > 0$, so may drop the absolute value signs.

$$\log y = \frac{x^2}{2} + 1 \iff y = e^{1+x^2/2}$$

S-12: This is a separable differential equation.

$$\frac{dy}{dx} = \frac{1}{(x^2+x)y} \iff y\,dy = \frac{dx}{x(x+1)} = \Big[\frac{1}{x} - \frac{1}{x+1}\Big]dx \iff \frac{y^2}{2} = \log\frac{x}{x+1} + C$$

To satisfy the initial condition $y(1) = 2$ we must choose C to obey

$$\frac{2^2}{2} = \log\frac{1}{1+1} + C \implies C = 2 - \log\frac{1}{2}$$

So

$$y(x) = \sqrt{2\Big(\log\frac{x}{x+1} - \log\frac{1}{2} + 2\Big)} = \sqrt{4 + 2\log\frac{2x}{x+1}}$$

Note that, to satisfy $y(1) = 2$, we need the positive square root.

S-13: This is a separable differential equation.

$$\begin{aligned}\frac{1+\sqrt{y^2-4}}{\tan x}y' = \frac{\sec x}{y} &\iff y\big[1+\sqrt{y^2-4}\big]\,dy = \sec x\ \tan x\,dx \\ &\iff \int y\big[1+\sqrt{y^2-4}\big]\,dy = \int \sec x\ \tan x\,dx \\ &\iff \frac{y^2}{2} + \frac{1}{3}\big[y^2-4\big]^{3/2} = \sec x + C\end{aligned}$$

To determine C we set $x = 0$ and $y = 2$.

$$\frac{2^2}{2} + \frac{1}{3}\big[2^2-4\big]^{3/2} = \sec 0 + C = 1 + C \Rightarrow C = 1$$

So the solution is

$$\frac{y^2}{2} + \frac{1}{3}\big[y^2-4\big]^{3/2} = \sec x + 1$$

S-14: The given differential equation is separable and we solve it accordingly.

$$\frac{dP}{dt} = -k\sqrt{P} \implies \frac{dP}{\sqrt{P}} = -k\,dt \implies \frac{\sqrt{P}}{1/2} = -kt + C$$

At $t = 0$, $P = 90,000$ so

$$2\sqrt{90,000} = -k \times 0 + C \implies C = 2 \times 300 = 600 \implies P(t) = \frac{1}{4}(600-kt)^2$$

The constant of proportionality is determined by

$$\begin{aligned}P(6) = 40,000 \implies 40,000 = \frac{1}{4}(600-6k)^2 \implies 200 = \frac{1}{2}(600-6k) \implies 600-6k = 400 \\ \implies k = \frac{200}{6}\end{aligned}$$

Subbing in the value of k, $P(t) = \frac{1}{4}\big(600 - \frac{200}{6}t\big)^2$ so that $P(t) = 10,000$ when

$$10,000 = \frac{1}{4}\Big(600 - \frac{200}{6}t\Big)^2 \implies \frac{1}{2}\Big(600 - \frac{200}{6}t\Big) = 100$$
$$\implies t = 12 \text{ weeks}$$

S-15: The given differential equation is separable and we solve it accordingly.

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -(mg + kv^2) \implies \frac{\mathrm{d}v}{\mathrm{d}t} = -g\Big(1 + \frac{k}{mg}v^2\Big) \implies \int \frac{\mathrm{d}v}{1 + \frac{k}{mg}v^2} = -g\int \mathrm{d}t$$

Substitute $u = \sqrt{\frac{k}{mg}}\,v$, $du = \sqrt{\frac{k}{mg}}\,\mathrm{d}v$

$$\sqrt{\frac{mg}{k}}\int \frac{du}{1+u^2} = -g\int \mathrm{d}t \implies \sqrt{\frac{mg}{k}}\tan^{-1} u = -gt + C$$
$$\implies \sqrt{\frac{mg}{k}}\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,v\right) = -gt + C$$

At $t = 0$, $v = v_0$ so $C = \sqrt{\frac{mg}{k}}\tan^{-1}\big(\sqrt{\frac{k}{mg}}\,v_0\big)$. At its highest point, the object has $v = 0$. This happens when t obeys

$$\sqrt{\frac{mg}{k}}\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,0\right) = -gt + C \implies -gt + C = 0 \implies t = \frac{C}{g} = \sqrt{\frac{m}{kg}}\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right)$$

S-16: (a) The given differential equation is separable and we solve it accordingly.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -k\,v^2 \implies \frac{\mathrm{d}v}{v^2} = -k\,\mathrm{d}t \implies \int \frac{\mathrm{d}v}{v^2} = -\int k\,\mathrm{d}t \implies -\frac{1}{v} = -kt + C$$

At $t = 0$, $v = 40$ so

$$-\frac{1}{40} = -k \times 0 + C \implies C = -\frac{1}{40} \implies v(t) = \frac{1}{kt - C} = \frac{1}{kt + 1/40} = \frac{40}{40kt + 1}$$

The constant of proportionality is determined by

$$v(10) = 20 \implies 20 = \frac{40}{40k \times 10 + 1} \implies \frac{1}{2} = \frac{1}{400k + 1} \implies 400k + 1 = 2$$
$$\implies k = \frac{1}{400}$$

(b) Subbing in the value of k,

$$v(t) = \frac{40}{40kt + 1} = \frac{40}{t/10 + 1} \implies v(t) = 5 \text{ when } 5 = \frac{40}{t/10 + 1} \implies \frac{t}{10} + 1 = 8$$
$$\implies t = 70\text{sec}$$

S-17: (a) The given differential equation is separable and we solve it accordingly.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(3-x)(2-x) \iff \frac{\mathrm{d}x}{(x-2)(x-3)} = k\mathrm{d}t \iff \int \left[\frac{1}{x-3} - \frac{1}{x-2}\right] \mathrm{d}x = \int k\mathrm{d}t$$
$$\iff \log\left|\frac{x-3}{x-2}\right| = kt + C$$
$$\iff \frac{x-3}{x-2} = De^{kt}$$

where $D = \pm e^C$. When $t = 0$, $x = 1$, forcing

$$\frac{1-3}{1-2} = De^0 \implies D = 2$$

Hence

$$\frac{x-3}{x-2} = 2e^{kt} \iff x-3 = 2e^{kt}(x-2) \iff x - 2e^{kt}x = 3 - 4e^{kt} \iff x(t) = \frac{3-4e^{kt}}{1-2e^{kt}}$$

(b)

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{3-4e^{kt}}{1-2e^{kt}} = \lim_{t\to\infty} \frac{3e^{-kt}-4}{e^{-kt}-2} = \frac{-4}{-2} = 2$$

S-18: (a) The given differential equation is separable and we solve it accordingly.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 4P - P^2 \implies \frac{\mathrm{d}P}{4P-P^2} = \mathrm{d}t \implies \frac{\mathrm{d}P}{P(4-P)} = \mathrm{d}t \implies \frac{1}{4}\left[\frac{1}{P} + \frac{1}{4-P}\right]\mathrm{d}P = \mathrm{d}t$$
$$\implies \frac{1}{4}\Big[\log|P| - \log|4-P|\Big] = t + C$$

When $t = 0$, $P = 2$, so $\frac{1}{4}\big[\log|2| - \log|2|\big] = C \implies C = 0$. So

$$\frac{1}{4}\log\left|\frac{P}{4-P}\right| = t$$

At time $t = 0$, $\frac{P}{4-P} = 1 > 0$. The ratio may not change sign at any finite time, because this could only happen if at some finite time P took either the value 0 or the value 4. But at this time $t = \frac{1}{4}\log\left|\frac{P}{4-P}\right|$ would have to be infinite. So $\frac{P}{4-P} > 0$ for all time and

$$\frac{1}{4}\log\frac{P}{4-P} = t \implies \log\frac{P}{4-P} = 4t \implies \frac{P}{4-P} = e^{4t}$$
$$\implies P = (4-P)e^{4t} \implies P + Pe^{4t} = 4e^{4t}$$
$$\implies P = \frac{4e^{4t}}{1+e^{4t}} = \frac{4}{1+e^{-4t}}$$

(b) At $t = \frac{1}{2}$, $P = \frac{4}{1+e^{-2}} = 3.523$,. As $t \to \infty$, $e^{-4t} \to 0$ and $P \to 4$.

S-19: The rate of change of speed at time t is $-kv(t)^2$ for some constant of proportionality k (to be determined). So $v(t)$ obeys the differential equation $\boxed{\dfrac{dv}{dt} = -kv^2}$. This is a separable differential equation, which we can solve in the usual way.

$$\frac{dv}{dt} = -kv^2 \implies \frac{dv}{v^2} = -k\,dt \implies \int \frac{dv}{v^2} = -kt + c \implies -\frac{1}{v} = c - kt$$

At time $t = 0$, $v = 400$, so $c = -\frac{1}{400}$ and $\frac{1}{v} = \frac{1}{400} + kt$ or $v = \frac{400}{400kt+1}$. At time $t = 1$, $v = 200$, so

$$200 = \frac{400}{400k+1} \implies 400k + 1 = 2 \implies k = \frac{1}{400} \implies \boxed{v = \frac{400}{t+1}}$$

The speed is 50 when $\frac{400}{t+1} = 50$ or $t + 1 = 8$ or $\boxed{t = 7}$.

S-20: (a) The given differential equation is separable and we solve it accordingly.

$$\begin{aligned}
\frac{dB}{dt} = (0.06 + 0.02\sin t)B \implies& \frac{dB}{B} = (0.06 + 0.02\sin t)\,dt \\
\implies& \log B(t) = 0.06t - 0.02\cos t + C' \\
\implies& B(t) = Ce^{0.06t - 0.02\cos t}
\end{aligned}$$

for arbitrary constants C' and $C = e^{C'} \geqslant 0$. (Note that the function $B(t) = 0$ obeys the differential equation so that $C = 0$ is allowed, even though it is not of the form $C = e^{C'}$.)

(b) We are told that

$$B(0) = 1000 \implies Ce^{0.06\times 0 - 0.02\cos 0} = Ce^{-0.02} = 1000 \implies C = 1000e^{0.02}$$

so that

$$B(2) = 1000e^{0.02}e^{0.06\times 2 - 0.02\cos 2} = \boxed{\$1159.89}$$

to the nearest cent. Note that $\cos 2$ is the cosine of 2 radians.

S-21: (a) The given differential equation is separable and we could solve it accordingly. In fact we have already done so. If we rewrite the equation in the form

$$\frac{dB}{dt} = a\Big(B - \frac{m}{a}\Big)$$

it is of the form covered by Theorem 2.4.4 in the CLP 101 notes. So that theorem tells us that the solution is

$$B(t) = \Big\{B(0) - \frac{m}{a}\Big\}\,e^{at} + \frac{m}{a}$$

In this problem we are told that $a = 0.02 = \frac{1}{50}$, so

$$B(t) = \{B(0) - 50m\}\,e^{t/50} + 50m = \{30000 - 50m\}\,e^{t/50} + 50m$$

(b) The solution of part (a) is independent of time if and only if $30000 - 50m = 0$. So we need

$$m = \frac{30000}{50} = \$600$$

S-22: By the Fundamental Theorem of Calculus

$$y'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \big(y(t)^2 - 3y(t) + 2\big)\sin t\, \mathrm{d}t = \big(y(x)^2 - 3y(x) + 2\big)\sin x$$

So $y(x)$ satisfies the differential equation $y' = \big(y^2 - 3y + 2\big)\sin x = (y-2)(y-1)\sin x$ and the initial equation $y(0) = 3$ (just substitute $x = 0$ into $(*)$). For $y \neq 1, 2$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = (y-2)(y-1)\sin x \iff & \frac{\mathrm{d}y}{(y-2)(y-1)} = \sin x\, \mathrm{d}x \\
\iff & \int \Big[\frac{1}{y-2} - \frac{1}{y-1}\Big]\mathrm{d}y = \int \sin x\, \mathrm{d}x \\
\iff & \log|y-2| - \log|y-1| = -\cos x + c \\
\iff & \left|\frac{y-2}{y-1}\right| = e^{c - \cos x}
\end{aligned}$$

The condition $y(0) = 3$ forces $\left|\frac{3-2}{3-1}\right| = e^{c-1}$ or $e^c = \frac{1}{2}e$ and $\left|\frac{y-2}{y-1}\right| = \frac{1}{2}e^{1-\cos x}$. Observe that, when $x = 0$, $\frac{y-2}{y-1} = \frac{1}{2} > 0$. Furthermore $\frac{1}{2}e^{1-\cos x}$, and hence $\left|\frac{y-2}{y-1}\right|$, can never take the value zero. As $y(x)$ varies continuously with x, $y(x)$ must remain larger than 2. Consquently, $\frac{y-2}{y-1}$ remains positive and we may drop the absolute value signs. Hence

$$\frac{y-2}{y-1} = \frac{1}{2}e^{1-\cos x}$$

Solving for y,

$$\begin{aligned}
\frac{y-2}{y-1} = \frac{1}{2}e^{1-\cos x} \iff & 2(y-2) = e^{1-\cos x}(y-1) \iff y\big(2 - e^{1-\cos x}\big) = 4 - e^{1-\cos x} \\
\iff & y = \frac{4 - e^{1-\cos x}}{2 - e^{1-\cos x}}
\end{aligned}$$

To avoid division by zero in the last step, we need

$$e^{1-\cos x} < 2 \iff 1 - \cos x < \log 2 \iff \cos x > 1 - \log 2$$

The largest allowed interval is $-\cos^{-1}(1 - \log 2) < x < \cos^{-1}(1 - \log 2) \approx 1.259$. As x approachs the end points of this interval, $e^{1-\cos x}$ approachs 2 and y approachs infinity.

