# Short Answer Questions

**1** (a) **Key:** $\int \sin^3 x \, dx = \int \sin^2 x \, \underbrace{\sin x \, dx}$

$\sin^2 x \downarrow (1-\cos^2 x)$, use $u = \cos x$

**Ans:** $\frac{4}{3}$

(b) **Key:** $x = \sqrt{7} \sin\theta$

**Ans:** $7\left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right)$ ~~[illegible]~~

(c) **Key:** improper, write with limits, $u = \ln t$

**Ans:** diverges

(d) **Key:** $t^2 + 6t + 8 = 0 \Rightarrow \boxed{(t+2)}(t+4) = 0$

makes integral improper because of the lower limit

- write as limit
- use partial fractions
- take the limit $a \to -2^+$

**Ans:** diverges

$\boxed{2.}$ Key: need $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Vert$$

$$\int_0^{\infty} a e^{-4x} = 1$$

Ans: $a = 4$

$\boxed{3.}$ Key: $\displaystyle A = \int_0^1 \frac{t}{\sqrt{4+t^2}}\,dt, \qquad u = 4+t^2$

Ans: $\sqrt{5} - 2$

$\boxed{4.}$ (Problem was modified)

Ans:

$$G'(x) = \cos(\cos(\cos x))$$

$$\to G'\left(\tfrac{\pi}{2}\right) = \cos(\cos(0)) = \cos(\underline{1})$$

$\boxed{5.}$ (a) $\dfrac{(-1)^n}{\sqrt{n}} \xrightarrow{n\to\infty} 0$ (similar to a HW question)

$$\Rightarrow \quad 1 + \frac{(-1)^n}{\sqrt{n}} \xrightarrow{n\to\infty} 1$$

(b) $$\frac{n^2+5}{\sqrt{3n^4+13n}} = \frac{\cancel{n^2}\left(1+\frac{5}{n^2}\right)}{\cancel{n^2}\sqrt{3+\frac{13}{n^3}}} \xrightarrow{n\to\infty} \frac{1}{\sqrt{3}}$$

(c) $(-1)^n \dfrac{n+1}{n}$

$n = \text{even}$: $\dfrac{n+1}{n} \xrightarrow{n\to\infty} 1$

DNE

$n = \text{odd}$: $-\dfrac{n+1}{n} \xrightarrow{n\to\infty} -1$

(d) $$\sum_{n=1}^{\infty} \frac{4}{n^2+2n} = \sum_{n=1}^{\infty} \frac{4}{n(n+2)} \underset{\text{fraction}}{\overset{\text{partial}}{=}} 2\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

Converges (e.g. by comparison or limit comparison with $(4)\sum \frac{1}{n^2}$)

nearby → suggest <u>telescopic</u>

<u>Ans:</u> $$\sum_{n=1}^{\infty} \frac{4}{n^2+2n} = \cancel{4}^{\,2}\cdot\frac{3}{\cancel{2}} = 3$$

(e) $\sum_{n=1}^{\infty} e \dfrac{4^{n-1}}{5^{2n}} = \sum_{n=1}^{\infty} e \dfrac{4^{n-1}}{(25)^n} = \sum_{n=1}^{\infty} \dfrac{e}{25}\left(\dfrac{4}{25}\right)^{n-1}$

$\overset{\text{geometric}}{=} \dfrac{e}{25} \dfrac{1}{1-\frac{4}{25}} = \dfrac{e}{\cancel{25}} \dfrac{\cancel{25}}{21} = \dfrac{e}{21}$

(f) $\sum_{n=0}^{\infty} \dfrac{2^n}{n!} = e^2 \quad \left( e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} \right)$

(g) $\sum_{n=1}^{\infty} \dfrac{3^n}{5^n n!} = \sum_{n=1}^{\infty} \dfrac{\left(\frac{3}{5}\right)^n}{n!} = \sum_{n=0}^{\infty} \dfrac{\left(\frac{3}{5}\right)^n}{n!} - \dfrac{\left(\frac{3}{5}\right)^0}{0!}$

