MATH 105 – 951

MATH 105 Final Exam Practice Problems

Short Answer Questions

1. Evaluate the following integrals or state that they diverge.

   (a) $\displaystyle\int_0^{\pi} \sin^3(x)dx$

   (b) $\displaystyle\int_0^{\frac{\sqrt{7}}{2}} \sqrt{7-x^2}dx$

   (c) $\displaystyle\int_2^{\infty} \frac{1}{t\ln(t)}dt$

   (d) $\displaystyle\int_{-2}^{0} \frac{1}{t^2+6t+8}dt$

2. Find the value of a which makes $f(x)$ a probability density function

$$f(x) = \begin{cases} ae^{-4x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

3. Find the area beneath the curve $y = \frac{t}{\sqrt{4+t^2}}$ from $t = 0$ to $t = 1$.

4. If $G(x) = \int_0^x \cos(\cos(\cos(t)))dt$, find $G'\left(\frac{\pi}{2}\right)$.

5. For each sequence/series, find the limit/sum or state that it diverges.

   (a) $\left\{1 + \frac{(-1)^n}{\sqrt{n}}\right\}$.

   (b) $\left\{\frac{n^2+5}{\sqrt{3n^4+13n}}\right\}$.

   (c) $\left\{(-1)^n \frac{n+1}{n}\right\}$.

   (d) $\displaystyle\sum_{k=1}^{\infty} \frac{4}{k^2+2k}$.

(e) $\displaystyle\sum_{k=1}^{\infty} e\frac{4^{k-1}}{5^{2k}}$.

(f) $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{k!}$.

(g) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{5^k k!}$

6. For each of the following, determine whether the series converges absolutely, conditionally or diverges.

   (a) $\displaystyle\sum_{k=7}^{\infty} \frac{k}{k^{\pi}-2}$.

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$.

   (c) $\displaystyle\sum_{k=4}^{\infty} \frac{e^{7/k}}{k^2}$.

   (d) $\displaystyle\sum_{k=10}^{\infty} (-1)^k e^{-k}$.

   (e) $\displaystyle\sum_{k=0}^{\infty} \cos(\pi k)$.

   (f) $\displaystyle\sum_{k=17}^{\infty} \frac{2^k \cos(k)}{k!}$.

   (g) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln(k)}$.

7. Given the function $f(x) = x^3 \sin(2x^2)$, use Maclaurin series to find $f^{(17)}(0)$, the seventeenth derivative of f at $x = 0$.

8. Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} n!(x-4)^n$.

9. Of the vectors $\langle 1, 2, 0\rangle$, $\langle -1, 0, 3\rangle$, and $\langle 4, -2, 3\rangle$, which two, if any, are perpendicular?

10. If $f(x, y) = 1 + 2x\sqrt{y}$, find $\frac{\partial^2 f}{\partial x \partial y}$.

11. Find the equation of the plane tangent to $f(x, y) = y \ln(x)$ at the point $(1, 4, 0)$.

12. If $f(x, y) = \sin(2x + 5y)$, find $\nabla f(x, y)$.

13. Find the rate of change $f(x, y) = \cos(xy) + 2xe^y$ in the direction $\langle 1, 2 \rangle$ at the point $(1, 0)$.

14. Find the maximum rate of change of $f(x, y) = \frac{y^2}{x}$ at the point $(2, 4)$ and a vector pointing in the direction in which it occurs.

Long Answer Questions

1. The lifetime T (in hours) of a lightbulb has probability density function

$$f(t) = \begin{cases} \frac{1}{100} e^{-t/100} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

   (a) Find the probability that the lightbulb will *not* burn out within the first 100 hours.

   (b) Find the expected lifetime of the lightbulb.

2. The distance X (in cm) between a dart's location on a dartboard and the bullseye (centre of the dartboard) has probability density function

$$f(x) = \begin{cases} \frac{3}{4000} \left(20x - x^2\right) & \text{if } 0 \leq x \leq 20 \\ 0 & \text{otherwise} \end{cases}$$

   Calculate the standard deviation in X, $\sigma(X)$.

3. Find power series representations for the following functions.

   (a) $\frac{x}{4+x^2}$

   (b) $\int x \cos(3x^4) dx$

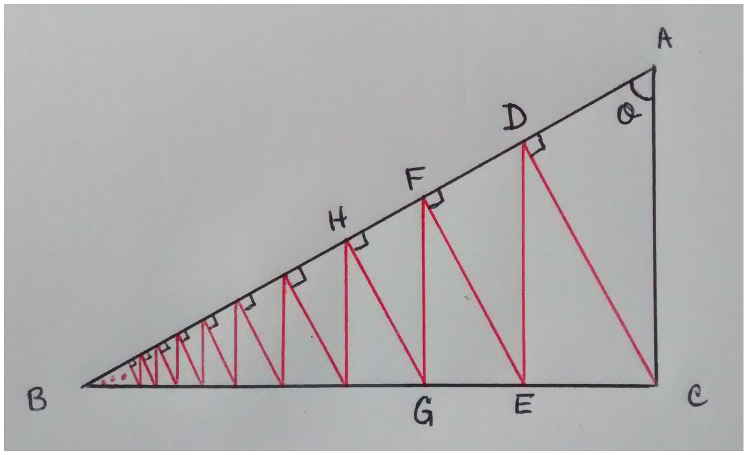
   (c) $\frac{e^{x^2} - 1}{x^2}$

4. Find the radius and the interval of convergence of the power series $\sum_{n=2}^{\infty} \frac{(x-5)^n}{n \ln n}$.

5. A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh, where $0 < r < 1$ is a constant. Suppose the ball is dropped from an initial height of 5m and that $r = \frac{3}{4}$. Assuming the ball continues to bounce indefinitely, find the total distance that it travels.

6. A right triangle ABC (shown below) is given with $\theta = \frac{\pi}{3}$ radians and $|AC| = 4$. CD is drawn perpendicular to AB, DE is drawn perpendicular to BC, EF is drawn perpendicular to AB, and this process is continued indefinitely. Find the total length of all of the perpendiculars

$$|CD| + |DE| + |EF| + |FG| + \cdots$$



7. Find the interval of convergence of the following power series.

   (a) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{10^k (x+2)^k}{k^2}$

   (b) $\displaystyle\sum_{k=1} \frac{k^2 x^k}{2 \cdot 4 \cdot 6 \cdots (2k)}$

8. Use a series to evaluate the limit.

$$\lim_{x \to 0} \frac{\sin(x) - x + \frac{1}{6}x^3}{x^5}$$

9. Let $f(x) = x^{-\frac{3}{2}}$.

   (a) Find the first 5 non-zero terms of Taylor series for $f(x)$ centred at $x = 1$.

   (b) Write down the full Taylor series for $f(x)$ centred at $x = 1$.

10. Find the critical points of the function $f(x, y) = e^y(y^2 - x^2)$ and classify them as being local maxima, local minima or saddle points.

11. Use the method of Lagrange multipliers to find the global maximum and the global minimum of the function $f(x,y) = x^2y$ subject to the constraint $x^2 + 2y^2 = 6$.

12. Find the global maximum and global minimum values of the function $f(x,y) = e^{2y+x}$ subject to the constraint $x^2 + y^2 \leq 4$.