S-23: Suppose that in a very short time interval $\mathrm{d}t$, the height of water in the tank changes by $\mathrm{d}h$ (which is negative). Then in this time interval the amount of the water in the tank decreases by $-\pi(3)^2\mathrm{d}h$. This must be the same as the amount of water that flows through the hole in this time interval, which is $\pi(0.01)^2 v(t)\,\mathrm{d}t = \pi(0.01)^2\sqrt{2gh(t)}\,\mathrm{d}t$. Thus

$$-\pi(3)^2\mathrm{d}h = \pi(0.1)^2\sqrt{2gh(t)}\,\mathrm{d}t \iff \frac{\mathrm{d}h}{h^{1/2}} = -\Big(\frac{0.01}{3}\Big)^2\sqrt{2g}\,\mathrm{d}t$$
$$\iff 2\sqrt{h} = -\Big(\frac{0.01}{3}\Big)^2\sqrt{2g}\,t + C$$

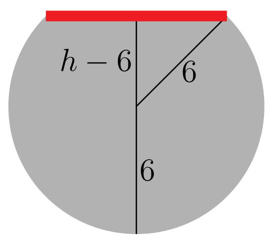
At time 0, the height is 6, so $C = 2\sqrt{6}$ and

$$2\sqrt{h} = -\Big(\frac{0.01}{3}\Big)^2\sqrt{2g}\,t + 2\sqrt{6}$$

The height drops to zero at time t obeying

$$0 = -\Big(\frac{0.01}{3}\Big)^2\sqrt{2g}\,t + 2\sqrt{6} \iff t = 2\Big(\frac{3}{0.01}\Big)^2\sqrt{\frac{3}{g}} = 180,000\sqrt{\frac{3}{g}} \approx 99,591\text{sec} \approx 27.66\text{hr}$$

S-24: Suppose that at time t, the mercury in the tank has height h, which is between 0 and 12 feet. At that time, the top surface of the mercury forms a circular disk of radius



$\sqrt{6^2-(h-6)^2}$. Now suppose that in a very short time interval $\mathrm{d}t$, the height of mercury in the tank changes by $\mathrm{d}h$ (which is negative). Then in this time interval the amount of the mercury in the tank decreases by $-\pi\big(\sqrt{6^2-(h-6)^2}\,\big)^2\mathrm{d}h$. (That's the volume of the red disk in the figure above.) This must be the same as the amount of mercury that flows through the hole in this time interval, which is, $\pi\big(\frac{1}{12}\big)^2 v\,\mathrm{d}t = \pi\big(\frac{1}{12}\big)^2\sqrt{2gh}\,\mathrm{d}t$. Thus

$$-\pi\big(\sqrt{6^2-(h-6)^2}\,\big)^2\mathrm{d}h = \pi\Big(\frac{1}{12}\Big)^2\sqrt{2gh}\,\mathrm{d}t \iff \big(h^2-12h\big)\mathrm{d}h = \frac{1}{144}\sqrt{2gh}\,\mathrm{d}t$$
$$\iff \big(h^{3/2}-12h^{1/2}\big)\mathrm{d}h = \frac{1}{144}\sqrt{2g}\,\mathrm{d}t$$
$$\iff \frac{h^{5/2}}{5/2} - 12\frac{h^{3/2}}{3/2} = \frac{1}{144}\sqrt{2g}\,t + C$$

At time 0, the height is 12, so $C = \frac{12^{5/2}}{5/2} - 12\frac{12^{3/2}}{3/2} = 12^{5/2}\big(\frac{2}{5}-\frac{2}{3}\big) = -\frac{4}{15}12^{5/2}$ and

$$\frac{h^{5/2}}{5/2} - 12\frac{h^{3/2}}{3/2} = \frac{1}{144}\sqrt{2g}\,t - \frac{4}{15}12^{5/2}$$

The height drops to zero at time t obeying

$$0 = \frac{1}{144}\sqrt{2g}\,t - \frac{4}{15}12^{5/2} \iff t = \frac{4\times 144}{15}\sqrt{\frac{12^5}{2g}} = 38.4\sqrt{\frac{124416}{g}} \approx 2,394\,\text{sec} \approx 0.665\,\text{hr}$$

S-25: (a) Setting $x = 0$ gives

$$f(0) = 3 + \int_0^0 \big(f(t) - 1\big)\big(f(t) - 2\big)\ \mathrm{d}t = 3$$

(b) By the fundamental theorem of calculus

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \big(f(t) - 1\big)\big(f(t) - 2\big)\ \mathrm{d}t = \big(f(x) - 1\big)\big(f(x) - 2\big)$$

Thus $y = f(x)$ obeys the differential equation $y' = (y-1)(y-2)$.

(c) If $y \neq 1, 2$,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = (y-1)(y-2) \iff \frac{\mathrm{d}y}{(y-1)(y-2)} = \mathrm{d}x &\iff \int \Big[\frac{1}{y-2} - \frac{1}{y-1}\Big]\mathrm{d}y = \int \mathrm{d}x \\ &\iff \log|y-2| - \log|y-1| = x + C \end{aligned}$$

Observe that $\frac{\mathrm{d}y}{\mathrm{d}x} = (y-1)(y-2) > 0$ for all $y \geqslant 2$. That is, $f(x)$ is increasing at all x for which $f(x) > 2$. As $f(0) = 3$, $f(x)$ increases for all $x \geqslant 0$ and $f(x) \geqslant 3$ for all $x \geqslant 0$. So we may drop the absolute value signs.

$$\log \frac{f(x) - 2}{f(x) - 1} = x + C \iff \frac{f(x) - 2}{f(x) - 1} = e^C e^x$$

At $x = 0$, $\frac{f(x)-2}{f(x)-1} = \frac{1}{2}$ so $e^C = \frac{1}{2}$ and

$$\begin{aligned} \frac{f(x) - 2}{f(x) - 1} = \frac{1}{2} e^x &\iff 2f(x) - 4 = [f(x) - 1]e^x \iff [2 - e^x]f(x) = 4 - e^x \\ &\iff f(x) = \frac{4 - e^x}{2 - e^x} \end{aligned}$$

S-26: Suppose that at time t (measured in hours starting at noon), the water in the tank has height y, which is between 0 and 2m. At that time, the top surface of the water forms a circular disk of radius $r = y^p$ and area $A(y) = \pi y^{2p}$. Thus, by Torricelli's law,

$$\begin{aligned} \pi y^{2p} \frac{\mathrm{d}y}{\mathrm{d}t} = -c\sqrt{y} &\iff -\frac{\pi}{c} y^{2p - \frac{1}{2}} \mathrm{d}y = \mathrm{d}t \\ &\iff -\frac{\pi}{c} \frac{y^{2p+\frac{1}{2}}}{2p + \frac{1}{2}} + d = t \end{aligned}$$

At time 0, the height is 2, so $d = \frac{\pi}{c} \frac{2^{2p+\frac{1}{2}}}{2p+\frac{1}{2}}$ and

$$t = \frac{\pi}{c} \left(\frac{2^{2p+\frac{1}{2}}}{2p + \frac{1}{2}} - \frac{y^{2p+\frac{1}{2}}}{2p + \frac{1}{2}} \right)$$

The time at which the height is 1 is obtained by subbing $y = 1$ into this formula and the time at which the height is 0 is obtained by subbing $y = 0$ into this formula. Thus the condition that the top half ($y = 2$ to $y = 1$) takes exactly the same amount of time to drain as the bottom half ($y = 1$ to $y = 0$) is

$$\frac{\pi}{c}\left(\frac{2^{2p+\frac{1}{2}}}{2p+\frac{1}{2}} - \frac{1^{2p+\frac{1}{2}}}{2p+\frac{1}{2}}\right) = \frac{\pi}{c}\left(\frac{2^{2p+\frac{1}{2}}}{2p+\frac{1}{2}} - \frac{0^{2p+\frac{1}{2}}}{2p+\frac{1}{2}}\right) - \frac{\pi}{c}\left(\frac{2^{2p+\frac{1}{2}}}{2p+\frac{1}{2}} - \frac{1^{2p+\frac{1}{2}}}{2p+\frac{1}{2}}\right)$$

or

$$\left(2^{2p+\frac{1}{2}} - 1^{2p+\frac{1}{2}}\right) = \left(2^{2p+\frac{1}{2}} - 0^{2p+\frac{1}{2}}\right) - \left(2^{2p+\frac{1}{2}} - 1^{2p+\frac{1}{2}}\right)$$

or

$$2^{2p+\frac{1}{2}} = 2 \implies 2p + \frac{1}{2} = 1 \implies p = \frac{1}{4}$$

Solutions to Exercises 3.1 — Jump to TABLE OF CONTENTS

S-1: Since $\left|\sin^3 k\right| \leqslant 1$ and $(k+1)! = (k+1)k!$,

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k!\sin^3 k}{(k+1)!} = \lim_{k\to\infty} \frac{\sin^3 k}{k+1} = 0$$

S-2: As $n \to \infty$ we have $\frac{1}{n} \to 0$ and hence $\sin\frac{1}{n} \to 0$. So the sequence $(-1)^n \sin\frac{1}{n}$ converges to 0.

S-3:

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{6n^2+5n}{n^2+1} + 3\cos(1/n^2)\right] &= \lim_{n\to\infty}\frac{6+\frac{5}{n}}{1+\frac{1}{n^2}} + 3\lim_{n\to\infty}\cos(1/n^2) = \frac{6+0}{1+0} + 3\cos(0)\\ &= 9\end{aligned}$$

S-4: Write $\frac{1}{n} = x$. Then

$$\begin{aligned}\log\left(\sin\frac{1}{n}\right) + \log(2n) &= \log\left(\sin\frac{1}{n}\right) + \log(n) + \log(2) = \log\left(n\sin\frac{1}{n}\right) + \log(2)\\ &= \log(2) + \log\left(\frac{\sin x}{x}\right)\end{aligned}$$

Note that $x \to 0$ as $n \to \infty$. Since, by l'Hôpital,

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$$

we have

$$\lim_{n\to\infty}\left\{\log\left(\sin\frac{1}{n}\right)+\log(2n)\right\}=\lim_{x\to 0}\left\{\log(2)+\log\left(\frac{\sin x}{x}\right)\right\}$$
$$=\log(2)+\log\left(\lim_{x\to 0}\frac{\sin x}{x}\right)$$
$$=\log(2)+\log(1)=\log(2)$$

S-5: (a), (c) Write $f(x)=x-\sqrt{3+\sin x}$. We want to show that $f(x)$ has exactly one zero. Because $-1\leqslant\sin x\leqslant 1$, we have that $\sqrt{2}\leqslant\sqrt{3+\sin x}\leqslant\sqrt{4}=2$. So if $x<\sqrt{2}$, we have $f(x)<0$ and if $x>2$, we have that $f(x)>0$. As $f(x)$ is continuous, it must take the value 0 at least once (for some x between $\sqrt{2}$, where $f(x)\leqslant 0$, and 2, where $f(x)\geqslant 0$). Since

$$f'(x)=1-\frac{\cos x}{2\sqrt{3+\sin x}}$$

and

$$\left|\frac{\cos x}{2\sqrt{3+\sin x}}\right|\leqslant\frac{1}{2\sqrt{2}}<1$$

we necessarily have $f'(x)>0$ for all x. That is, $f(x)$ is a strictly increasing function, and so can take the values 0 for at most one value of x.

(b) Subtracting $L=\sqrt{3+\sin L}$ from $a_{n+1}=\sqrt{3+\sin a_n}$ gives

$$a_{n+1}-L=\sqrt{3+\sin a_n}-\sqrt{3+\sin L}=\left[\sqrt{3+\sin a_n}-\sqrt{3+\sin L}\right]\frac{\sqrt{3+\sin a_n}+\sqrt{3+\sin L}}{\sqrt{3+\sin a_n}+\sqrt{3+\sin L}}$$
$$=\frac{\sin a_n-\sin L}{\sqrt{3+\sin a_n}+\sqrt{3+\sin L}}$$

Now

$$\big|\sin a_n-\sin L\big|=\left|\int_L^{a_n}\cos t\,\mathrm{d}t\right|\leqslant|a_n-L|\qquad\text{since }|\cos t|\leqslant 1$$

and

$$\sqrt{3+\sin a_n}+\sqrt{3+\sin L}\geqslant\sqrt{2}+\sqrt{2}\qquad\text{since }\sin a_n,\sin L\geqslant -1$$

so that

$$\big|a_{n+1}-L\big|=\left|\frac{\sin a_n-\sin L}{\sqrt{3+\sin a_n}+\sqrt{3+\sin L}}\right|\leqslant\frac{|a_n-L|}{2\sqrt{2}}$$

Thus the distance from a_n to L decreases by a factor of at least $\frac{1}{2\sqrt{2}}$ every time the index n increases by one and

$$\lim_{n\to\infty}\big|a_n-L\big|=0$$

Solutions to Exercises 3.2 — Jump to TABLE OF CONTENTS

S-1: This series is $\dfrac{1}{8^7}+\dfrac{1}{8^8}+\dfrac{1}{8^9}+\cdots$. We recognize that this is a geometric series $a+ar+ar^2+ar^3+\cdots$, with ratio $r=1/8$ and first term $a=\frac{1}{8^7}$. We know this converges to:

$$\frac{a}{1-r}=\frac{1/8^7}{1-1/8}=\frac{1}{7\times 8^6}$$

S-2: We recognize that this is a geometric series $a+ar+ar^2+ar^3+\cdots$, with ratio $r=1/3$ and first term $a=1$. We know this converges to:

$$\frac{a}{1-r}=\frac{1}{1-1/3}=\frac{3}{2}$$

S-3: We recognize this as a telescoping series. When we compute the n^{th} partial sum, i.e. the sum of of the first n terms, successive terms cancel and only the first half of the first term, $\left(\frac{6}{k^2}-\frac{6}{(k+1)^2}\right)\Big|_{k=1}$, and the second half of the n^{th} term, $\left(\frac{6}{k^2}-\frac{6}{(k+1)^2}\right)\Big|_{k=n}$, survive.

$$s_n=\sum_{k=1}^{n}\left(\frac{6}{k^2}-\frac{6}{(k+1)^2}\right)=\frac{6}{1^2}-\frac{6}{(n+1)^2}$$

Therefore, we can see directly that the sequence of partial sums $\{s_n\}$ is convergent:

$$\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(\frac{6}{1^2}-\frac{6}{(n+1)^2}\right)=6$$

By definition the series is also convergent with limit 6.

S-4: The Nth partial sum is

$$\begin{aligned}s_N=\sum_{n=3}^{N}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+1}\right)\right)&=\left(\cos\left(\frac{\pi}{3}\right)-\cos\left(\frac{\pi}{4}\right)\right)+\left(\cos\left(\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{5}\right)\right)\\&\qquad+\cdots+\left(\cos\left(\frac{\pi}{N}\right)-\cos\left(\frac{\pi}{N+1}\right)\right)\\&=\cos\left(\frac{\pi}{3}\right)-\cos\left(\frac{\pi}{N+1}\right).\end{aligned}$$

As $N\to\infty$, the argument $\frac{\pi}{N+1}$ converges to 0, and $\cos x$ is continuous at $x=0$. By definition, the value of the series is

$$\lim_{N\to\infty}s_N=\lim_{N\to\infty}\left(\cos\left(\frac{\pi}{3}\right)-\cos\left(\frac{\pi}{N+1}\right)\right)=\cos\left(\frac{\pi}{3}\right)-\cos(0)=-\frac{1}{2}$$

S-5: (a) Since

$$\begin{aligned} s_{n-1} &= a_1 + a_2 + \cdots + a_{n-1} \\ s_n &= a_1 + a_2 + \cdots + a_{n-1} + a_n \end{aligned}$$

we can find a_n by subtracting:

$$\begin{aligned} a_n &= s_n - s_{n-1} \\ &= \frac{1+3n}{5+4n} - \frac{1+3(n-1)}{5+4(n-1)} = \frac{3n+1}{4n+5} - \frac{3n-2}{4n+1} \\ &= \frac{(3n+1)(4n+1)-(3n-2)(4n+5)}{(4n+1)(4n+5)} \\ &= \frac{11}{16n^2+24n+5} \end{aligned}$$

(b) Since,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1+3n}{5+4n} = \lim_{n\to\infty} \frac{1/n+3}{5/n+4} = \frac{0+3}{0+4} = \frac{3}{4}$$

the series converges to $\frac{3}{4}$, by definition.