$\underset{\text{before}}{\overset{\text{as}}{=}} \boxed{e^{\frac{3}{5}} - 1}$

$\boxed{6.}$ (a) $\overset{\text{large } n}{\approx} \dfrac{n}{n^{\pi \approx 3.14}} \approx \dfrac{1}{n^{2.14}} \to \sum \dfrac{1}{n^{\underset{\vee 1}{(\pi-1)}}}$ converging p series

<u>Ans</u>: Converges by limit comparison

(b) <u>Ans</u>: converges by the Ratio Test (limit $= \frac{1}{2}$)

(c) Key: Integral Test

$$\int_4^\infty \frac{e^{\frac{7}{x}}}{x^2}\,dx \xrightarrow[du=-\frac{7}{x^2}dx]{u=\frac{7}{x}} \left[\lim_{a\to\infty}\int -\frac{1}{7}e^u\,du\right]_4^a$$

$$= -\frac{1}{7}\lim_{a\to\infty} e^{\frac{7}{x}}\Big|_4^\infty$$

$$= -\frac{1}{7}\left(1-e^{7/4}\right) = \frac{e^{7/4}-1}{7}$$

$\Rightarrow$ series $\boxed{\text{converges by I.T.}}$

(d) $\sum_{n=10}^{\infty}(-1)^n e^{-n} = \sum_{n=10} \frac{(-1)^n}{e^n}$

Ans: Converges (A. S. T.)

Also converges absolutely since $\sum \frac{1}{e^n} = \sum\left(\frac{1}{e}\right)^n$ $<1$

(e) Key: $\cos(n\pi) = \{-1, 1, -1, 1, \ldots\} = (-1)^n$

$(-1)^n \longrightarrow$ no limit

$\longrightarrow$ the series diverges (Divergence Test)

(f) $\displaystyle\sum_{n=17}^{\infty} \frac{2^n \cos(n)}{n!}$ → not always positive

↓

Consider

$$\sum \left| \frac{2^n \cos(n)}{n!} \right| = \sum \frac{2^n}{n!} |\cos n|$$

$|\cos(n)| \le 1$

$$\le \sum \frac{2^n}{n!} \rightarrow \text{Ratio Test} \quad \frac{\frac{\cancel{2^{n+1}}\,2}{\cancel{(n+1)!}\,(n+1)}}{\frac{\cancel{2^n}}{\cancel{n!}}} = \frac{2}{n+1} \xrightarrow{n\to\infty} 0$$

→ Converges

⇝ original converges absolutely ⇒ converges

(g) → converges (A. S. T.)

→ not absolutely ( $\sum \frac{1}{n \ln n}$ diverges

⇙ Conditionally

↙ Integral test)

$\boxed{7.}$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\sin(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x^2)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 2^{2n+1} x^{4n+2}$$

$$x^3 \sin(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 2^{2n+1} x^{4n+5}$$

$$f^{(17)}(0) = c_{17} \cdot (17!)$$

$$= \underbrace{\boxed{\frac{(-1)^3}{7!} 2^7} \cdot (17)!}_{\text{Ans}}$$

$$\boxed{\begin{aligned} 4n+5 &= 17 \\ 4n &= 12 \\ n &= 3 \end{aligned}}$$

$\boxed{8.}$ Ratio Test:

$$\left| \frac{\cancel{(n+1)!}^{(n+1)} \; (x-4)^{\cancel{n+1}}}{\cancel{n!} \; \cancel{(x-4)^n}} \right| = (n+1)\,|x-4|$$

Notice that for $\underline{x=4}$: $\lim = 0$

$x \neq 4$: $\lim = \infty$

$\rightsquigarrow$ series only converges for $\boxed{x=4}$

$(R=0)$

$\boxed{9.}$ Denote: $\vec{v}_1 = \langle 1, 2, 0 \rangle$

$\vec{v}_2 = \langle -1, 0, 3 \rangle$

$\vec{v}_3 = \langle 4, -2, 3 \rangle$

Check: $\vec{v}_1 \cdot \vec{v}_2 = -1 \neq 0$

$\vec{v}_1 \cdot \vec{v}_3 = 4 - 4 + 0 = 0 \Rightarrow \vec{v}_1 \perp \vec{v}_3$

