① (a) $\sum_{n=7}^{\infty} \frac{(-1)^n}{n\ln(n)}$

doesn't matter for Convergence/divergence

* $(-1)^n \cdot \underbrace{\frac{1}{n\ln(n)}}_{b_n \geq 0}$: alternating series

Check: $\frac{1}{n\ln n}$ → > 0 ✓

$\frac{1}{n\ln n} > \frac{1}{n+1} \frac{1}{\ln(n+1)} \Rightarrow b_n > b_{n+1}$ ✓

$\xrightarrow{n\to\infty} 0$ ✓

$\Rightarrow$ by the Alternating Series Test, the series converges

* (both positive + negative terms → the question of { conditional / absolute } Convergence makes sense

form $\sum_{n=7}^{\infty} \left| \frac{(-1)^n}{n\ln(n)} \right| = \sum_{n=7}^{\infty} \frac{1}{n\ln(n)}$ ⇝ integral test

Check: $\int_7^{\infty} \frac{1}{x\ln x} dx = \lim_{a\to\infty} \int_7^{a} \frac{1}{\ln x} \frac{1}{x} dx$, $u = \ln x$, $du = \frac{1}{x} dx$

$$= \lim_{a\to\infty} \Big[\underbrace{\int \frac{1}{u}\,du}_{= \ln u \,=\, \ln(\ln x)} \Big]_7^a = \infty$$

$\downarrow$

Integral diverges

$\Rightarrow$ series diverges (notice the conditions $\boxed{f(x) = \frac{1}{x\ln x}}$ are satisfied:
- positive
- continuous
- decreasing)

$\Rightarrow$ not absolutely convergent,

so ⊝ convergent (as an alternating series)

⊝ conditionally convergent (by definition)

(b) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{4n^2-2}{9n^2+4}}$; notice $a_n = \dfrac{4n^2-2}{9n^2+4} = \dfrac{\cancel{n^2}\left(4-\frac{2}{n^2}\right)}{\cancel{n^2}\left(9+\frac{4}{n^2}\right)}$

$\longrightarrow \dfrac{4}{9} \neq 0$

$\Rightarrow$ the series diverges by the $\boxed{\text{Divergence Test}}$

* terms positive (don't care about abs/cond. convergence)

(c) $\displaystyle\sum_{n=3}^{\infty} \frac{5+n}{n^2\sqrt{4n-1}}$

positive terms

Notice that (for large n) $\dfrac{5+n}{n^2\sqrt{4n-1}}$ $\underset{\text{like}}{\overset{\text{behaves}}{\approx}}$ $\dfrac{n}{n^2\, n^{1/2}} = \dfrac{1}{n^{3/2}}$

use as b_n

Limit Comparison:

$$\lim_{n\to\infty} \frac{\overbrace{\dfrac{5+n}{n^2\sqrt{4n-1}}}^{a_n}}{\underbrace{\dfrac{1}{n^{3/2}}}_{b_n}} = \lim_{n\to\infty} \frac{\dfrac{n\left(\frac{5}{n}+1\right)}{n^2\sqrt{n(4-\frac{1}{n})}}}{\dfrac{1}{n^{3/2}}}$$

$$= \lim_{n\to\infty} \frac{\dfrac{\cancel{n}\left(\frac{5}{n}+1\right)}{\cancel{n^{5/2}}\sqrt{4-\frac{1}{n}}}}{\dfrac{1}{\cancel{n^{3/2}}}} = \frac{1}{2} > 0 \Rightarrow \text{limit comparison}$$

says $\displaystyle\sum_{n=3}^{\infty} \frac{5+n}{n^2\sqrt{4n-1}}$ behaves like $\displaystyle\sum \frac{1}{n^{3/2}}$

which converges (p-series, $p > 1$)

again, positive terms

$\Rightarrow$ abs / cond convergence

Not relevant

(2) $$\sum_{n=1}^{\infty} \underbrace{\frac{(x-7)^n}{2n}}_{a_n}$$

(a) Ratio Test : $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-7)^{n+1}}{2(n+1)}}{\frac{(x-7)^n}{2n}}\right| =$$

$$= \lim_{n\to\infty} |x-7| \frac{2n}{2(n+1)} = |x-7| \underset{\text{impose}}{<} 1 \Rightarrow \boxed{R=1}$$

center = 7

$$\Rightarrow R_{(\text{radius of convergence})} = 1$$

For <u>interval</u> of convergence, check endpoints:

$(6, 8)$

$$-1 < x-7 < 1$$

$$\downarrow$$

$$6 < x-7 < 8$$ → preliminary estimate on the interval of convergence

- <u>$x=8$</u>: $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$ diverges

- <u>$x=6$</u>: $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$ ⇝ alternating series, $b_n = \frac{1}{2n}$ $\to 0$ ✓, $\downarrow$ ✓, $\to 0$ ✓

⇝ convergence

→ interval of convergence $\boxed{6 \le x < 8}$

(b)

1st Solution:

(A)

set $y = x-7$

$$\boxed{S(x)} := \sum_{n=1}^{\infty} \frac{(x-7)^n}{2n} \overset{\downarrow}{=} \frac{1}{2}\sum_{n=1}^{\infty} \frac{y^n}{n}$$

want to find

$$\Rightarrow \quad S'(x) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\cancel{n}\, y^{n-1}}{\cancel{n}} = \frac{1}{2}\sum_{n=0}^{\infty} y^n$$

$$\overset{|y|<1}{=} \frac{1}{2}\,\frac{1}{1-y} = -\frac{1}{2}\,\frac{1}{y-1}$$

$$\overset{\int \dots dy}{\Rightarrow} \quad S(x) = -\frac{1}{2}\int \frac{1}{y-1}\,dy = -\frac{1}{2}\ln|y-1| + C$$

$$\overset{y\to x}{=} -\frac{1}{2}\ln|x-8| + C \quad (B)$$

To find C:

Setting $x=7$ in (A) (see top)

$$S(7) = \sum_{n=1}^{\infty} \frac{0}{2n} = 0$$

Now, use (B): $x = 7 \Rightarrow \quad 0 = S(7) = -\frac{1}{2}\cancelto{0}{\ln|1|} + C \Rightarrow C = 0$

$$\Rightarrow \quad \boxed{S(x) = -\frac{1}{2}\ln|x-8|}\;, \quad \text{for } -8 < x < +8$$

since R for $\sum_{n=1}^{\infty} \frac{(x-7)^n}{2n}$ is 1

2nd solution

(b) $$\sum_{n=1}^{\infty} \frac{(x-7)^n}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(x-7)^n}{n}$$

Consider $$\sum_{n=1}^{\infty} \frac{y^n}{n} \overset{*}{=} \sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1}$$

$$* \sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1} \underset{\substack{n=0\\ \to k=1}}{\overset{k=n+1}{=\!=}} \sum_{k=1}^{\infty} \frac{y^k}{k} \underset{\text{variable}}{\overset{k=\text{dummy}}{=\!=}} \sum_{n=1}^{\infty} \frac{y^n}{n}$$

But, starting from

$$\frac{1}{1-y} \overset{|y|<1}{=} \sum_{n=0}^{\infty} y^n$$

$$\Big\Downarrow \int dy$$

$$\int \frac{1}{1-y}\,dy = \int \sum_{n=0}^{\infty} y^n\,dy$$

$$\Rightarrow -\int \frac{1}{y-1}\,dy = \sum_{n=0}^{\infty} \int y^n\,dy$$

$$\Rightarrow -\ln|y-1| = \sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1}$$

$$\Rightarrow -\frac{1}{2}\ln|y-1| + c = \frac{1}{2}\sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{y^n}{n} = \sum_{n=1}^{\infty} \frac{y^n}{2n}$$

$$\xrightarrow{y=x-7} \sum_{n=1}^{\infty} \frac{(x-7)^n}{2n} = -\frac{1}{2}\ln|x-8| + c \underset{c=0}{\overset{\text{e.g. plug in } 7}{\Longrightarrow}} \sum_{n=1}^{\infty} \frac{(x-7)^n}{2n} = -\frac{1}{2}\ln|x-8|$$

(3) Sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2-n}{2^n} =: S$

Sln: Notice that $\displaystyle\frac{n^2-n}{2^n} = n(n-1)\left(\frac{1}{2}\right)^n$

suggests $\sim n(n-1)\,x^n$

Also observe: $(x^n)' = n\,x^{n-1}$

$(x^n)'' = \underline{n(n-1)}\,x^{n-2}$

so one can already guess the series has something to do with the geometric series (and $x = \frac{1}{2}$ plugged in)

$$\frac{1}{1-x} \overset{|x|<1}{=} \sum_{n=0}^{\infty} x^n \overset{d/dx}{\Rightarrow} \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\,x^{n-1}$$

$$\overset{d/dx}{\Rightarrow} \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)\,x^{n-2}$$

$$\overset{\cdot x^2}{\Rightarrow} \frac{2x^2}{(1-x)^3} = \underbrace{\sum_{n=2}^{\infty} n(n-1)\,x^n}_{S} \overset{\boxed{x=\frac{1}{2}}}{\Rightarrow} \boxed{S = 4}$$