S-6:

$$\sum_{n=2}^{\infty} \frac{3\cdot 4^{n+1}}{8\cdot 5^n} = \frac{3}{2}\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^n = \frac{3}{2}\sum_{n=2}^{\infty} r^n\Big|_{r=4/5}$$

This is a geometric series whose first term (in this case the $n=2$ term) is $a = \frac{3}{2}r^2$ and whose ratio $r = \frac{4}{5}$. So

$$\sum_{n=2}^{\infty} \frac{3\cdot 4^{n+1}}{8\cdot 5^n} = \frac{3}{2}\frac{r^2}{1-r}\Big|_{r=4/5} = \frac{3}{2}\frac{16/25}{1/5} = \frac{24}{5}$$

S-7: The number is $0.2 + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10^4} + \cdots = \frac{1}{5} + \frac{3}{10^2}\sum_{n=0}^{\infty} 10^{-n}$. The geometric series sums to

$$\frac{3}{10^2}\frac{1}{1-\frac{1}{10}} = \frac{3}{10(10-1)} = \frac{1}{30}$$

so the fraction is

$$\frac{1}{5} + \frac{1}{30} = \frac{7}{30}$$

S-8: The number is $2 + \frac{65}{10^2} + \frac{65}{10^4} + \frac{65}{10^6} + \cdots = 2 + \frac{65}{10^2} \sum_{n=0}^{\infty} 10^{-2n}$. The geometric series sums to

$$\frac{65}{10^2} \frac{1}{1 - \frac{1}{100}} = \frac{65}{99}$$

so the fraction is

$$2 + \frac{65}{99} = \frac{263}{99}$$

S-9: The number

$$0.\overline{321} = 0.321321321\ldots = \frac{321}{1000} + \frac{321}{10^6} + \frac{321}{10^9} + \cdots = \sum_{n=1}^{\infty} 321 \times 10^{-3n}$$

This is a geometric series with first term $a = \frac{321}{1000}$ and ratio $r = \frac{1}{1000}$ and so sums to

$$\frac{a}{1-r} = \frac{321}{1000} \frac{1}{1 - \frac{1}{1000}} = \frac{321}{1000} \frac{1}{\frac{999}{1000}} = \frac{321}{999} = \frac{107}{333}$$

S-10: We split the sum into two parts. The first part is a geometric series with first term $a = \frac{2^{2+1}}{3^2} = \frac{8}{9}$ and ratio $r = \frac{2}{3}$.

$$\sum_{n=2}^{\infty} \frac{2^{n+1}}{3^n} = \frac{a}{1-r} = \frac{8/9}{1-2/3} = \frac{8}{3}$$

The rest telescopes.

$$\sum_{n=2}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) = \overbrace{\left(\frac{1}{3} - \frac{1}{5} \right)}^{n=2} + \overbrace{\left(\frac{1}{5} - \frac{1}{7} \right)}^{n=3} + \overbrace{\left(\frac{1}{7} - \frac{1}{9} \right)}^{n=4} + \cdots = \frac{1}{3}$$

So

$$\sum_{n=2}^{\infty} \left(\frac{2^{n+1}}{3^n} + \frac{1}{2n-1} - \frac{1}{2n+1} \right) = \frac{8}{3} + \frac{1}{3} = 3$$

S-11: We split the sum into two parts.

$$\sum_{n=1}^{\infty} \left[\left(\frac{1}{3} \right)^n + \left(-\frac{2}{5} \right)^{n-1} \right] = \sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^n + \sum_{n=1}^{\infty} \left(-\frac{2}{5} \right)^{n-1}$$

Both are geometric series. The first has first term $a = \left(\frac{1}{3}\right)^{n=1} = \frac{1}{3}$ and ratio $\frac{1}{3}$. The second has first term $a = \left(-\frac{2}{5}\right)^{n-1}\big|_{n=1} = 1$ and ratio $-\frac{2}{5}$. So

$$\sum_{n=1}^{\infty} \left[\left(\frac{1}{3} \right)^n + \left(-\frac{2}{5} \right)^{n-1} \right] = \frac{1/3}{1-1/3} + \frac{1}{1-(-2/5)} = \frac{1}{2} + \frac{5}{7} = \frac{17}{14}$$

S-12: We split the sum into two parts.

$$\sum_{n=0}^{\infty} \frac{1+3^{n+1}}{4^n} = \sum_{n=0}^{\infty} \frac{1}{4^n} + \sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n}$$

Both are geometric series. The first has first term $a = \frac{1}{4^n}\Big|_{n=0} = 1$ and ratio $r = \frac{1}{4}$. The second has first term $a = \frac{3^{n+1}}{4^n}\Big|_{n=0} = 3$ and ratio $r = \frac{3}{4}$. So

$$\sum_{n=0}^{\infty} \frac{1+3^{n+1}}{4^n} = \frac{1}{1-1/4} + \frac{3}{1-3/4} = \frac{4}{3} + 12 = 13\frac{1}{3}$$

Solutions to Exercises 3.3 — Jump to TABLE OF CONTENTS

S-1: The limit

$$\lim_{n\to\infty} \frac{n^2}{3n^2+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{3+1/(n\sqrt{n})} = \frac{1}{3} \neq 0$$

is nonzero, so the series diverges by the divergence test.

S-2: When n is very large, the term 2^n dominates the numerator, and the term 3^n dominates the denominator. So when n is very large $a_n \approx \frac{2^n}{3^n}$. Therefore we should take $b_n = \dfrac{2^n}{3^n}$. Note that, with this choice of b_n,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n+n}{3^n+1}\,\frac{3^n}{2^n} = \lim_{n\to\infty} \frac{1+n/2^n}{1+1/3^n} = 1$$

as desired.

S-3: (a) In general false. The harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges by the p–test with $p = 1$.

(b) Be careful. You were not told that the a_n's are positive. So this is false in general. If $a_n = (-1)^n \frac{1}{n}$, then $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ is again the harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$, which diverges.

(c) In general false. Take, for example, $a_n = 0$ and $b_n = 1$.

S-4: This precise question was asked on a 2014 final exam. Note that the n^{th} term in the series is $a_n = \frac{5^k}{4^k+3^k}$ and does *not* depend on n! There are two possibilities. Either this was intentional (and the instructor was being particularly nasty) or it was a typo and the intention was to have $a_n = \frac{5^n}{4^n+3^n}$. In both cases, the limit

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5^k}{4^k+3^k} = \frac{5^k}{4^k+3^k} \neq 0$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5^n}{4^n+3^n} = \lim_{n\to\infty} \frac{(5/4)^n}{1+(3/4)^n} = +\infty \neq 0$$

is nonzero, so the series diverges by the divergence test.

S-5: Let $f(x) = \dfrac{1}{x+\frac{1}{2}}$. Then $f(x)$ is positive and decreases as x increases. So, by the integral test, which is Theorem 3.3.5 in the CLP 101 notes, the given series converges if and only if the integral $\int_0^\infty \frac{1}{x+\frac{1}{2}}\,\mathrm{d}x$ converges. Since

$$\begin{aligned}\int_0^\infty \frac{1}{x+\frac{1}{2}}\,\mathrm{d}x = \lim_{R\to\infty}\int_0^R \frac{1}{x+\frac{1}{2}}\,\mathrm{d}x &= \lim_{R\to\infty}\left\{\log\left(x+\frac{1}{2}\right)\Big|_{x=0}^{x=R}\right\}\\ &= \lim_{R\to\infty}\left\{\log\left(R+\frac{1}{2}\right)-\log\frac{1}{2}\right\}\end{aligned}$$

diverges, the series diverges.

S-6: Let $f(x) = \dfrac{5}{x(\log x)^{3/2}}$. Then $f(x)$ is positive and decreases as x increases. So the sum $\displaystyle\sum_3^\infty f(n)$ and the integral $\displaystyle\int_3^\infty f(x)\,\mathrm{d}x$ either both converge or both diverge, by the integral test, which is Theorem 3.3.5 in the CLP 101 notes. For the integral, we use the substitution $u = \log x$, $\mathrm{d}u = \frac{\mathrm{d}x}{x}$ to get

$$\int_3^\infty \frac{5\,\mathrm{d}x}{x(\log x)^{3/2}} = \int_{\log 3}^\infty \frac{5\,\mathrm{d}u}{u^{3/2}}$$

which converges by the p–test (which is Example 1.12.8 in the CLP 101 notes) with $p = \frac{3}{2} > 1$.

S-7: Since

$$\int_2^\infty \frac{1}{x(\log x)^p}\mathrm{d}x = \lim_{R\to\infty}\int_2^R \frac{1}{(\log x)^p}\frac{\mathrm{d}x}{x} = \lim_{R\to\infty}\int_{\log 2}^{\log R}\frac{1}{u^p}\mathrm{d}u \quad \text{with } u=\log x,\ \mathrm{d}u = \frac{\mathrm{d}x}{x}$$

converges if and only if $p > 1$, the same is true for the series, by the integral test, which is Theorem 3.3.5 in the CLP 101 notes.

S-8: Set $f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}}$. For $x \geqslant 1$, this function is positive and decreasing (since it is the product of the two positive decreasing functions $e^{-\sqrt{x}}$ and $\frac{1}{\sqrt{x}}$). We use the integral test with this finction. Using the substitution $u = \sqrt{x}$, so that $\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$, we see that

$$\begin{aligned}
\int_1^\infty f(x)\,\mathrm{d}x &= \lim_{R\to\infty} \int_1^R \left(\frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x \right) \\
&= \lim_{R\to\infty} \left(\int_1^{\sqrt{R}} e^{-u} \cdot 2\,\mathrm{d}u \right) \\
&= \lim_{R\to\infty} \left(-2e^{-u} \Big|_1^{\sqrt{R}} \right) \\
&= \lim_{R\to\infty} \left(-2e^{-\sqrt{R}} + 2e^{-\sqrt{1}} \right) = 0 + 2e^{-1},
\end{aligned}$$

and so this improper integral converges. By the integral test, the given series also converges.

S-9: We first develop some intuition. For very large n, $3n^2$ dominates 7 so that

$$\frac{\sqrt{3n^2-7}}{n^3} \approx \frac{\sqrt{3n^2}}{n^3} = \frac{\sqrt{3}}{n^2}$$

The series $\sum\limits_{n=2}^{\infty} \frac{1}{n^2}$ converges by the p–test with $p = 2$, so we expect the given series to converge too.

To verify that our intuition is correct, it suffices to observe that

$$0 < a_n = \frac{\sqrt{3n^2-7}}{n^3} < \frac{\sqrt{3n^2}}{n^3} = \frac{\sqrt{3}}{n^2} = c_n$$

for all $n \geqslant 2$. As the series $\sum\limits_{n=2}^{\infty} c_n$ converges, the comparison test says that $\sum\limits_{n=2}^{\infty} a_n$ converges too.

S-10: We first develop some intuition. For very large k, k^4 dominates 1 so that the numerator $\sqrt[3]{k^4+1} \approx \sqrt[3]{k^4} = k^{4/3}$, and k^5 dominates 9 so that the denominator $\sqrt{k^5+9} \approx \sqrt{k^5} = k^{5/2}$ and the summand

$$\frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}} \approx \frac{k^{4/3}}{k^{5/2}} = \frac{1}{k^{7/6}}$$

The series $\sum\limits_{n=1}^{\infty} \frac{1}{k^{7/6}}$ converges by the p–test with $p = \frac{7}{6} > 1$, so we expect the given series to converge too.

To verify that our intuition is correct, we apply the limit comparison test with

$$a_k = \frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}} \quad \text{and} \quad b_k = \frac{1}{k^{7/6}} = \frac{k^{4/3}}{k^{5/2}}$$

which is valid since

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt[3]{k^4+1}/k^{4/3}}{\sqrt{k^5+9}/k^{5/2}} = \lim_{k\to\infty} \frac{\sqrt[3]{1+1/k^4}}{\sqrt{1+9/k^5}} = 1$$

exists. Since the series $\sum_{k=1}^{\infty} b_k$ is a convergent p–series (with ratio $p = \frac{7}{6} > 1$), the given series converges.

S-11: We first develop some intuition. For very large n, $2n$ dominates 7 so that

$$\frac{n^4 2^{n/3}}{(2n+7)^4} \approx \frac{n^4 2^{n/3}}{(2n)^4} = \frac{1}{16} 2^{n/3}$$

The series $\sum_{n=1}^{\infty} 2^{n/3}$ is a geometric series with ratio $r = 2^{1/3} > 1$ and so diverges. We expect the given series to diverge too.

To verify that our intuition is correct, we apply the limit comparison test with

$$a_n = \frac{n^4 2^{n/3}}{(2n+7)^4} \quad \text{and} \quad b_n = 2^{n/3}$$

which is valid since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4}{(2n+7)^4} = \lim_{n\to\infty} \frac{1}{(2+7/n)^4} = \frac{1}{2^4}$$

exists and is nonzero. Since the series $\sum_{n=1}^{\infty} b_n$ is a divergent geometric series (with ratio $r = 2^{1/3} > 1$), the given series diverges.

(It is possible to use the plain comparison test as well. One needs to show something like $a_n = \frac{n^4 2^{n/3}}{(2n+7)^4} \geqslant \frac{n^4 2^{n/3}}{(2n+7n)^4} = \frac{1}{9^4} b_n$.)

Alternatively, one could show

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^{n/3}}{(2+7/n)^4} = \infty$$

directly, and thus conclude that the given series diverges by the divergence test.

Alternatively, one can apply the ratio test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^4 2^{(n+1)/3}/(2(n+1)+7)^4}{n^4 2^{n/3}/(2n+7)^4}\right| \\
&= \lim_{n\to\infty}\frac{(n+1)^4(2n+7)^4}{n^4(2n+9)^4}\,\frac{2^{(n+1)/3}}{2^{n/3}} \\
&= \lim_{n\to\infty}\frac{(1+1/n)^4(2+7/n)^4}{(2+9/n)^4}\cdot 2^{1/3} = 1\cdot 2^{1/3} > 1.
\end{aligned}$$

S-12: We first develop some intuition. For very large n, n^2 dominates $\sin n$ and n^6 dominates n^2 so that

$$\frac{n^2-\sin n}{n^6+n^2} \approx \frac{n^2}{n^6} = \frac{1}{n^4}$$

The series $\sum_{n=1}^{\infty}\frac{1}{n^4}$ converges by the p–test with $p=4>1$. We expect the given series to converge too.

To verify that our intuition is correct, we apply the limit comparison test with

$$a_n = \frac{n^2-\sin n}{n^6+n^2} \quad \text{and} \quad b_n = \frac{1}{n^4}$$

which is valid since

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{(n^2-\sin n)}{n^6+n^2}\frac{n^4}{1} = \lim_{n\to\infty}\frac{n^6-n^4\sin n}{n^6+n^2} = \lim_{n\to\infty}\frac{1-n^{-2}\sin n}{1+n^{-4}} = 1$$

exists and is nonzero. Since the series $\sum_{n=1}^{\infty} b_n$ converges, the given series converges absolutely.

S-13: You might think that this series converges by the alternating series test. But you would be wrong. The problem is that the n^{th} term does not converge to zero as $n\to\infty$, so that the series actually diverges by the divergence test. To verify that the n^{th} term does not converge to zero as $n\to\infty$ let's write $a_n = \frac{(2n)!}{(n^2+1)(n!)^2}$ (i.e. a_n is the n^{th} term without the sign) and check to see whether a_{n+1} is bigger than or smaller than a_n.

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(2n+2)!}{((n+1)^2+1)((n+1)!)^2}\,\frac{(n^2+1)(n!)^2}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2}\,\frac{n^2+1}{(n+1)^2+1} \\
&= \frac{2(2n+1)}{(n+1)}\,\frac{1+1/n^2}{(1+1/n)^2+1/n^2} = 4\frac{1+1/2n}{1+1/n}\,\frac{1+1/n^2}{(1+1/n)^2+1/n^2}
\end{aligned}$$

So

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 4$$

and, in particular, for large n, $a_{n+1} > a_n$. Thus, for large n, a_n increases with n and so cannot converge to 0. So the series diverges by the divergence test.