$\vec{v}_2 \cdot \vec{v}_3 = 2 \neq 0$

$\boxed{10.}$ $\frac{\partial f}{\partial x} = 2\sqrt{y} \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \cancel{2}\frac{1}{\cancel{2}\sqrt{y}}$

$\boxed{11.}$

$$z - z_0 = f_x\big|_p (x - x_0) + f_y\big|_p (y - y_0)$$

$$z_0 = 0$$

$$f_x = \frac{y}{x} \Rightarrow f_x\big|_p = 4$$

$$f_y = \ln x \Rightarrow f_y|_p = \ln 1 = 0$$

$$\Rightarrow \quad z = 4\,(x - x_0)$$

$\boxed{12.}$

$$\vec{\nabla} f = \langle f_x, f_y \rangle = \langle 2\cos(2x+5y),\ 5\cos(2x+5y) \rangle$$

$\boxed{13.}$

$\langle 1, 2 \rangle$ not unit length $(|\cdot| = \sqrt{5})$

$$\Rightarrow \quad \vec{u} = \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$$

$$\boxed{D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}}$$

$$\vec{\nabla} f\big|_{(1,0)} = \langle -y\sin(xy) + 2e^y,\ -x\sin(xy) + 2xe^y \rangle\big|_{(1,0)} = \langle 2, 2 \rangle$$

$$D_{\vec{u}} f = \frac{1}{\sqrt{5}}\left(\langle 2,2 \rangle \cdot \langle 1,2 \rangle\right) = \frac{6}{\sqrt{5}}$$

$\boxed{14}$ $\rightarrow$ direction of the gradient gives maximum

$$\vec{\nabla}f = \left\langle -\left(\frac{y}{x}\right)^2, \frac{2y}{x} \right\rangle$$

$$\vec{\nabla}f\Big|_{(2,4)} = \langle -4, 4 \rangle$$

$$\text{max rate} = D_{\frac{\vec{\nabla}f}{|\vec{\nabla}f|}} f = \vec{\nabla}f \cdot \frac{1}{|\vec{\nabla}f|}\vec{\nabla}f =$$

$$= \frac{1}{\cancel{|\vec{\nabla}f|}} \cdot |\vec{\nabla}f|^{\cancel{2}} = 4\sqrt{2}$$

## Long Answer Questions

1. (a) The probability the lightbulb will <u>not</u> burn out within the first 100 hours is equal to the probability it will burn out at some point after 100 hrs:

$$\mathbb{P}(100 \leq T < \infty) = \int_{100}^{\infty} \frac{1}{100} e^{-t/100}\, dt = -\frac{100}{100} e^{-t/100} \Big|_{100}^{\infty}$$

$$= \left( \lim_{t \to \infty} -e^{-t/100} \right) - \left(-e^{-1}\right) = \boxed{\frac{1}{e}}$$

(b) $$\mathbb{E}(T) = \int_{-\infty}^{\infty} t \cdot f(t)\, dt = \int_{0}^{\infty} t \cdot \frac{1}{100} e^{-t/100}\, dt$$

$$\boxed{\begin{array}{ll} \text{IBP:} & \\ u = t & dv = \frac{1}{100} e^{-t/100}\, dt \\ du = dt & \\ & v = -\frac{100}{100} e^{-t/100} \end{array}}$$

$$= -te^{-t/100} \Big|_{0}^{\infty} - \int_{0}^{\infty} -e^{-t/100}\, dt$$

$$= \Big( \lim_{t \to \infty} \underbrace{\underset{-\infty \cdot 0}{-te^{-t/100}}}_{\text{indeterminate form}} \Big) - 0 - \left( 100 e^{-t/100} \Big|_{0}^{\infty} \right)$$

