MATH 105 – Review Problems: Integrals and Series

Brief Solutions

Integrals

(a)

$$\int \sin^3(u)\cos^2(u)du$$

Note: $3 =$ odd power. Single a $\sin(u)$ out, use $\sin^2(u) = 1 - \cos^2(u)$ then sub $v = \cos(u)$.

Answer: $\frac{\cos^5 u}{5} - \frac{\cos^3 u}{3} + C$

(b)

$$\int \frac{e^x}{1+e^{2x}}dx$$

$e^{2x} = (e^x)^2$, use sub $u = e^x, du = e^x dx$.

Answer: $\arctan(e^x) + C$

(c)

$$\int y^2\sqrt{1+y^3}dy$$

$u = 1 + y^3$ since $\frac{1}{3}du = y^2 dy$, which appears already

Answer: $\frac{2}{9}(1+y^3)^{3/2} + C$

(d)

$$\int_1^\infty \frac{\ln(x)}{x^{101}}dx$$

Improper so $\int_1^\infty \frac{\ln(x)}{x^{101}}dx = \lim_{a\to\infty}\int_1^a \frac{\ln(x)}{x^{101}}dx$. Then IBP (with $u = \ln(x), dv = x^{-101}dx$. You need L'Hospital for the limit.)

(Alternative (harder) solution $u = \ln(x), x = e^u$, then one IBP)

Answer: $\frac{1}{10^4}$.

(e)

$$\int \frac{x}{\sqrt{1-x^4}}dx$$

First $u = x^2, \frac{1}{2}du = xdx$ to simplify, then trigonometric substitution $u = \sin(\theta)$. Remember to go back: $\theta \to u \to x$

Answer: $\frac{1}{2}\arcsin(x^2) + C$

(f)

$$\int \frac{1}{x^2\sqrt{16-x^2}}dx$$

Trigonometric Substitution $x = 4\cos(x)$
(better than $u = 4\sin(x)$ for this problem since you don't have to use an antiderivative that was not discussed in the course; $u = 4\sin(x)$ works too if you know $\int \frac{1}{\sin^2(y)}dy$ BUT don't learn it.) We need the triangle to go back to x.

Answer: $-\frac{\sqrt{16-x^2}}{16x} + C$

(g)

$$\int \cos(\sqrt{x})dx$$

Start with $u = \sqrt{x}$. Bit tricky but you should only have u's in the new integral. Then IBP.

Answer: $2(\sqrt{x}\sin(\sqrt{x}) + \cos(\sqrt{x})) + C$

(h) (ignore; ended up being more challenging that intended; the antiderivative of $\sec(x)$ was required, don't learn it)

Series: Converges or diverges?

1.

$$\sum_{n=1}^{\infty} \frac{2\ln n}{n^6}$$

converges by the Comparison Test, comparing with $\sum_{n=1}^{\infty} \frac{2n}{n^6} = 2\sum_{n=1}^{\infty} \frac{1}{n^5}$. To see why we can compare with the latter check that $\ln x < x$, (hint: do monotonicity analysis on $f(x) = \ln x - x$). *alternative solution:* use the Integral Test (check the decreasing property); the corresponding integral is almost the same as (d) above.

2.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3+1}{n^3-7}$$

diverges by the Divergence Test, since for even n $a_n = \frac{n^3+1}{n^3-7} \to 1$, while for odd n $a_n = (-1)\frac{n^3+1}{n^3-7} \to -1$ so a_n doesn't have a limit.

3.

$$\sum_{n=1}^{\infty} \frac{7}{n5^n}$$

converges by the Comparison Test, since $\frac{7}{n5^n} \leq \frac{7}{5^n}$ and the series $\sum_{n=1}^{\infty} \frac{7}{5^n} = 7\sum_{n=1}^{\infty} \frac{1}{5^n} = 7\sum_{n=1}^{\infty} (\frac{1}{5})^n$ which is a converging geometric series. Alternatively, one can use the Ratio Test (limit $= \frac{1}{5} < 1$)

4.

$$\sum_{n=1}^{\infty} \frac{2n^2}{9n^2-7}$$

diverges by the Divergence Test, since $a_n = \frac{2n^2}{9n^2-7} \to \frac{2}{9} \neq 0$

5.

$$\sum_{n=1}^{\infty} \frac{1}{4+\sqrt[4]{n^3}}$$

diverges by the Limit Comparison Test; notice that for large n the fraction behaves like $\frac{1}{n^{3/4}}$ whose corresponding series diverges (p-series with $p<1$). Since we cannot use the Comparison Test (check that the inequality is in the unhelpful direction), the problem calls for the Limit Comparison Test. Check that $\lim_n \frac{\frac{1}{4+\sqrt[4]{n^3}}}{\frac{1}{\sqrt[4]{n^3}}} = 1 > 0$ hence by the Limit Comparison test our series also diverges.

6.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3 2^n}{n!}$$

converges by the Ratio Test (limit is 0)

7.

$$\sum_{n=1}^{\infty} \frac{10+9^n}{5+8^n}$$

diverges by the Divergence Test, since $a_n = \frac{10+9^n}{5+8^n} \to \infty$ (which you can see either by using L' Hospital's rule or by diving both numerator by 9^n and the denominator by 8^n)

8.

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos n}{n^5}$$

converges; we will show that this series is *absolutely convergent*, thus is convergent. Consider the series of the absolute values which becomes $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^5}$. Now, by the Comparison Test (can now use since my series has non-negative terms) since $0 \leq |\cos n| \leq 1$ the series of the absolute values converges by comparison to a converging p-series ($p=5$). We have shown that the series converges absolutely and hence converges.