S-14: This series converges by the alternating series test. It also converges absolutely by the integral test. For the details, see Example 3.3.7 (with $p = 101 > 1$) in the CLP101 notes.

S-15: (a) For large n, $n^2 \gg 1$ and so $\sqrt{n^2+1} \approx \sqrt{n^2} = n$. This suggests that we apply the limit comparison test with $a_n = \frac{1}{\sqrt{n^2+1}}$ and $b_n = \frac{1}{n}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\sqrt{n^2+1}}{1/n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^2}} = 1$$

and since $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, the given series diverges.

(b) Since $\cos(n\pi) = (-1)^n$, the given series converges by the alternating series test. To check that $a_n = \frac{n}{2^n}$ decreases to 0 as n tends to infinity, note that

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)2^{-(n+1)}}{n2^{-n}} = \Big(1+\frac{1}{n}\Big)\frac{1}{2}$$

is smaller than 1 (so that $a_{n+1} \leqslant a_n$) for all $n \geqslant 1$, and is smaller than $\frac{3}{4}$ (so $a_{n+1} \leqslant \frac{3}{4}a_n$) for all $n \geqslant 2$.

S-16: For large k, $k^4 \gg 2k^3 - 2$ and $k^5 \gg k^2 + k$ so $\frac{k^4-2k^3+2}{k^5+k^2+k} \approx \frac{k^4}{k^5} = \frac{1}{k}$. This suggests that we apply the limit comparison test with $a_k = \frac{k^4-2k^3+2}{k^5+k^2+k}$ and $b_k = \frac{1}{k}$. Since

$$\begin{aligned}\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{k^4-2k^3+2}{k^5+k^2+k}\,\frac{k}{1} = \lim_{k\to\infty} \frac{k^5-2k^4+k^2}{k^5+k^2+k} = \lim_{k\to\infty} \frac{1-2/k+1/k^3}{1+1/k^3+1/k^4}\\ &= 1\end{aligned}$$

and since $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges, by the p–test with $p = 1$, the given series diverges.

S-17: (a) For large n, $n^2 \gg n+1$ and so the numerator $n^2+n+1 \approx n^2$. For large n, $n^5 \gg n$ and so the denominator $n^5 - n \approx n^5$. So, for large n, $\frac{n^2+n+1}{n^5-n} \approx \frac{n^2}{n^5} = \frac{1}{n^3}$. This suggests that we apply the limit comparison test with $a_n = \frac{n^2+n+1}{n^5-n}$ and $b_n = \frac{1}{n^3}$. Since

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{(n^2+n+1)/(n^5-n)}{1/n^3} = \lim_{n\to\infty} \frac{n^5+n^4+n^3}{n^5-n} = \lim_{n\to\infty} \frac{1+1/n+1/n^2}{1-1/n^4}\\ &= 1\end{aligned}$$

and since $\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ converges, by the p–test with $p = 3 > 1$, the given series converges.

(b) For large m, $3m \gg |\sin\sqrt{m}|$ and so $\frac{3m+\sin\sqrt{m}}{m^2} \approx \frac{3m}{m^2} = \frac{3}{m}$. This suggests that we apply the limit comparison test with $a_m = \frac{3m+\sin\sqrt{m}}{m^2}$ and $b_m = \frac{1}{m}$. (We could also use $b_m = \frac{3}{m}$.)

Since

$$\lim_{m\to\infty}\frac{a_m}{b_m}=\lim_{m\to\infty}\frac{(3m+\sin\sqrt{m})/m^2}{1/m}=\lim_{m\to\infty}\frac{3m+\sin\sqrt{m}}{m}=\lim_{m\to\infty}3+\frac{\sin\sqrt{m}}{m}$$
$$=3$$

and since $\sum\limits_{m=1}^{\infty}\frac{1}{m}$ diverges, by the p–test with $p=1$, the given series diverges.

S-18: This is a geometric series with first term $a=\frac{6}{7^2}$ and ratio $r=\frac{1}{7}$. As $|r|<1$, the series converges and takes the value

$$\frac{a}{1-r}=\frac{6/7^2}{1-1/7}=\frac{6/7^2}{6/7}=\frac{1}{7}$$

S-19: (a) The given series is

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots=\sum_{n=1}^{\infty}a_n \text{ with } a_n=\frac{1}{2n-1}$$

First we'll develop some intuition by observing that, for very large n, $a_n\approx\frac{1}{2n}$. We know that the series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ diverges by the p–test with $p=1$. So let's apply the limit comparison test with $b_n=\frac{1}{n}$. Since

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n}{2n-1}=\lim_{n\to\infty}\frac{1}{2-\frac{1}{n}}=\frac{1}{2}$$

the series $\sum\limits_{n=1}^{\infty}a_n$ converges if and only if the series $\sum\limits_{n=1}^{\infty}b_n$ converges. So the given series diverges.

(a, again) The series

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots\geqslant\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\cdots$$
$$=\frac{1}{2}\Big(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots\Big)$$

The series in the brackets is the harmonic series which we know diverges, by the p–test with $p=1$. So the series on the right hand side diverges. By the convergence test, the series on the left hand side diverges too.

(b) We'll use the ratio test with $a_n=\frac{(2n+1)}{2^{2n+1}}$. Since

$$\frac{a_{n+1}}{a_n}=\frac{(2n+3)}{2^{2n+3}}\frac{2^{2n+1}}{(2n+1)}=\frac{1}{4}\frac{(2n+3)}{(2n+1)}=\frac{1}{4}\frac{(2+3/n)}{(2+1/n)}\to\frac{1}{4}<1 \text{ as } n\to\infty$$

the series converges.

S-20: (a) For very large k, $k \ll k^2$ so that $a_n = \frac{\sqrt[3]{k}}{k^2-k} \approx \frac{\sqrt[3]{k}}{k^2} = \frac{1}{k^{5/3}}$. So we apply the limiting convergence test with $b_k = \frac{1}{k^{5/3}}$. Since

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt[3]{k}/(k^2-k)}{1/k^{5/3}} = \lim_{k\to\infty} \frac{k^2}{k^2-k} = \lim_{k\to\infty} \frac{1}{1-1/k} = 1$$

and $\sum\limits_{k=1}^{\infty} \frac{1}{k^{5/3}}$ converges by the p–test with $p = \frac{5}{3} > 1$, the given series converges by the limiting comparison test.

(b) The k^{th} term in this series is $a_k = \frac{k^{10}10^k(k!)^2}{(2k)!}$. So

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{(k+1)^{10}10^{k+1}((k+1)!)^2}{(2k+2)!} \frac{(2k)!}{k^{10}10^k(k!)^2} = 10\Big(\frac{k+1}{k}\Big)^{10} \frac{(k+1)^2}{(2k+2)(2k+1)} \\
&= 10\Big(1+\frac{1}{k}\Big)^{10} \frac{(1+1/k)^2}{(2+2/k)(2+1/k)}
\end{aligned}$$

As k tends to ∞, this converges to $10 \times 1 \times \frac{1}{2\times 2} > 1$. So the series diverges by the ratio test.

(c) We'll use the integal test. The k^{th} term in the series is $a_k = \frac{1}{k(\log k)(\log\log k)} = f(k)$ with $f(x) = \frac{1}{x(\log x)(\log\log x)}$, which is continuous, positive and decreasing for $x \geqslant 3$. Since

$$\begin{aligned}
\int_3^\infty f(x)\,\mathrm{d}x = \int_3^\infty \frac{\mathrm{d}x}{x(\log x)(\log\log x)} &= \lim_{R\to\infty} \int_3^R \frac{\mathrm{d}x}{x(\log x)(\log\log x)} \\
&= \lim_{R\to\infty} \int_{\log 3}^{\log R} \frac{\mathrm{d}y}{y\log y} \qquad \text{with } y = \log x,\ \mathrm{d}y = \frac{\mathrm{d}x}{x} \\
&= \lim_{R\to\infty} \int_{\log\log 3}^{\log\log R} \frac{\mathrm{d}t}{t} \qquad \text{with } t = \log y,\ \mathrm{d}t = \frac{\mathrm{d}y}{y} \\
&= \Big[\log t\Big]_{\log\log 3}^{\log\log R}
\end{aligned}$$

diverges as $R \to \infty$, the series is divergent.

S-21: For large n, the numerator $n^3 - 4 \approx n^3$ and the denominator $2n^5 - 6n \approx 2n^5$, so the nth term is approximately $\frac{n^3}{2n^5} = \frac{1}{2n^2}$. So we apply the limit comparison test with $a_n = \frac{n^3-4}{2n^5-6n}$ and $b_n = \frac{1}{n^2}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^3-4)/(2n^5-6n)}{1/n^2} = \lim_{n\to\infty} \frac{1-\frac{4}{n^3}}{2-\frac{6}{n^4}} = \frac{1}{2},$$

the given series $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if the series $\sum\limits_{n=1}^{\infty} b_n$ converges. Since the series $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \frac{1}{n^p}\Big|_{p=2}$ is a convergent p-series, both series converge.

S-22: By the alternating series test, the error introduced when we approximate the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 10^n}$ by $\sum_{n=1}^{N} \frac{(-1)^n}{n \cdot 10^n}$ is at most the magnitude of the first omitted term $\frac{1}{(N+1)10^{(N+1)}}$. By trial and error, we find that this expression becomes smaller than 10^{-6} when $N+1 \geqslant 6$. So the smallest allowable value is $N = 5$.

S-23: The sequence $\{\frac{1}{n^2}\}$ decreases to zero as n increases to infinity. So, by the alternating series error bound, which is given in Theorem 3.3.14 in the CLP 101 notes, $\frac{\pi^2}{12} - S_N$ lies between zero and the first omitted term, $\frac{(-1)^N}{(N+1)^2}$. We therefore need $\frac{1}{(N+1)^2} \leqslant 10^{-6}$, which is equivalent to $N+1 \geqslant 10^3$ and $N \geqslant 999$.

S-24: The error introduced when we approximate S by the n^{th} partial sum S_n lies between 0 and the first term dropped, which is $\frac{(-1)^{n+2}}{(2n+3)^2}$. So we need the smallest positive integer n obeying

$$\frac{1}{(2n+3)^2} \leqslant \frac{1}{100} \iff (2n+3)^2 \geqslant 100 \iff 2n+3 \geqslant 10 \iff n \geqslant \frac{7}{2}$$

So we need $n = 4$ and then

$$S_4 = \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{9^2}$$

S-25: (a) There are plenty of powers/factorials. So let's try the ratio test with $a_n = \frac{n^n}{9^n n!}$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{9^{n+1}(n+1)!} \frac{9^n n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n\, 9\, (n+1)} = \lim_{n\to\infty} \frac{(1+1/n)^n}{9} = \frac{e}{9}$$

Here we have used that $\lim_{n\to\infty} (1+1/n)^n = e$. See Example **??** in the CLP100 notes, with $x = \frac{1}{n}$ and $a = 1$. As $e < 9$, our series converges.

(b) We know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the p–test with $p = 2$, and also that $\log n \geqslant 2$ for all $n \geqslant e^2$. So let's use the limit comparison test with $a_n = \frac{1}{n^{\log n}}$ and $b_n = \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^{\log n}} \frac{n^2}{1} = \lim_{n\to\infty} \frac{1}{n^{\log n - 2}} = 0$$

So our series converges, by the limit comparison test.

S-26: (a)

- Our first task is to identify the potential sources of impropriety for this integral.
- The domain of integration extends to $+\infty$. On the domain of integration the denominator is never zero so the integrand is continuous. Thus the only problem is at $+\infty$.

- Our second task is to develop some intuition about the behavior of the integrand for very large x. When x is very large:
  - $|\sin x| \leqslant 1 \ll x$, so that the numerator $x + \sin x \approx x$, and
  - $1 \ll x^2$, so that denominator $1 + x^2 \approx x^2$, and
  - the integrand $\frac{x+\sin x}{1+x^2} \approx \frac{x}{x^2} = \frac{1}{x}$
- Now, since $\int_2^\infty \frac{\mathrm{d}x}{x}$ diverges, we would expect $\int_2^\infty \frac{x+\sin x}{1+x^2}\,\mathrm{d}x$ to diverge too.
- Our final task is to verify that our intuition is correct. To do so, we set

$$f(x) = \frac{x + \sin x}{1 + x^2} \qquad\qquad g(x) = \frac{1}{x}$$

  and compute

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x + \sin x}{1 + x^2} \div \frac{1}{x} \\
&= \lim_{x\to\infty} \frac{(1 + \sin x/x)x}{(1/x^2 + 1)x^2} \times x \\
&= \lim_{x\to\infty} \frac{1 + \sin x/x}{1/x^2 + 1} \\
&= 1
\end{aligned}$$

- Since $\int_2^\infty g(x)\,\mathrm{d}x = \int_2^\infty \frac{\mathrm{d}x}{x}$ diverges, by Example 1.12.8 in the CLP 101 notes[3], with $p = 1$, Theorem 1.12.22(b) in the CLP 101 notes now tells us that $\int_2^\infty f(x)\,\mathrm{d}x = \int_2^\infty \frac{x+\sin x}{e^{-x}+x^2}\,\mathrm{d}x$ diverges too.

(a again) Since $\int_2^\infty \frac{1}{1+x^2}\,\mathrm{d}x \leqslant \int_2^\infty \frac{1}{x^2}\,\mathrm{d}x$ converges, by the p–test with $p = 2$, and $\frac{|\sin x|}{1+x^2} \leqslant \frac{1}{1+x^2}$, the integral $\int_2^\infty \frac{\sin x}{1+x^2}\,\mathrm{d}x$ converges. Hence $\int_2^\infty \frac{x+\sin x}{1+x^2}\,\mathrm{d}x$ converges if and only if $\int_2^\infty \frac{x}{1+x^2}\,\mathrm{d}x$ converges. But

$$\int_2^\infty \frac{x}{1+x^2}\,\mathrm{d}x = \lim_{r\to\infty} \int_2^r \frac{x}{1+x^2}\,\mathrm{d}x = \lim_{r\to\infty} \Big[\tfrac{1}{2}\log(1+x^2)\Big]_2^r$$

diverges, so $\int_2^\infty \frac{x+\sin x}{1+x^2}\,\mathrm{d}x$ diverges.

(b) The problem is that $f(x) = \frac{x+\sin x}{1+x^2}$ is not a decreasing function. To see this, compute the derivative:

$$f'(x) = \frac{(1+\cos x)(1+x^2) - (x+\sin x)(2x)}{(1+x^2)^2} = \frac{(\cos x - 1)x^2 - 2x\sin x + 1 + \cos x}{(1+x^2)^2}$$

If $x = 2m\pi$, the numerator is $0 - 0 + 1 + 1 > 0$.

(c) Set $a_n = \frac{n+\sin n}{1+n^2}$. We first try to develop some intuition about the behaviour of a_n for large n and then we confirm that our intuition was correct.