$$= \left( \lim_{t \to \infty} \frac{-t}{e^{t/100}} \right) - \left( \lim_{t \to \infty} 100 e^{-t/100} \right) + 100$$

$$\overset{LH}{=} \left( \lim_{t \to \infty} \frac{-1}{\left(\frac{e^{t/100}}{100}\right)} \right) + 100$$

$$= \boxed{100 \text{ hours.}}$$

2. $\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{20} \frac{3}{4000}\cdot(20x^2 - x^3)\,dx$

$$= \frac{3}{4000}\left(\frac{20}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^{20}$$

$$= \frac{3}{4000}\left(\frac{20\cdot 20^3}{3} - \frac{20^4}{4}\right) - 0$$

$$= \frac{3}{4000}\cdot\frac{20^4}{12} = \frac{1}{4\cancel{000}}\cdot\frac{16\cancel{0000}}{4} = 10.$$

$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{20} \frac{3}{4000}\left(20x^3 - x^4\right)dx$

$$= \frac{3}{4000}\left(\frac{20}{4}x^4 - \frac{1}{5}x^5\right)\Big|_0^{20}$$

$$= \frac{3}{4000}\left(\frac{20\cdot 20^4}{4} - \frac{20^5}{5}\right) - 0$$

$$= \frac{3}{4000}\left(\frac{20^5}{20}\right) = \frac{3\cdot 16\cancel{0000}}{4\cancel{000}} = 120.$$

$$\Rightarrow Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = 120 - 100 = 20.$$

$$\Rightarrow \sigma(X) = \sqrt{Var(X)} = \boxed{\sqrt{20}}$$

3 (a) $$\frac{x}{4+x^2} = \frac{x}{4}\cdot\frac{1}{1+x^2/4} = \frac{x}{4}\cdot\frac{1}{1-\left(\frac{-x^2}{4}\right)}.$$

$$= \frac{x}{4}\cdot\sum_{n=0}^{\infty}\left(\frac{-x^2}{4}\right)^n$$

$$= \frac{x}{4}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{4^n} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{4^{n+1}}}$$

(b) $$\int x\cos(3x^4)\,dx$$

$$= \int x\cdot\left(\sum_{n=0}^{\infty}\frac{(-1)^n(3x^4)^{2n}}{(2n)!}\right)dx$$

$$= \int\sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n} x^{8n+1}}{(2n)!}\,dx$$

$$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n 3^{2n}}{(2n)!}\int x^{8n+1}\,dx\right)$$

$$= \boxed{\left(\sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n}}{(2n)!}\cdot\frac{x^{8n+2}}{8n+2}\right) + C}$$

(c) $$\frac{e^{x^2}-1}{x^2} = \frac{\left(\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}\right) - 1}{x^2}$$

$$= \frac{\left(\sum_{n=\cancel{0}\,1}^{\infty} \frac{x^{2n}}{n!}\right) \cancel{-1}}{x^2}.$$

when $n=0$, $\frac{x^{2n}}{n!} = 1$

$$= \frac{\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}}{x^2} = \boxed{\sum_{n=1}^{\infty} \frac{x^{2n-2}}{n!}}$$

4. $$\sum_{n=2}^{\infty} \frac{(x-5)^n}{n\ln(n)}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-5)^{n+1}}{(n+1)\ln(n+1)}}{\frac{(x-5)^n}{n\ln(n)}}\right| = \left|\frac{(x-5)^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n\ln(n)}{(x-5)^n}\right|$$

$$= |x-5| \cdot \left(\frac{n}{n+1}\right) \cdot \left(\frac{\ln(n)}{\ln(n+1)}\right).$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-5| \cdot \left(\lim_{n\to\infty} \frac{n}{n+1}\right) \cdot \left(\lim_{n\to\infty} \underbrace{\frac{\ln(n)}{\ln(n+1)}}_{\infty/\infty}\right)$$

$$\overset{L'H}{=} |x-5| \cdot \left(\lim_{n\to\infty} \frac{1/n}{1/(n+1)}\right)$$

$$= |x-5| \cdot \left(\lim_{n\to\infty} \frac{n+1}{n}\right)$$

$$= |x-5| < 1 \Rightarrow \boxed{R=1}$$

$|x-5|<1 \iff -1<x-5<1 \iff 4<x<6$

When $x=4$: $\displaystyle\sum_{n=2}^{\infty} \frac{(4-5)^n}{n\ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$