3 To change the lower limit of integration from 1 to 2, just apply Theorem 1.12.20 in the CLP 101 notes.

- *Step 1: Develop intuition.* When $n \gg 1$, the numerator $n + \sin n \approx n$, and the denominator $1 + n^2 \approx n^2$ so that $a_n \approx \frac{n}{n^2} = \frac{1}{n}$ and it looks like our series should diverge by the p–test (Example 3.3.6 in the CLP 101 notes) with $p = 1$.
- *Step 2: Verify intuition.* To confirm our intuition we set $b_n = \frac{1}{n}$ and compute the limit
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+\sin n}{1+n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n[n+\sin n]}{1+n^2} = \lim_{n\to\infty} \frac{1+\frac{\sin n}{n}}{\frac{1}{n^2}+1} = 1$$
We already know that the series $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges by the p–test with $p = 1$. So our series diverges by the limit comparison test, Theorem 3.3.11 in the CLP 101 notes.

(c again) Since $\left|\frac{\sin n}{1+n^2}\right| \leqslant \frac{1}{n^2}$ and the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p–test with $p = 2$, the series $\sum\limits_{n=1}^{\infty} \frac{\sin n}{1+n^2}$ converges. Hence $\sum\limits_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$ converges if and only if the series $\sum\limits_{n=1}^{\infty} \frac{n}{1+n^2}$ converges. Now $f(x) = \frac{x}{1+x^2}$ is a continuous, positive, decreasing function on $[1, \infty)$ since

$$f'(x) = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

is negative for all $x > 1$. We saw in part (a) that the integral $\int_2^\infty \frac{x}{1+x^2}\,\mathrm{d}x$ diverges. So the integral $\int_1^\infty \frac{x}{1+x^2}\,\mathrm{d}x$ diverges too and the sum $\sum\limits_{n=1}^{\infty} \frac{n}{1+n^2}$ diverges by the integral test. So $\sum\limits_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$ diverges.

S-27: Note that $\frac{e^{-\sqrt{x}}}{\sqrt{x}}$ decreases as x increases. Hence, for every $n \geqslant 1$,

$$\frac{e^{-\sqrt{n}}}{\sqrt{n}} = \int_{n-1}^{n} \frac{e^{-\sqrt{n}}}{\sqrt{n}}\,\mathrm{d}x \leqslant \int_{n-1}^{n} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$$

and, for every $N \geqslant 1$,

$$E_N = \sum_{n=N+1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} \leqslant \sum_{n=N+1}^{\infty} \int_{n-1}^{n} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = \int_{N}^{N+1} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x + \int_{N+1}^{N+2} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x + \cdots$$
$$= \int_{N}^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$$

Substituting $y = \sqrt{x}$, $dy = \frac{1}{2}\frac{\mathrm{d}x}{\sqrt{x}}$,

$$\int_{N}^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = 2\int_{\sqrt{N}}^{\infty} e^{-y}\,dy = -2e^{-y}\Big|_{\sqrt{N}}^{\infty} = 2e^{-\sqrt{N}}$$

This shows that $\sum_{n=N+1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converges and is between 0 and $2e^{-\sqrt{N}}$. Since $E_{14} = 2e^{-\sqrt{14}} = 0.047$, we may truncate the series at $n = 14$.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} &= \sum_{n=1}^{14} \frac{e^{-\sqrt{n}}}{\sqrt{n}} + E_{14} \\
&= \quad 0.3679 + 0.1719 + 0.1021 + 0.0677 + 0.0478 \\
&\quad + 0.0352 + 0.0268 + 0.0209 + 0.0166 + 0.0134 \\
&\quad + 0.0109 + 0.0090 + 0.0075 + 0.0063 + E_{14} \\
&= 0.9042 + E_{14}
\end{aligned}$$

The sum is between 0.9035 and 0.9535. This even allows for a roundoff error of 0.00005 in each term.

S-28: Since $\sum\limits_{n=1}^{\infty} a_n$, converges a_n must converge to zero as $n \to \infty$. In particular there must be some integer N such that $\frac{1}{2} > a_n \geqslant 0$ for all $n > N$. Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a_n}{1-a_n} &= \sum_{n=1}^{N} \frac{a_n}{1-a_n} + \sum_{n=N+1}^{\infty} \frac{a_n}{1-a_n} \leqslant \sum_{n=1}^{N} \frac{a_n}{1-a_n} + \sum_{n=N+1}^{\infty} \frac{a_n}{1/2} \\
&= \sum_{n=1}^{N} \frac{a_n}{1-a_n} + 2\sum_{n=N+1}^{\infty} a_n \qquad \leqslant \sum_{n=1}^{N} \frac{a_n}{1-a_n} + 2\sum_{n=1}^{\infty} a_n
\end{aligned}$$

is finite.

S-29: By the divergence test, the fact that $\sum\limits_{n=0}^{\infty} (1-a_n)$ converges guarantees that $\lim\limits_{n\to\infty} (1-a_n) = 0$, or equivalently, that $\lim\limits_{n\to\infty} a_n = 1$. So, by the divergence test, a second time, the fact that

$$\lim_{n\to\infty} 2^n a_n = +\infty$$

guarantees that $\sum\limits_{n=0}^{\infty} 2^n a_n$ diverges too.

S-30: By the divergence test, the fact that $\sum\limits_{n=1}^{\infty} \dfrac{na_n - 2n + 1}{n+1}$ converges guarantees that $\lim\limits_{n\to\infty} \frac{na_n-2n+1}{n+1} = 0$, or equivalently, that

$$0 = \lim_{n\to\infty} \frac{n}{n+1} a_n - \lim_{n\to\infty} \frac{2n-1}{n+1} = \lim_{n\to\infty} a_n - 2 \iff \lim_{n\to\infty} a_n = 2$$

The series of interest can be written $-\log a_1 + \sum\limits_{n=1}^{\infty} \big[\log(a_n) - \log(a_{n+1})\big]$ which looks like a telescoping series. So we'll compute the partial sum

$$\begin{aligned}
S_N &= -\log a_1 + \sum_{n=1}^{N} \big[\log(a_n) - \log(a_{n+1})\big] \\
&= -\log a_1 + \big[\log(a_1) - \log(a_2)\big] + \big[\log(a_2) - \log(a_3)\big] + \cdots + \big[\log(a_N) - \log(a_{N+1})\big] \\
&= -\log(a_{N+1})
\end{aligned}$$

and then take the limit $N \to \infty$

$$-\log a_1 + \sum_{n=1}^{\infty} \big[\log(a_n) - \log(a_{n+1})\big] = \lim_{N\to\infty} S_N = -\lim_{N\to\infty} \log(a_{N+1}) = -\log 2 = \log \frac{1}{2}$$

S-31: We are told that $\sum_{n=1}^{\infty} a_n$ converges. Thus we must have that $\lim_{n\to\infty} a_n = 0$. In particular, there is an index N such that $0 \leqslant a_n \leqslant 1$ for all $n \geqslant N$. Then $a_n^2 \leqslant a_n$ for all $n \geqslant N$ and

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{N-1} a_n^2 + \sum_{n=N}^{\infty} a_n^2 \leqslant \sum_{n=1}^{N-1} a_n^2 + \sum_{n=N}^{\infty} a_n < \infty$$

Thus $\sum_{n=1}^{\infty} a_n^2$ converges.

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S-1: False. For example if $b_n = \frac{1}{n}$, then $\sum\limits_{n=1}^{\infty} (-1)^{n+1} b_n = \sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges by the alternating series test, but $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges by the p–test.

S-2: The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{9n+5}$ converges by the alternating series test. On the other hand the series $\sum\limits_{n=1}^{\infty} \left|\frac{(-1)^n}{9n+5}\right| = \sum_{n=1}^{\infty} \frac{1}{9n+5}$ diverges by the limiting comparison test with $b_n = \frac{1}{n}$. So the given series is conditionally convergent.

S-3: Note that $(-1)^{2n+1} = (-1) \cdot (-1)^{2n} = -1$. So we can simplify

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{1+n} = -\sum_{n=1}^{\infty} \frac{1}{1+n}$$

Since $\dfrac{1}{1+n} \geqslant \dfrac{1}{n+n} = \dfrac{1}{2n}$, $\sum\limits_{n=1}^{\infty} \dfrac{1}{1+n}$ diverges by the comparison test with the divergent harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$. The extra overall factor of -1 in the original series does not change the conclusion of divergence.

S-4: Since

$$\lim_{n\to\infty}\frac{1+4^n}{3+2^{2n}}=\lim_{n\to\infty}\frac{1+4^n}{3+4^n}=1$$

the alternating series test cannot be used. Indeed, $\lim_{n\to\infty}(-1)^{n-1}\dfrac{1+4^n}{3+2^{2n}}$ does not exist (for very large n, $(-1)^{n-1}\frac{1+4^n}{3+2^{2n}}$ alternates between a number close to $+1$ and a number close to -1) so the divergence test says that the series diverges. (Note that "none of the above" cannot possibly be the correct answer — *every* series either converges absolutely, converges conditionally, or diverges.)

S-5: First, we'll develop some intuition. For very large n

$$\left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right|\approx\left|\frac{\sqrt{n}\cos(n)}{n^2}\right|=\left|\frac{\cos(n)}{n^{3/2}}\right|\leqslant\frac{1}{n^{3/2}}$$

since $|\cos(n)|\leqslant 1$ for all n. By the p–test, the series $\sum\limits_{n=5}^{\infty}\dfrac{1}{n^p}$ converges for all $p>1$. So we would expect the given series to converge absolutely.

Now, to confirm that our intuition is correct, we'll set $a_n=\left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right|$ and apply the limit comparison test with the comparison series having n^{th} term $b_n=\frac{1}{n^p}$. We'll choose a specific p shortly. Since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n}&=\lim_{n\to\infty}\frac{\left|\sqrt{n}\,\cos n/(n^2-1)\right|}{1/n^p}=\lim_{n\to\infty}\frac{n^{p+1/2}|\cos n|}{n^2(1-1/n^2)}=\lim_{n\to\infty}\frac{|\cos n|}{n^{3/2-p}(1-1/n^2)}\\&=0\quad\text{if }p<\frac{3}{2}\end{aligned}$$

the limit comparison test says that if $p<\frac{3}{2}$ and the series $\sum\limits_{n=5}^{\infty}b_n$ converges (which is the case if $p>1$) then the series $\sum\limits_{n=5}^{\infty}\left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right|$ also converges. So choosing any $1<p<\frac{3}{2}$, for example $p=\frac{5}{4}$, we conclude that the given series converges absolutely.

S-6: (a) We need to show that $\sum\limits_{n=1}^{\infty}24n^2e^{-n^3}$ converges. If we replace n by x in the summand, we get $f(x)=24x^2e^{-x^3}$, which we can integate. (Just substitute $u=x^3$.) So let's try the integral test. First, we have to check that $f(x)$ is positive and decreasing. It is certainly positive. To determine if it is dereasing, we compute

$$\frac{\mathrm{d}f}{\mathrm{d}x}=48xe^{-x^3}-24\times 3x^4e^{-x^3}=24x(2-3x^3)e^{-x^3}$$

which is negative for $x\geqslant 1$. Therefore $f(x)$ is decreasing for $x\geqslant 1$, and the integral test applies. The substitution $u=x^3$, $\mathrm{d}u=3x^2\,\mathrm{d}x$, yields

$$\int f(x)\,\mathrm{d}x=\int 24x^2e^{-x^3}\,\mathrm{d}x=\int 8e^{-u}\,\mathrm{d}u=-8e^{-u}+C=-8e^{-x^3}+C.$$

Therefore

$$\int_1^\infty f(x)\,\mathrm{d}x = \lim_{R\to\infty}\int_1^R f(x)\,\mathrm{d}x = \lim_{R\to\infty}\left[-8e^{-x^3}\right]_1^R$$
$$= \lim_{R\to\infty}(-8e^{-R^3}+8e^{-1}) = 8e^{-1}$$

Since the integral is convergent, the series $\sum\limits_{n=1}^{\infty} 24n^2e^{-n^3}$ converges and the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}24n^2e^{-n^3}$ converges absolutely.

Alternatively, we can use the ratio test with $a_n = 24n^2e^{-n^3}$. We calculate

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{24(n+1)^2e^{-(n+1)^3}}{24n^2e^{-n^3}}\right|$$
$$= \lim_{n\to\infty}\left(\frac{(n+1)^2}{n^2}\frac{e^{n^3}}{e^{(n+1)^3}}\right)$$
$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 e^{-(3n^2+3n+1)} = 1\cdot 0 = 0 < 1,$$

and therefore the series converges absolutely.

Alternatively, alternatively, we can use the limiting comparison test. First a little intuition building. Recall that we need to show that $\sum\limits_{n=1}^{\infty} 24n^2e^{-n^3}$ converges. The n^{th} term in this series is

$$a_n = 24n^2e^{-n^3} = \frac{24n^2}{e^{n^3}}$$

It is a ratio with both the numerator and denominator growing with n. A good rule of thumb is that exponentials grow a lot faster than powers. For example, if $n = 10$ the numerator is $2400 = 2.4\times 10^3$ and the denominator is about 2×10^{434}. So we would guess that a_n tends to zero as $n\to\infty$. The question is "does a_n tend to zero fast enough with n that our series converges?". For example, we know that $\sum_{n=1}^\infty \frac{1}{n^2}$ converges (by the p–test with $p=2$). So if a_n tends to zero faster than $\frac{1}{n^2}$ does, our series will converge. So let's try the limiting convergence test with $a_n = 24n^2e^{-n^3} = \frac{24n^2}{e^{n^3}}$ and $b_n = \frac{1}{n^2}$.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{24n^2e^{-n^3}}{1/n^2} = \lim_{n\to\infty}\frac{24n^4}{e^{n^3}}$$

By l'Hôpital's rule, twice,

$$\begin{aligned}
\lim_{x\to\infty}\frac{24x^4}{e^{x^3}} &= \lim_{x\to\infty}\frac{4\times 24x^3}{3x^2e^{x^3}} && \text{by l'Hôpital}\\
&= \lim_{x\to\infty}\frac{32x}{e^{x^3}} && \text{just cleaning up}\\
&= \lim_{x\to\infty}\frac{32}{3x^2e^{x^3}} && \text{by l'Hôpital, again}\\
&= 0
\end{aligned}$$

That's it. The limiting convergence test now tells us that $\sum_{n=1}^{\infty} a_n$ converges.

(b) In part (a) we saw that $24n^2e^{-n^3}$ is positive and decreasing. The limit of this sequence equals 0 (as can be shown with l'Hôpital's Rule, just as we did at the end of part (a)). Therefore, we can use the alternating series test, so that the error made in approximating the infinite sum $S = \sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty}(-1)^{n-1}24n^2e^{-n^3}$ by the sum, $S_N = \sum\limits_{n=1}^{N} a_n$, of its first N terms lies between 0 and the first omitted term, a_{N+1}. If we use 5 terms, the error satisfies

$$|S - S_5| \leqslant |a_6| = 24\times 36e^{-6^3}$$

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S-1: (a) We apply the ratio test for the series whose k^{th} term is $a_k = (-1)^k2^{k+1}x^k$. Then

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{(-1)^{k+1}2^{k+2}x^{k+1}}{(-1)^k2^{k+1}x^k}\right|\\
&= \lim_{k\to\infty}|2x| = |2x|
\end{aligned}$$

Therefore, by the ratio test, the series converges for all x obeying $|2x| < 1$, i.e. $|x| < \frac{1}{2}$, and diverges for all x obeying $|2x| > 1$, i.e. $|x| > \frac{1}{2}$. So the radius of convergence is $R = \frac{1}{2}$.

Alternatively, one can set $A_k = (-1)^k2^{k+1}$ and compute

$$A = \lim_{k\to\infty}\left|\frac{A_{k+1}}{A_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}2^{k+2}}{(-1)^k2^{k+1}}\right| = \lim_{k\to\infty} 2 = 2$$

so that $R = \frac{1}{A} = \frac{1}{2}$, again.

(b) The series is

$$\sum_{k=0}^{\infty}(-1)^k2^{k+1}x^k = 2\sum_{k=0}^{\infty}(-2x)^k = 2\sum_{k=0}^{\infty}r^k\Big|_{r=-2x} = 2\times\frac{1}{1-r} = \frac{2}{1+2x}$$

for all $|r| = |2x| < 1$, i.e. all $|x| < \frac{1}{2}$.

S-2: We apply the ratio test for the series whose k^{th} term is $a_k = \frac{x^k}{10^{k+1}(k+1)!}$. Then

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{x^{k+1}}{10^{k+2}(k+2)!}\ \frac{10^{k+1}(k+1)!}{x^k}\right| \\
&= \lim_{k\to\infty}\frac{1}{10(k+2)}|x| = 0 < 1
\end{aligned}$$

for all x. Therefore, by the ratio test, the series converges for all x and the radius of convergence is $R = \infty$.

Alternatively, one can set $A_k = \frac{1}{10^{k+1}(k+1)!}$ and compute $A = \lim_{k\to\infty}\left|\frac{A_{k+1}}{A_k}\right| = 0$, so that R is again $+\infty$.

S-3: We apply the ratio test with $a_n = \frac{(x-2)^n}{n^2+1}$. Since

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2+1}\frac{n^2+1}{(x-2)^n}\right| \\
&= \lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1}|x-2| \\
&= \lim_{n\to\infty}\frac{1+1/n^2}{(1+1/n)^2+1/n^2}|x-2| \\
&= |x-2|
\end{aligned}$$

the series converges if $|x-2| < 1$ and diverges if $|x-2| > 1$. So the radius of convergence is 1.

S-4: We apply the ratio test for the series whose n^{th} term is $a_n = \frac{(-1)^n(x+2)^n}{\sqrt{n}}$. Then

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{\sqrt{n+1}}\ \frac{\sqrt{n}}{(x+2)^n}\right| \\
&= \lim_{n\to\infty}|x+2|\ \frac{\sqrt{n}}{\sqrt{n+1}} \\
&= \lim_{n\to\infty}|x+2|\ \frac{1}{\sqrt{1+1/n}} \\
&= |x+2|
\end{aligned}$$

So the series must converge when $|x+2| < 1$ and must diverge when $|x+2| > 1$. When $x+2 = 1$, the series reduces to

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

which converges by the alternating series test. When $x+2 = -1$, the series reduces to

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

which diverges by the p–series test with $p = \frac{1}{2}$. So the interval of convergence is $-1 < x + 2 \leqslant 1$ or $(-3, -1]$.

S-5: We apply the ratio test for the series whose n^{th} term is $a_n = \frac{(-1)^n}{n+1}\left(\frac{x+1}{3}\right)^n$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{n+2}\left(\frac{x+1}{3}\right)^{n+1}}{\frac{(-1)^n}{n+1}\left(\frac{x+1}{3}\right)^n}\right| \\ &= \lim_{n\to\infty}\frac{n+1}{n+2}\left|\frac{x+1}{3}\right| = \frac{|x+1|}{3}.\end{aligned}$$

Therefore, by the ratio test, the series converges when $\frac{|x+1|}{3} < 1$ and diverges when $\frac{|x+1|}{3} > 1$. In particular, it converges when

$$|x+1| < 3 \iff -3 < x+1 < 3 \iff -4 < x < 2$$

and the radius of convergence is $R = 3$. (Alternatively, one can set $A_n = \frac{(-1)^n}{(n+1)3^n}$ and compute $A = \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \frac{1}{3}$, so that $R = \frac{1}{A} = 3$.)

Next, we consider the endpoints 2 and -4. At $x = 2$, i.e. $x + 1 = 3$, the series is simply $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$, which is an alternating series: the signs alternate, and the unsigned terms decrease to zero. Therefore the series converges at $x = 2$ by the alternating series test.

At $x = -4$ the series is

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left(\frac{-4+1}{3}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(-1)^n = \sum_{n=0}^{\infty}\frac{1}{n+1},$$

since $(-1)^n \cdot (-1)^n = (-1)^{2n} = \left((-1)^2\right)^n = 1$. This series diverges, either by comparison or limit comparison with the harmonic series (the p-series with $p = 1$). (For that matter, it is exactly equal to the standard harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, re-indexed to start at $n = 0$.)

In summary, the interval of convergence is $-4 < x \leqslant 2$, or simply $(-4, 2]$.

S-6: We first apply the ratio test with $a_n = \frac{(x-2)^n}{n^{4/5}(5^n-4)}$. Since

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^{4/5}(5^{n+1}-4)}\frac{n^{4/5}(5^n-4)}{(x-2)^n}\right| \\ &= \lim_{n\to\infty}\frac{n^{4/5}(5^n-4)}{(n+1)^{4/5}(5^{n+1}-4)}|x-2| \\ &= \lim_{n\to\infty}\frac{(1-4/5^n)}{(1+1/n)^{4/5}(5-4/5^n)}|x-2| \\ &= \frac{|x-2|}{5}\end{aligned}$$

the series converges if $|x-2| < 5$ and diverges if $|x-2| > 5$. When $x-2 = +5$, i.e. $x = 7$, the series reduces to $\sum_{n=1}^{\infty} \frac{5^n}{n^{4/5}(5^n-4)} = \sum_{n=1}^{\infty} \frac{1}{n^{4/5}(1-4/5^n)}$ which diverges by the limit comparison test with $b_n = \frac{1}{n^{4/5}}$. When $x-2 = -5$, i.e. $x = -3$, the series reduces to $\sum_{n=1}^{\infty} \frac{(-5)^n}{n^{4/5}(5^n-4)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{4/5}(1-4/5^n)}$ which converges by the alternating series test. So the interval of convergence is $-3 \leqslant x < 7$ or $[-3, 7)$.

S-7: We apply the ratio test with $a_n = \frac{(x+2)^n}{n^2}$. Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x+2)^{n+1}}{(n+1)^2}}{\frac{(x+2)^n}{n^2}}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2}|x+2| = \lim_{n\to\infty}\frac{1}{(1+1/n)^2}|x+2| = |x+2|$$

we have convergence for

$$|x+2| < 1 \iff -1 < x+2 < 1 \iff -3 < x < -1$$

and divergence for $|x+2| > 1$. For $|x+2| = 1$, i.e. for $x+2 = \pm 1$, i.e. for $x = -3, -1$, the series reduces to $\sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^2}$, which converges absolutely, because $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p = 2 > 1$. So the given series converges if and only if $-3 \leqslant x \leqslant -1$.

S-8: We apply the ratio test with $a_n = \frac{4^n}{n}(x-1)^n$. Since

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{4^{n+1}(x-1)^{n+1}/(n+1)}{4^n(x-1)^n/n}\right| \\ &= \lim_{n\to\infty} 4|x-1|\frac{n}{n+1} \\ &= 4|x-1|\lim_{n\to\infty}\frac{n}{n+1} = 4|x-1|\cdot 1.\end{aligned}$$

the series converges if

$$4|x-1| < 1 \iff -1 < 4(x-1) < 1 \iff -\frac{1}{4} < x-1 < \frac{1}{4} \iff \frac{3}{4} < x < \frac{5}{4}$$

and diverges if $4|x-1| > 1$. Checking the right endpoint $x = \frac{5}{4}$, we see that

$$\sum_{n=1}^{\infty}\frac{4^n}{n}\left(\frac{5}{4}-1\right)^n = \sum_{n=1}^{\infty}\frac{1}{n}$$

is the divergent harmonic series. At the left endpoint $x = \frac{3}{4}$,

$$\sum_{n=1}^{\infty}\frac{4^n}{n}\left(\frac{3}{4}-1\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

converges by the alternating series test. Therefore the interval of convergence of the original series is $\frac{3}{4} \leqslant x < \frac{5}{4}$, or $\left[\frac{3}{4}, \frac{5}{4}\right)$.

S-9: We apply the ratio test with $a_n = (-1)^n \frac{(x-1)^n}{2^n(n+2)}$. Since

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{2^{n+1}(n+3)}\,\frac{2^n(n+2)}{(x-1)^n}\right| \\
&= \lim_{n\to\infty}\frac{|x-1|}{2}\frac{n+2}{n+3} \\
&= \frac{|x-1|}{2}\lim_{n\to\infty}\frac{1+2/n}{1+3/n} = \frac{|x-1|}{2}
\end{aligned}$$

the series converges if

$$\frac{|x-1|}{2} < 1 \iff |x-1| < 2 \iff -2 < (x-1) < 2 \iff -1 < x < 3$$

and diverges if $|x-1| > 2$. So the series has radius of convergence 2. Checking the left endpoint $x = -1$, so that $\frac{x-1}{2} = -1$, we see that

$$\sum_{n=0}^{\infty}(-1)^n\frac{(-1-1)^n}{2^n(n+2)} = \sum_{n=0}^{\infty}\frac{1}{n+2}$$

is the divergent harmonic series. At the right endpoint $x = 3$, so that $\frac{x-1}{2} = +1$ and

$$\sum_{n=0}^{\infty}(-1)^n\frac{(3-1)^n}{2^n(n+2)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}$$

converges by the alternating series test. Therefore the interval of convergence of the original series is $-1 < x \leqslant 3$, or $\big(-1, 3\big]$.

S-10: We apply the ratio test with $a_n = (-1)^n n^2(x-a)^{2n}$. Since

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^2(x-a)^{2(n+1)}}{(-1)^n n^2(x-a)^{2n}}\right| \\
&= \lim_{n\to\infty}|x-a|^2\frac{(n+1)^2}{n^2} \\
&= |x-a|^2\lim_{n\to\infty}\big(1+{}^1/_n\big)^2 = |x-a|^2\cdot 1.
\end{aligned}$$

the series converges if

$$|x-a|^2 < 1 \iff |x-a| < 1 \iff -1 < x-a < 1 \iff a-1 < x < a+1$$

and diverges if $|x-a| > 1$. Checking both endpoints $x - a = \pm 1$, we see that

$$\sum_{n=1}^{\infty}(-1)^n n^2(x-a)^{2n}\Big|_{x-a=\pm 1} = \sum_{n=1}^{\infty}(-1)^n n^2$$

fails the divergence test — the n^{th} term does not converge to zero as $n\to\infty$. Therefore the interval of convergence of the original series is $a-1 < x < a+1$, or $\big(a-1, a+1\big)$.

S-11: (a) We apply the ratio test for the series whose k^{th} term is $A_k = \frac{(x+1)^k}{k^2 9^k}$. Then

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{A_{k+1}}{A_k}\right| &= \lim_{k\to\infty}\left|\frac{(x+1)^{k+1}}{(k+1)^2 9^{k+1}}\ \frac{k^2 9^k}{(x+1)^k}\right| \\
&= \lim_{k\to\infty}|x+1|\ \frac{1}{9}\ \frac{k^2}{(k+1)^2} \\
&= \lim_{k\to\infty}|x+1|\ \frac{1}{9}\ \frac{1}{(1+1/k)^2} \\
&= \frac{|x+1|}{9}
\end{aligned}$$

So the series must converge when $|x+1| < 9$ and must diverge when $|x+1| > 9$. When $x+1 = \pm 9$, the series reduces to

$$\sum_{k=1}^{\infty}\frac{(\pm 9)^k}{k^2 9^k} = \sum_{k=1}^{\infty}\frac{(\pm 1)^k}{k^2}$$

which converges (since, by the p–test, $\sum_{k=1}^{\infty}\frac{1}{k^p}$ converges for any $p > 1$). So the interval of covnergence is $|x+1| \leqslant 9$ or $-10 \leqslant x \leqslant 8$ or $[-10, 8]$.

(b) The partial sum

$$\sum_{k=1}^{N}\left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}}\right) = \left(\frac{a_1}{a_2} - \frac{a_2}{a_3}\right) + \left(\frac{a_2}{a_3} - \frac{a_3}{a_4}\right) + \cdots + \left(\frac{a_N}{a_{N+1}} - \frac{a_{N+1}}{a_{N+2}}\right) = \frac{a_1}{a_2} - \frac{a_{N+1}}{a_{N+2}}$$

We are told that $\sum\limits_{k=1}^{\infty}\left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}}\right) = \frac{a_1}{a_2}$. This means that the above partial sum converges to $\frac{a_1}{a_2}$ as $N \to \infty$, or equivalently, that

$$\lim_{N\to\infty}\frac{a_{N+1}}{a_{N+2}} = 0$$

and hence that

$$\lim_{k\to\infty}\frac{|a_{k+1}(x-1)^{k+1}|}{|a_k(x-1)^k|} = |x-1|\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|}$$

is infinite for any $x \neq 1$. So, by the ratio test, this series converges only for $x = 1$.

S-12: Using the geometric series $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$,

$$\frac{x^3}{1-x} = x^3\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+3} = \sum_{n=3}^{\infty} x^n$$

S-13: We apply the ratio test for the series whose n^{th} term is either $a_n = \frac{x^n}{3^{2n}\log n}$ or $a_n = \left|\frac{x^n}{3^{2n}\log n}\right|$. For both series

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{3^{2(n+1)}\log(n+1)}\ \frac{3^{2n}\log n}{x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{x\log n}{3^2\log(n+1)}\right| = \lim_{n\to\infty}\left|\frac{x\log n}{3^2[\log(n)+\log(1+1/n)]}\right| \\
&= \lim_{n\to\infty}\left|\frac{x}{3^2[1+\log(1+1/n)/\log(n)]}\right| \\
&= \frac{|x|}{9}
\end{aligned}$$

Therefore, by the ratio test, our series converges absolutely when $|x| < 9$ and diverges when $|x| > 9$.

For $x = -9$, $\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{3^{2n}\log n} = \sum_{n=2}^{\infty}\frac{(-1)^n}{\log n}$ which converges by the alternating series test.

For $x = +9$, $\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{3^{2n}\log n} = \sum_{n=2}^{\infty}\frac{1}{\log n}$ which is the same series as $\displaystyle\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{\log n}\right|$. We shall shortly show that $n \geqslant \log n$, and hence $\frac{1}{\log n} \geqslant \frac{1}{n}$ for all $n \geqslant 1$. This implies that the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\log n}$ diverges by comparison with the divergent series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^p}\bigg|_{p=1}$. This yelds both divergence for $x = 9$ and also the failure of absolute convergence for $x = -9$.

Finally, we show that $n - \log n > 0$, for all $n \geqslant 1$. Set $f(x) = x - \log x$. Then $f(1) = 1 > 0$ and

$$f'(x) = 1 - \frac{1}{x} \geqslant 0 \qquad \text{for all } x \geqslant 1$$

So $f(x)$ is (strictly) positive when $x = 1$ and is increasing for all $x \geqslant 1$. So $f(x)$ is (strictly) positive for all $x \geqslant 1$.