Since $\frac{1}{n\ln(n)}$ is decreasing ($n\ln(n)$ is increasing), positive, and $\displaystyle\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$, this series converges by the Alternating Series Test.

When $x=6$: $\displaystyle\sum_{n=2}^{\infty} \frac{(6-5)^n}{n\ln(n)} = \sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

Again, $\frac{1}{n\ln(n)}$ is positive and decreasing. Also

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx \left(\begin{array}{l} u=\ln(x) \\ du=\frac{1}{x}dx \end{array}\right) = \int_{\ln(2)}^{\infty} \frac{1}{u}\,du = \ln|u| \Big|_{\ln(2)}^{\infty}$$

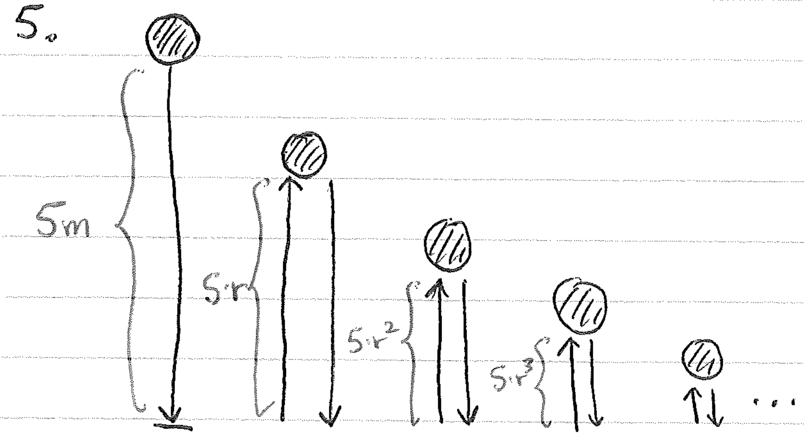
$$= \left(\lim_{u\to\infty} \ln|u|\right) - \ln(\ln 2)$$

$$= \infty \quad \therefore \text{ diverges.}$$

So by Integral Test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ also diverges.

$\therefore$ interval of convergence is $\boxed{[4,6)}$

5.




$5 + 2\cdot 5r + 2\cdot 5r^2 + 2\cdot 5r^3 + \cdots$

When the ball first bounces, it rebounds to the height $5\cdot r$. It then falls this height and rebounds to the height $r\cdot(5r) = 5r^2$. It, again, falls this height and rebounds to the height $r\cdot(5r^2) = 5r^3$. When it bounces for the $n^{th}$ time, it rebounds to the height $5r^n$

So, the total distance travelled by the ball is.

$$5 + \sum_{n=1}^{\infty} 2\cdot 5r^n$$

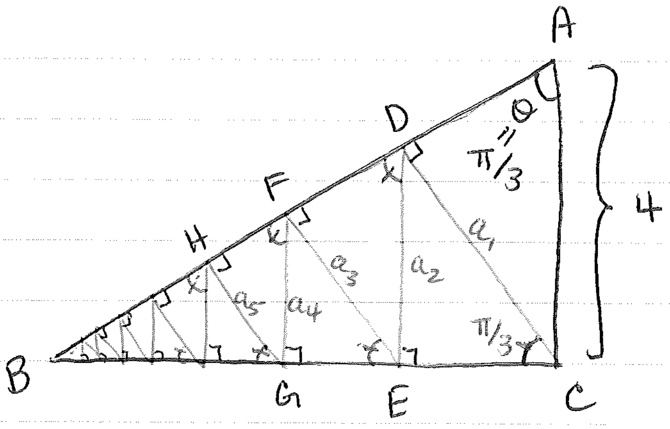
$$= 5 + \sum_{n=1}^{\infty} \underbrace{10r}_{a} \cdot r^{n-1}$$

$$= 5 + \frac{10r}{1-r}$$

$(r = 3/4)$

$$= 5 + \frac{30/4}{1-3/4} = 5 + 30 = \boxed{35\ m}$$

6.