S-14: (a) Applying $\frac{1}{1+r} = \sum\limits_{n=0}^{\infty}(-1)^n r^n$, with $r = x^3$, gives

$$\int \frac{1}{1+x^3}\,\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n\int x^{3n}\,\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+1}}{3n+1} + C$$

(b) By part (a),

$$\int_0^{1/4}\frac{1}{1+x^3}\,\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+1}}{3n+1}\bigg|_0^{1/4} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(3n+1)4^{3n+1}}$$

This is an alternating series with successively smaller terms that converge to zero as $n \to \infty$. So truncating it introduces an error no larger than the magnitude of the first dropped term. We want that first dropped term to obey

$$\frac{1}{(3n+1)4^{3n+1}} < 10^{-5} = \frac{1}{10^5}$$

So let's check the first few terms.

$$\begin{aligned}
\left.\frac{1}{(3n+1)4^{3n+1}}\right|_{n=0} &= \frac{1}{4} > \frac{1}{10^5} \\
\left.\frac{1}{(3n+1)4^{3n+1}}\right|_{n=1} &= \frac{1}{4^5} > \frac{1}{10^5} \\
\left.\frac{1}{(3n+1)4^{3n+1}}\right|_{n=2} &= \frac{1}{7\times 4^7} = \frac{1}{7\times 2^{14}} = \frac{1}{7\times 16\times 1024} = \frac{1}{112\times 1024} < \frac{1}{10^5}
\end{aligned}$$

So we need to keep two terms (the $n = 0$ and $n = 1$ terms).

S-15: (a) Differentiating both sides of

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

gives

$$\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$

Now multiplying both sides by x gives

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

as desired.

(b) Differentiating both sides of the conclusion of part (a) gives

$$\sum_{n=0}^{\infty} n^2x^{n-1} = \frac{(1-x)^2 - 2x(x-1)}{(1-x)^4} = \frac{(1-x)(1-x+2x)}{(1-x)^4} = \frac{1+x}{(1-x)^3}$$

Now multiplying both sides by x gives

$$\sum_{n=0}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3}$$

We know that differentiation preserves the radius of convergence of power series. So this series has radius of convergence 1 (the radius of convergence of the original geometric series). At $x = \pm 1$ the series diverges by the divergence test. So the series converges for $-1 < x < 1$.

S-16: By the divergence test, the fact that $\sum_{n=0}^{\infty}(1-b_n)$ converges guarantees that $\lim_{n\to\infty}(1-b_n)=0$, or equivalently, that $\lim_{n\to\infty}b_n=1$. So, by equation (3.5.2) in the CLP 101 notes, the radius of convergence is

$$R=\left[\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|\right]^{-1}=\left[\frac{1}{1}\right]^{-1}=1 \tag{3.6.1}$$

S-17: (a) We know that the radius of convergence R obeys

$$\frac{1}{R}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{n}{n+1}\,\frac{(n+1)a_{n+1}}{na_n}=1\frac{C}{C}=1$$

because we are told that $\lim_{n\to\infty}na_n=C$. So $R=1$.

(b) Just knowing that the radius of convergence is 1, we know that the series converges for $|x|<1$ and diverges for $|x|>1$. That leaves $x\pm1$.

When $x=+1$, the series reduces to $\sum_{n=1}^{\infty}a_n$. We are told that na_n *decreases* to $C>0$. So $a_n\geqslant\frac{C}{n}$. By the comparison test with the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges by the p–test with $p=1$, our series diverges when $x=1$.

When $x=-1$, the series reduces to $\sum_{n=1}^{\infty}(-1)^n a_n$. We are told that na_n *decreases* to $C>0$. So $a_n>0$ and a_n converges to 0 as $n\to\infty$. Consequently $\sum_{n=1}^{\infty}(-1)^n a_n$ converges by the alternating series test.

In conclusion $\sum_{n=1}^{\infty}a_nx^n$ converges when $-1\leqslant x<1$.

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S-1: Substituting $y=3x$ into the exponential series

$$e^y=\sum_{n=0}^{\infty}\frac{y^n}{n!}$$

gives

$$e^{3x}=\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}=\sum_{n=0}^{\infty}\frac{3^n\,x^n}{n!}$$

so that c_5, the coefficient of x^5, which appears only in the $n=5$ term, is $c_5=\frac{3^5}{5!}$

S-2: We just need to substitute $y = x^3$ into the known Maclaurin series for $\sin y$, to get the Maclaurin series for $\sin(x^3)$, and then multiply the result by x^2.

$$\begin{aligned}
\sin y &= y - \frac{y^3}{3!} + \cdots \\
\sin(x^3) &= x^3 - \frac{x^9}{3!} + \cdots \\
x^2 \sin(x^3) &= x^5 - \frac{x^{11}}{3!} + \cdots
\end{aligned}$$

so $a = 1$ and $b = -\frac{1}{3!} = -\frac{1}{6}$.

S-3: Substituting $y = 2x$ into $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ gives

$$f(x) = \frac{1}{2x-1} = -\frac{1}{1-2x} = -\sum_{n=0}^{\infty} (2x)^n = -\sum_{n=0}^{\infty} 2^n x^n$$

S-4: Substituting first $y = -x$ and then $y = 2x$ into $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ gives

$$\begin{aligned}
\frac{1}{1-(-x)} &= \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \\
\frac{1}{1-(2x)} &= \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n
\end{aligned}$$

Hence

$$\begin{aligned}
f(x) &= \frac{3}{x+1} - \frac{1}{2x-1} = \frac{3}{1-(-x)} + \frac{1}{1-2x} = 3\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} 2^n x^n \\
&= \sum_{n=0}^{\infty} \big(3(-1)^n + 2^n\big) x^n
\end{aligned}$$

So $b_n = 3(-1)^n + 2^n$.

S-5: Recall that

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \cdots$$

Setting $y = -x^2$, we have

$$\begin{aligned}
e^{-x^2} &= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots \\
e^{-x^2} - 1 &= -x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots \\
\frac{e^{-x^2} - 1}{x} &= -x + \frac{x^3}{2} - \frac{x^5}{6} + \cdots \\
\int \frac{e^{-x^2} - 1}{x}\,\mathrm{d}x &= C - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{36} + \cdots
\end{aligned}$$

S-6: Recall that

$$\arctan(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{2n+1}$$

Setting $y = 2x$, we have

$$\begin{aligned}
\int x^4 \arctan(2x)\,\mathrm{d}x &= \sum_{n=0}^{\infty} (-1)^n \int x^4 \frac{(2x)^{2n+1}}{2n+1}\,\mathrm{d}x = \sum_{n=0}^{\infty} (-1)^n \int \frac{2^{2n+1} x^{2n+5}}{2n+1}\,\mathrm{d}x \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+6}}{(2n+1)(2n+6)} + C = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n+6}}{(2n+1)(n+3)} + C
\end{aligned}$$

S-7: Since

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log(1+2t) = \frac{2}{1+2t} = 2\sum_{n=0}^{\infty} (-2t)^n \qquad \text{if } |2t| < 1 \text{ i.e. } |t| < \frac{1}{2}$$

and $f(0) = 0$, we have

$$f(x) = \int_0^x f'(t)\,\mathrm{d}t = 2\sum_{n=0}^{\infty} \int_0^x (-1)^n 2^n t^n\,\mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n 2^{n+1} \frac{x^{n+1}}{n+1} \qquad \text{for all } |x| < \frac{1}{2}$$

S-8: Since $3^n = \left(\sqrt{3}\right)^{2n} = \frac{1}{\sqrt{3}}\left(\sqrt{3}\right)^{2n+1}$

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} &= \sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\left(\sqrt{3}\right)^{2n+1}} = \sqrt{3} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\bigg|_{x=\frac{1}{\sqrt{3}}} = \sqrt{3} \arctan \frac{1}{\sqrt{3}} \\
&= \sqrt{3}\,\frac{\pi}{6} = \frac{\pi}{2\sqrt{3}}
\end{aligned}$$

S-9: Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. So

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \Big[\sum_{n=0}^{\infty} \frac{x^n}{n!} \Big]_{x=-1} = \Big[e^x \Big]_{x=-1} = e^{-1}$$

S-10: Recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. So

$$\sum_{k=0}^{\infty} \frac{1}{e^k k!} = \Big[\sum_{k=0}^{\infty} \frac{x^k}{k!} \Big]_{x=1/e} = \Big[e^x \Big]_{x=1/e} = e^{1/e}$$

S-11: Recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. So

$$\sum_{k=0}^{\infty} \frac{1}{\pi^k k!} = \Big[\sum_{k=0}^{\infty} \frac{x^k}{k!} \Big]_{x=1/\pi} = \Big[e^x \Big]_{x=1/\pi} = e^{1/\pi}$$

This series differs from the given one only in that it starts with $k = 0$ while the given series starts with $k = 1$. So

$$\sum_{k=1}^{\infty} \frac{1}{\pi^k k!} = \sum_{k=0}^{\infty} \frac{1}{\pi^k k!} - \underbrace{1}_{k=0} = e^{1/\pi} - 1$$

S-12: Recall that

$$\log(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

(To get from the first sum to the second sum we substituted $n = k + 1$. If you don't see why the two sums are equal, write out the first few terms of each.) So

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n} = \Big[\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \Big]_{x=1/2} = \Big[\log(1+x) \Big]_{x=1/2} = \log(3/2)$$

S-13: Write

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n+2}{n!} e^n &= \sum_{n=1}^{\infty} \frac{n}{n!} e^n + \sum_{n=1}^{\infty} \frac{2}{n!} e^n \\
&= \sum_{n=1}^{\infty} \frac{e^n}{(n-1)!} + 2 \sum_{n=1}^{\infty} \frac{e^n}{n!} \\
&= e \sum_{n=1}^{\infty} \frac{e^{n-1}}{(n-1)!} + 2 \sum_{n=1}^{\infty} \frac{e^n}{n!} \\
&= e \sum_{n=0}^{\infty} \frac{e^n}{n!} + 2 \sum_{n=1}^{\infty} \frac{e^n}{n!}
\end{aligned}$$

Recall that $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$. So

$$\sum_{n=1}^{\infty} \frac{n+2}{n!} e^n = e\Big[\sum_{n=0}^{\infty} \frac{x^n}{n!}\Big]_{x=e} + 2\Big[\sum_{n=1}^{\infty} \frac{x^n}{n!}\Big]_{x=e} = e\Big[e^x\Big]_{x=e} + 2\Big[e^x - 1\Big]_{x=e} = e^{e+1} + 2(e^e - 1)$$
$$= (e+2)e^e - 2$$

S-14: Substituting $y = -3x^3$ into $\frac{1}{1-y} = \sum\limits_{n=0}^{\infty} y^n$ gives

$$\frac{\mathrm{d}f}{\mathrm{d}x} = x\frac{1}{1+3x^3} = x\sum_{n=0}^{\infty}\big(-3x^3\big)^n = \sum_{n=0}^{\infty}(-1)^n 3^n x^{3n+1}$$

Now integrating,

$$f(x) = \sum_{n=0}^{\infty}(-1)^n 3^n \frac{x^{3n+2}}{3n+2} + C$$

To have $f(0) = 1$, we need $C = 1$. So, finally

$$f(x) = 1 + \sum_{n=0}^{\infty}(-1)^n \frac{3^n}{3n+2} x^{3n+2}$$

S-15: (a) Using the geometric series expansion with $r = t^4$,

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \implies \frac{1}{1+t^4} = \sum_{n=0}^{\infty}\left(-t^4\right)^n$$

Substituting this into our integral,

$$I(x) = \int_0^x \frac{1}{1+t^4}\,dt = \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{4n}\,dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{4n+1}$$

(b) Substituting in $x = \frac{1}{2}$.

$$I(1/2) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(4n+1)2^{4n+1}}$$
$$= \tfrac{1}{2} - \frac{1}{5\times 2^5} + \frac{1}{9\times 2^9} - \frac{1}{13\times 2^{13}} + \cdots$$
$$= 0.5 - 0.00625 + 0.000217 - 0.0000094 + \cdots = \boxed{0.493967} - 0.0000094 + \cdots$$

See part (c) for the error analysis.

(c) The series for $I(x)$ is an alternating series (that is, the sign alternates) with successively smaller terms that converge to zero. So the error introduced by truncating the series is between zero and the first omitted term. In this case, the first omitted term was negative (-0.0000094). So the exact value of $I(1/2)$ is the approximate value found in part (b) plus a negative number whose magnitude is smaller than $0.00001 = 10^{-5}$. So the approximate value of part (b) is $\boxed{\text{larger}}$ than the true value of $I(1/2)$.

S-16: Expanding the exponential using its Maclaurin series,

$$\begin{aligned} I &= \int_0^1 x^4 e^{-x^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \int_0^1 x^4 \frac{(-x^2)^n}{n!}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n+4}\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+5)} = \underbrace{\frac{1}{5}}_{n=0} - \underbrace{\frac{1}{7}}_{n=1} + \underbrace{\frac{1}{18}}_{n=2} - \underbrace{\frac{1}{3!(11)}}_{n=3} + \cdots \end{aligned}$$

The signs of successive terms in this series alternate. Futhermore the magnitude of the n^{th} term decreases with n. Hence, by the alternating series test, I lies between $\frac{1}{5} - \frac{1}{7} + \frac{1}{18}$ and $\frac{1}{5} - \frac{1}{7} + \frac{1}{18} - \frac{1}{3!(11)}$. So

$$|I - a| \leqslant \frac{1}{3!(11)} = \frac{1}{66}$$

S-17: Expanding the exponential using its Taylor series,

$$\begin{aligned} I &= \int_0^{\frac{1}{2}} x^2 e^{-x^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \int_0^{\frac{1}{2}} x^2 \frac{(-x^2)^n}{n!}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{\frac{1}{2}} x^{2n+2}\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)} \frac{1}{2^{2n+3}} \end{aligned}$$

The signs of successive terms in this series alternate. Futhermore the magnitude of the n^{th} term decreases with n. Hence, by the alternating series test, I lies between $\sum\limits_{n=0}^{N} \frac{(-1)^n}{n!(2n+3)} \frac{1}{2^{2n+3}}$ and $\sum\limits_{n=0}^{N+1} \frac{(-1)^n}{n!(2n+3)} \frac{1}{2^{2n+3}}$, for every N. The first few terms are, to five decimal places,

n	0	1	2	3
$\frac{(-1)^n}{n!(2n+3)} \frac{1}{2^{2n+3}}$	0.04167	-0.00625	0.00056	-0.00004

Allowing for a roundoff error of 0.000005 in each of these, I must be between

$$0.04167 - 0.00625 + 0.00056 + 0.000005 \times 3 = 0.035995$$

and

$$0.04167 - 0.00625 + 0.00056 - 0.00004 - 0.000005 \times 4 = 0.035920$$

S-18: We use that

$$\log(1+y) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{y^n}{n} \qquad \text{for all } -1 < y \leqslant 1$$

with $y = \frac{x-2}{2}$ to give

$$\log(x) = \log(2+x-2) = \log 2 + \log\left(1 + \frac{x-2}{2}\right) = \log 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\, 2^n}(x-2)^n$$

It converges when $-1 < y = \frac{x-2}{2} \leqslant 1$, or equivalently, $0 < x \leqslant 4$.