Let $a_n$ be the length of the $n^{th}$ perpendicular

(so $a_1 = |CD|$, $a_2 = |DE|, \ldots$)

Then $\sin\left(\frac{\pi}{3}\right) = \frac{a_1}{4} \Rightarrow a_1 = 4 \cdot \sin\left(\frac{\pi}{3}\right)$

and $\sin\left(\frac{\pi}{3}\right) = \frac{a_2}{a_1} \Rightarrow a_2 = a_1 \sin\left(\frac{\pi}{3}\right) = 4\left(\sin\left(\frac{\pi}{3}\right)\right)^2$

Similarly, $\sin\left(\frac{\pi}{3}\right) = \frac{a_3}{a_2} \Rightarrow a_3 = a_2 \sin\left(\frac{\pi}{3}\right) = 4 \cdot \left(\sin\left(\frac{\pi}{3}\right)\right)^3$

and, in general,

$$\sin\left(\frac{\pi}{3}\right) = \frac{a_n}{a_{n-1}} \Rightarrow a_n = 4 \cdot \left(\sin\left(\frac{\pi}{3}\right)\right)^n$$

Now, $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, so $a_n = 4 \cdot \left(\frac{\sqrt{3}}{2}\right)^n$

$\therefore$ Length of all of the perpendiculars $= \sum_{n=1}^{\infty} 4 \cdot \left(\frac{\sqrt{3}}{2}\right)^n = \sum_{n=1}^{\infty} 4 \cdot \frac{\sqrt{3}}{2} \cdot \left(\frac{\sqrt{3}}{2}\right)^{n-1}$

$$= \frac{4 \cdot \sqrt{3}}{2} \cdot \frac{1}{1 - \sqrt{3}/2} = \boxed{\frac{4\sqrt{3}}{2 - \sqrt{3}}}$$

(8)

P7 (a) Ratio Test :

$$\left| \frac{\frac{(-1)^{n+1}\, 10 \cancel{10^{n+1}} \cancel{(x+2)^{n+1}}}{(n+1)^2}}{\frac{(-1)^n \cancel{10^n} \cancel{(x+2)^n}}{n^2}} \right| = 10\,|x+2| \underbrace{\left(\frac{n}{n+1}\right)^2}_{\downarrow\, n\to\infty \atop 1} \longrightarrow 10\,|x+2|$$

Impose $10\,|x+2| < 1 \Rightarrow |x+2| < \frac{1}{10}$ ← R

$$-2 - \frac{1}{10} < x < \frac{1}{10} - 2$$

$$\frac{-21}{10} < x < \frac{-19}{10}$$

- $x = -\frac{21}{10}$ : $\sum_{n=1}^{\infty} \frac{(-1)^n\, 10^n \left(-\frac{1}{10}\right)^n}{n^2} = \sum_{n=1} \frac{\cancel{(-1)^n} \cancel{(-1)^n}\, \cancel{10^n} \frac{1}{\cancel{10^n}}}{n^2}$ converges

- $x = \frac{-19}{10}$ : $\sum_{n=1}^{\infty} \frac{(-1)^n\, \cancel{10^n} \frac{1}{\cancel{10^n}}}{n^2}$ converges (A. S. T.)

$\Rightarrow$ interval of convergence is

$$\boxed{-\frac{21}{10} \le x \le \frac{-19}{10}}$$

(b) Ratio Test :

$$\left| \frac{\dfrac{(n+1)^2 \, \cancel{x^{n+1}}}{\cancel{2\cdot 4\cdot 6 \cdots (2n)}(2n+2)}}{\dfrac{n^2 \, \cancel{x^n}}{\cancel{2\cdot 4 \cdots (2n)}}} \right| = \underbrace{\left(\frac{n+1}{n}\right)^2}_{\downarrow \, 1} \frac{1}{2(n+1)} |x| \xrightarrow{n\to\infty} 0$$

$\Rightarrow$ $R = \infty$ (and interval of convergence all of $\mathbb{R}$)

$\boxed{88}$ (*) $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \underbrace{\frac{x^3}{3!}}_{\substack{\shortparallel \\ 6}} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$\frac{\sin x - x + \frac{1}{6} x^3}{x^5} \overset{(*)}{=} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x^5}$$

$$\Rightarrow \lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \frac{1}{5!}$$

$\boxed{9.}$ $f(x) = x^{-\frac{3}{2}}$

(a)

| | |
|---|---|
| $f(x) = x^{-\frac{3}{2}}$ | $f(1) = 1$ |
| $f'(x) = \left(-\frac{3}{2}\right) x^{-\frac{5}{2}}$ | $f'(1) = -\frac{3}{2}$ |
| $f''(x) = \left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) x^{-\frac{7}{2}}$ | $f''(1) = +\frac{3\cdot 5}{2^2}$ |
| $f'''(x) = \left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right) x^{-\frac{9}{2}}$ | $f'''(1) = -\frac{3\cdot 5\cdot 7}{2^3}$ |
| $f^{(4)}(x) = \left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right) x^{-\frac{11}{2}}$ | $f^{(4)}(1) = +\frac{3\cdot 5\cdot 7\cdot 9}{2^4}$ |

$$T_5(x) = 1 - \frac{3}{2\cdot 1!}(x-1) + \frac{3\cdot 5}{2!\,2^2}(x-1)^2 - \frac{3\cdot 5\cdot 7}{3!\,2^3}(x-1)^3 + \frac{3\cdot 5\cdot 7\cdot 9}{2!\,2^4}(x-1)^4$$

(b) $$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}\, 1\cdot 3\cdot 5\cdot 7 \cdots (2n+1)\, (x-1)^n$$

(Check: $n=0$, $n=1$, $n=2$, $n=3$, $n=4$

$$1 - \frac{1}{2\,1!}(1\cdot 3)(x-1) + \frac{1\cdot 3\cdot 5}{2^2\,2!}(x-1)^2 - \frac{1\cdot 3\cdot 5\cdot 7}{2^3\,3!}(x-1)^3 + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2^4\,4!}(x-1)^4$$
)

**10.** $f(x,y) = e^y(y^2 - x^2)$

$$f_y = e^y(y^2-x^2) + e^y(2y) = e^y(y^2+2y-x^2)$$

$$f_x = e^y(-2x)$$

CP: $\begin{cases} f_y = 0 \\ f_x = 0 \end{cases} \Rightarrow \left.\begin{matrix} y^2+2y-x^2=0 \\ -2x=0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} y(y+2)=0 \\ x=0 \end{matrix}\right\}$

$$\boxed{(0,0),\ (0,-2)}$$

- $f_{yy} = e^y(y^2+2y-x^2) + e^y(2y+2)$

$$= e^y\left[y^2+4y+2-x^2\right]$$

- $f_{xx} = -2e^y$
- $f_{xy} = (-2x)e^y$

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$(0,0)$: $(-2)\cdot 2 - 0^2 = -4 < 0$ saddle

$(0,-2)$: $(-2e^{-2})\cdot e^{-2}\cdot \underbrace{(4-8+2)}_{-2} - 0^2 > 0$

$+$

$f_{xx}(0,-2) = -2e^{-2} < 0$

$\downarrow$

$\boxed{(0,-2) \text{ local max}}$

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$\boxed{11.}$ $f(x,y) = x^2 y$

define $g(x,y) = x^2 + 2y^2 - 6$

(Constraint: $g(x,y) = 0$)

$$\vec{\nabla} f = \langle 2xy, x^2 \rangle$$

$$\vec{\nabla} g = \langle 2x, 4y \rangle$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow \langle 2xy, x^2 \rangle = \langle 2\lambda x, 4\lambda y \rangle$$

$$\left.\begin{array}{l} \cancel{2}xy = \cancel{2}\lambda x \Rightarrow x(y-\lambda) = 0 \\ x^2 = 4\lambda y \end{array}\right\}$$