S-19: (a) Using the Taylor series expansion of e^x with $x = -t$,

$$e^{-t} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \implies e^{-t} - 1 = \sum_{n=1}^{\infty} (-1)^n \frac{t^n}{n!} \implies \frac{e^{-t}-1}{t} = \sum_{n=1}^{\infty} (-1)^n \frac{t^{n-1}}{n!}$$

Substituting this into our integral,

$$I(x) = \int_0^x \frac{e^{-t}-1}{t}\, \mathrm{d}t = \sum_{n=1}^{\infty} (-1)^n \int_0^x \frac{t^{n-1}}{n!}\, \mathrm{d}t = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n\, n!}$$

(b) Substituting in $x = 1$.

$$\begin{aligned} I(1) &= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n\, n!} \\ &= -1 + \frac{1}{2\,2!} - \frac{1}{3\,3!} + \frac{1}{4\,4!} - \frac{1}{5\,5!} + \cdots \\ &= -1 + 0.25 - 0.0556 + 0.0104 - 0.0017 + \cdots = -0.80 \end{aligned}$$

See part (c) for the error analysis.

(c) The series for $I(x)$ is an alternating series (that is, the sign alternates) with successively smaller terms that converge to zero. So the error introduced by truncating the series is no larger than the first omitted term. So the magnitude of $-\frac{1}{5\,5!} + \cdots$ is no larger than 0.0017. Allowing for a roundoff error of at most 0.0001 in each of the two terms $-0.0556 + 0.0104$

$$I(1) = -1 + 0.25 - 0.0556 + 0.0104 \pm 0.0019 = -0.7952 \pm 0.0019$$

S-20: (a) Using the Taylor series expansion of $\sin x$ with $x = t$,

$$\sin t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \implies \frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$$

So

$$\Sigma(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t = \sum_{n=0}^{\infty}(-1)^n \int_0^x \frac{t^{2n}}{(2n+1)!}\,\mathrm{d}t = \sum_{n=0}^{\infty}(-1)^n\,\frac{x^{2n+1}}{(2n+1)(2n+1)!}$$

(b) The critical points of $\Sigma(x)$ are the solutions of $\Sigma'(x) = 0$. By the fundamental theorem of calculus $\Sigma'(x) = \frac{\sin x}{x}$, so the critical points of $\Sigma(x)$ are $x = \pm\pi, \pm 2\pi, \cdots$. The absolute maximum occurs at $x = \pi$.

(c) Substituting in $x = \pi$,

$$\begin{aligned}
\Sigma(\pi) &= \sum_{n=0}^{\infty}(-1)^n\,\frac{\pi^{2n+1}}{(2n+1)(2n+1)!} \\
&= \pi - \frac{\pi^3}{3\,3!} + \frac{\pi^5}{5\,5!} - \frac{\pi^7}{7\,7!} + \cdots \\
&= 3.1416 - 1.7226 + 0.5100 - 0.0856 + 0.0091 - 0.0007 + \cdots
\end{aligned}$$

The series for $\Sigma(\pi)$ is an alternating series (that is, the sign alternates) with successively smaller terms that converge to zero. So the error introduced by truncating the series is no larger than the first omitted term. So

$$\Sigma(\pi) = 3.1416 - 1.7226 + 0.5100 - 0.0856 + 0.0091 = 1.8525$$

with an error of magnitude at most $0.0007 + 0.0005$ (the 0.0005 is the maximum possible accumulated roundoff error in all five retained terms).

S-21: (a) Using the Taylor series expansion of $\cos x$ with $x = t$,

$$\begin{aligned}
\cos t &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!} \\
\frac{\cos t - 1}{t^2} &= -\frac{1}{2!} + \frac{t^2}{4!} - \frac{t^4}{6!} + \cdots \quad = \sum_{n=1}^{\infty}(-1)^n\frac{t^{2n-2}}{(2n)!} \\
I(x) = \int_0^x \frac{\cos t - 1}{t^2}\,\mathrm{d}t &= -\frac{x}{2!} + \frac{x^3}{4!3} - \frac{x^5}{6!5} + \cdots = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n)!(2n-1)}
\end{aligned}$$

(b), (c) Substituting in $x = 1$,

$$\begin{aligned}
I(1) &= -\frac{1}{2} + \frac{1}{4!3} - \frac{1}{6!5} + \cdots \\
&= -0.5 + 0.0139 - 0.0003 - \cdots \\
&= -0.486 \pm 0.001
\end{aligned}$$

The series for $I(1)$ is an alternating series with decreasing successive terms that converge to zero. So approximating $I(1)$ by $-\frac{1}{2} + \frac{1}{4!3}$ introduces an error between 0 and $-\frac{1}{6!5}$. Hence $I(1) < -\frac{1}{2} + \frac{1}{4!3}$.

S-22: (a) Using the Taylor series expansions of $\sin x$ and $\cos x$ with $x = t$,

$$\begin{aligned}
\cos t &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots \\
\sin t &= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \\
t\sin t &= \quad\; t^2 - \frac{t^4}{3!} + \frac{t^6}{5!} - \frac{t^8}{7!} + \cdots \\
\cos t + t\sin t - 1 &= \left(1 - \frac{1}{2!}\right)t^2 - \left(\frac{1}{3!} - \frac{1}{4!}\right)t^4 + \left(\frac{1}{5!} - \frac{1}{6!}\right)t^6 - \left(\frac{1}{7!} - \frac{1}{8!}\right)t^8 + \cdots \\
&= \frac{1}{2!}t^2 - \frac{3}{4!}t^4 + \frac{5}{6!}t^6 - \frac{7}{8!}t^8 + \cdots \quad \left(\text{use } 1 = \frac{2}{2!},\ \frac{1}{3!} = \frac{4}{4!},\ \frac{1}{5!} = \frac{6}{6!},\ \frac{1}{7!} = \frac{8}{8!},\ \cdots\right) \\
\frac{\cos t + t\sin t - 1}{t^2} &= \frac{1}{2!} - \frac{3}{4!}t^2 + \frac{5}{6!}t^4 - \frac{7}{8!}t^6 + \cdots \\
I(x) &= \frac{1}{2!}x - \frac{1}{4!}x^3 + \frac{1}{6!}x^5 - \frac{1}{8!}x^8 + \cdots = \frac{1 - \cos x}{x}
\end{aligned}$$

(b) $I(1) = \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} - \frac{1}{8!} + \cdots = 0.5 - 0.041\dot{6} + 0.00139 - 0.000024 + \cdots = \boxed{0.460}$. The error analysis is in part (c).

(c) The series for $I(1)$ is an alternating series with decreasing successive terms that convege to zero. So approximating $I(1)$ by $\frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!}$ introduces an error between 0 and $-\frac{1}{8!}$. So $I(1) < \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} < 0.460$.

S-23: (a) Substituting $x = -t$ into the known power series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$, we see that:

$$\begin{aligned}
e^{-t} &= 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots \\
1 - e^{-t} &= t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} + \cdots \\
\frac{1 - e^{-t}}{t} &= 1 - \frac{t}{2!} + \frac{t^2}{3!} - \frac{t^3}{4!} + \cdots \\
\int \frac{1 - e^{-t}}{t}\, dt &= C + x - \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} - \frac{x^4}{4 \cdot 4!} + \cdots
\end{aligned}$$

Finally, $f(0) = 0$ (since $f(0)$ is an integral from 0 to 0) and so $C = 0$. Therefore

$$f(x) = \int_0^x \frac{1 - e^{-t}}{t}\, dt = x - \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} - \frac{x^4}{4 \cdot 4!} + \cdots .$$

We can also do this calculation entirely in summation notation: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, and so

$$\begin{aligned}
e^{-t} &= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n!} \\
1 - e^{-t} &= -\sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^n}{n!} \\
\frac{1 - e^{-t}}{t} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{n-1}}{n!} \\
f(x) = \int_0^x \frac{1 - e^{-t}}{t}\,\mathrm{d}t &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n \cdot n!}
\end{aligned}$$

(b) We set $a_n = A_n x^n = \frac{(-1)^{n-1}}{n \cdot n!} x^n$ and apply the ratio test. Since

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(-1)^n x^{n+1} / ((n+1)\cdot(n+1)!)}{(-1)^{n-1} x^n / (n \cdot n!)}\right| \\
&= \lim_{n\to\infty} \left(\frac{|x|^{n+1}}{|x|^n} \frac{n \cdot n!}{(n+1)\cdot(n+1)!}\right) \\
&= \lim_{n\to\infty} \left(|x| \frac{n}{(n+1)^2}\right) \qquad \text{since } (n+1)! = (n+1)\,n! \\
&= 0
\end{aligned}$$

This is smaller than 1 no matter what x is. So the series converges for all x.

S-24:

$$\begin{aligned}
& e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \geqslant 1 + x \qquad \text{for all } x \geqslant 0 \\
\implies\ & e^x - 1 \geqslant x \\
\implies\ & \frac{x^3}{e^x - 1} \leqslant \frac{x^3}{x} = x^2 \\
\implies\ & \int_0^1 \frac{x^3}{e^x - 1}\,\mathrm{d}x \leqslant \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}
\end{aligned}$$

S-25: Using the Maclaurin series expansions of $\cos x$ and e^x,

$$\begin{aligned}
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \\
1 - \cos x &= \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
1 + x - e^x &= -\frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \\
\frac{1-\cos x}{1+x-e^x} &= \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots}{-\frac{x^2}{2!} - \frac{x^3}{3!} + \cdots} = \frac{\frac{1}{2!} - \frac{x^2}{4!} + \cdots}{-\frac{1}{2!} - \frac{x}{3!} + \cdots}
\end{aligned}$$

we have

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0} \frac{\frac{1}{2!} - \frac{x^2}{4!} + \cdots}{-\frac{1}{2!} - \frac{x}{3!} + \cdots} = \frac{\frac{1}{2!}}{-\frac{1}{2!}} = -1$$

S-26: Using the Maclaurin series expansion of $\sin x$,

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\sin x - x + \frac{x^3}{6} &= \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\frac{\sin x - x + \frac{x^3}{6}}{x^5} &= \frac{1}{5!} - \frac{x^2}{7!} + \cdots
\end{aligned}$$

we have

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \cdots \right) = \frac{1}{5!} = \frac{1}{120}$$

S-27: Using the Maclaurin series expansions of $\cos x$,

$$\begin{aligned}
&& \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \\
\implies && 1 - x^2 - \cos x &= -\frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots \\
\implies && \frac{1 - x^2 - \cos x}{x^{5/2}} &= -\frac{x^{-1/2}}{2} - \frac{x^{3/2}}{4!} + \frac{x^{7/2}}{6!} - \frac{x^{11/2}}{8!} + \cdots \\
\implies && \int_0^1 \frac{1 - x^2 - \cos x}{x^{5/2}}\, \mathrm{d}x &= -1 - \frac{2}{5 \times 4!} + \frac{2}{9 \times 6!} - \frac{2}{13 \times 8!} + \cdots
\end{aligned}$$

This is an alternating series with successive terms decreasing. So the error introduced by truncating is between 0 and the first term dropped. So

$$\int_0^1 \frac{1 - x^2 - \cos x}{x^{5/2}}\, \mathrm{d}x = -1 - \frac{1}{60} + \frac{1}{3240} - \frac{1}{262080} + \cdots = -1 - \frac{1}{60} + E = -\frac{61}{60} + E$$

with the error E between 0 and $\frac{1}{3240} < 0.00031$ and $-\frac{61}{60} = -1.017$ with an additional error of at most 0.00034.

S-28: (a) The naive strategy is to set $a_n = \frac{x^{2n}}{(2n)!}$ and apply the ratio test. Since

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{(2n)!}}\right| \\
&= \lim_{n\to\infty}\frac{x^2}{(2n+2)(2n+1)} \qquad \text{since } (2n+2)! = (2n+2)(2n+1)\,(2n)! \\
&= 0
\end{aligned}$$

This is smaller than 1 no matter what x is. So the series converges for all x.

Alternatively, the sneaky way is to observe that both $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ and $e^{-x} = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}$ are known to converge for all x. So

$$\frac{1}{2}\big(e^x + e^{-x}\big) = \sum_{n\text{ even}}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

also converges for all x.

(b) Recall that $e^x = \sum\limits_{n=0}^{\infty}\frac{x^n}{n!}$, Hence

$$\begin{aligned}
e &= \sum_{n=0}^{\infty}\frac{1}{n!} \\
e^{-1} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} \\
e + e^{-1} &= \sum_{n=0}^{\infty}\frac{1+(-1)^n}{n!} = 2\sum_{n\text{ even}}\frac{1}{n!} = 2\sum_{n=0}^{\infty}\frac{1}{(2n)!}
\end{aligned}$$

Hence $\sum\limits_{n=0}^{\infty}\frac{1}{(2n)!} = \frac{1}{2}\big(e + \frac{1}{e}\big)$.

S-29: (a) We know that $e^x = \sum\limits_{n=0}^{\infty}\frac{x^n}{n!}$ for all x. Replacing x by $-x$, we also have $e^{-x} = \sum\limits_{n=0}^{\infty}\frac{(-x)^n}{n!}$ for all x and hence

$$\cosh(x) = \frac{1}{2}\big[e^x + e^{-x}\big] = \frac{1}{2}\Big[\sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\Big] = \sum_{\substack{n=0\\ n\text{ even}}}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

for all x. In particular, the interval of convergence is all of $\mathbb{R}$.

(b) Using the power series expansion of part (a),

$$\cosh(2) = 1 + \frac{2^2}{2!} + \frac{2^4}{4!} + \sum_{n=3}^{\infty} \frac{x^{2n}}{(2n)!} = 3\frac{2}{3} + \sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!}$$

So it suffices to show that $\sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} \leqslant 0.1$. Let's write $b_n = \frac{2^{2n}}{(2n)!}$. The first term in $\sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!}$ is

$$b_3 = \frac{2^6}{6!} = \frac{2^6}{6 \times 5 \times 4 \times 3 \times 2} = \frac{4}{45}$$

The ratio between successive terms in $\sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!}$ is

$$\frac{b_{n+1}}{b_n} = \frac{2^{2n+2}/2^{2n}}{(2n+2)!/(2n)!} = \frac{4}{(2n+2)(2n+1)} \leqslant \frac{4}{8 \times 7} = \frac{1}{14} \qquad \text{for all } n \geqslant 3$$

Hence

$$\sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} \leqslant \overbrace{\frac{4}{45}}^{b_3} + \overbrace{\frac{4}{45} \times \frac{1}{14}}^{b_4 \leqslant} + \overbrace{\frac{4}{45} \times \frac{1}{14^2}}^{b_5 \leqslant} + \overbrace{\frac{4}{45} \times \frac{1}{14^3}}^{b_6 \leqslant} + \cdots = \frac{4}{45} \frac{1}{1 - \frac{1}{14}} = \frac{4}{45} \frac{14}{13} = \frac{56}{585} < \frac{1}{10}$$

(c) Comparing

$$\cosh(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(t^2)^n}{(2n)!} \qquad \text{and} \qquad e^{\frac{1}{2}t^2} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}t^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(t^2)^n}{2^n n!}$$

we see that it suffices to show that $(2n)! \geqslant 2^n n!$. Now. for all $n \geqslant 1$,

$$\begin{aligned} (2n)! &= \overbrace{1 \times 2 \times \cdots \times n}^{n \text{ factors}} \overbrace{(n+1) \times (n+2) \times \cdots \times 2n}^{n \text{ factors}} \\ &\geqslant \overbrace{1 \times 2 \times \cdots \times n}^{n \text{ factors}} \overbrace{2 \times 2 \times \cdots \times 2}^{n \text{ factors}} \\ &= 2^n \, n! \end{aligned}$$