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$f_x = 2xy$ $\quad\|\quad$ $g_x = 2x$

$f_y = x^2$ $\quad\|\quad$ $g_y = 4y$

$x^2 = 4\lambda y \quad (*)$

$\boxed{2xy = 2\lambda x}$

If $\boxed{x = 0 \rightsquigarrow \lambda = 0 \text{ or } y = 0}$

$\lambda = 0 \xrightarrow[\text{by constraint}]{\text{but}} y = \pm\sqrt{3}$ $\quad \boxed{(0, \sqrt{3})}\ \boxed{(0, -\sqrt{3})} \to f(\ldots) = 0,\ f(\ldots) = 0$

$y = 0 \to$ not on constraint

If $x \neq 0$ : $y = \lambda \xrightarrow{x^2 = 4\lambda y} x^2 = 4\lambda^2$

$\xrightarrow{\text{constraint}} 4\lambda^2 + 2\lambda^2 = 6$

$\Rightarrow \lambda = \pm 1$ $\quad 4 = f(\ldots) = f(\ldots) \to$ max

$\boxed{\lambda = 1} \to \boxed{y = 1} \overset{*}{\Rightarrow} x = \pm 2 \to \boxed{(2, 1)}, \boxed{(-2, 1)}$

$\boxed{\lambda = -1} \to \boxed{y = -1} \overset{*}{\Rightarrow} x = \pm 2 \to \boxed{(2, -1)}, \boxed{(-2, -1)}$

Min $\leftarrow f(\ldots) = -4$

**12.** $f(x,y) = e^{2y+x}$, on $x^2+y^2 \leq 4$

Break up into two problems:

P1 on $x^2+y^2<4$ (inside)

find CPs: $f_x = e^{2y+x} > 0$ always

$f_y = 2e^{2y+x} > 0$ -||-

$\Rightarrow$ $\boxed{\text{no CPs}}$

P2 on $x^2+y^2=4 \rightsquigarrow g(x,y) = x^2+y^2-4$

constraint $g(x,y)=0$

$$\rightsquigarrow \vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$$

$$< e^{2y+x}, 2e^{2y+x} > = < 2\lambda x, 2\lambda y >$$

$$e^{2y+x} = 2\lambda x \quad (1)$$

$$2e^{2y+x} = 2\lambda y \quad (2)$$

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(1) $\times y$: $\quad y e^{2y+x} = 2\lambda xy$

(2) $\times x$: $\quad 2x e^{2y+x} = 2\lambda xy$

$\Rightarrow \quad y \cancel{e^{2y+x}} = 2x \cancel{e^{2y+x}} \quad (e^{2y+x} > 0 \text{ always})$

$\Rightarrow \quad \boxed{y = 2x}$

Use the constraint: $\quad y^2 + x^2 = 4$

$\downarrow$

$4x^2 + x^2 = 4$

$\Rightarrow \quad x^2 = \frac{4}{5} \Rightarrow x = \pm\sqrt{\frac{4}{5}}$

$\rightsquigarrow \quad \boxed{\left(\sqrt{\frac{4}{5}}, 2\sqrt{\frac{4}{5}}\right)} \rightsquigarrow f(\cdot,\cdot) = e^{4\sqrt{\frac{4}{5}}} \rightsquigarrow \text{max}$

$\boxed{\left(-\sqrt{\frac{4}{5}}, -2\sqrt{\frac{4}{5}}\right)} \rightsquigarrow f(\cdot,\cdot) = e^{-4\sqrt{\frac{4}{5}}} \rightsquigarrow \text{min}$