Method of Lagrange Multipliers

To $\underbrace{\text{maximize / minimize}}_{\text{global max/min}}$ $f(x,y)$

subject to the constraint $g(x,y) = 0$

(with $\vec{\nabla} g \neq 0$)

if $=0$ picture doesn't work

(i) Find all values of x, y, λ

s.t.

$$\left.\begin{array}{l} \vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y) \\ \boxed{\text{and}} \\ g(x,y) = 0 \end{array}\right\}$$

(ii) Evaluate $f(x,y)$ at all points (x,y) found in (a) (make a list of the results).

$\boxed{\text{Largest}}$ value $\rightarrow$ max

$\boxed{\text{smallest}}$ -ll- $\rightarrow$ min

Ex 1 Find the max/min of $f(x,y) = 4x+6y$

subject to the constraint $x^2+y^2 = 13$

Sln: Rewrite constraint as $x^2+y^2-13=0$,

define $g(x,y) = x^2+y^2-13$

ws now the constraint is of the form $\boxed{g(x,y)=0}$

$$f(x,y) = 4x+6y \longrightarrow f_x = 4, \quad f_y = 6$$

$$\vec{\nabla} f = \langle 4, 6 \rangle$$

Similarly, $\vec{\nabla} g = \langle 2x, 2y \rangle$

→ Want to find x, y, λ s.t. $\left\{ \begin{array}{l} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g(x,y) = 0 \end{array} \right\}$

$$\Rightarrow \begin{array}{l} 4 = 2\lambda x \\ 6 = 2\lambda y \end{array} \xRightarrow[\text{can divide by } \lambda]{\lambda \neq 0 \text{ (check!)}} \left. \begin{array}{l} x = \frac{4}{2\lambda} = \frac{2}{\lambda} \\ y = \frac{6}{2\lambda} = \frac{3}{\lambda} \end{array} \right\} \quad \boxed{\lambda = ???}$$

$$0 = g(x,y) = \left(\frac{2}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 - 13$$

$$= \frac{4}{\lambda^2} + \frac{9}{\lambda^2} - 13$$

$$\Rightarrow \quad 13 = \frac{13}{\lambda^2} \Rightarrow \lambda^2 = 1 \Rightarrow \boxed{\lambda = \pm 1}$$

$\underline{\lambda = 1}$: $x = 2$, $y = 3$ $\rightsquigarrow$ point $(2,3)$

$\underline{\lambda = -1}$: $x = -2$, $y = -3$ $\longrightarrow$ point $(-2,-3)$

end of Step 1

Plug ~~into~~ $(2,3)$, $(-2,-3)$ into $f(x,y)$:

$f(2,3) = 4 \cdot 2 + 6 \cdot 3 = 26$ $\rightsquigarrow$ global max

$f(-2,-3) = 4(-2) + 6(-3) = -26$ $\rightsquigarrow$ global min

end of Step 2

Ex 2 Find the max/min values of

$f(x,y) = e^{xy}$ subject to the constraint $x^2+3y^2=16$

Sln: $f(x,y) = e^{xy}$, $g(x,y) = x^2+3y^2-16$

(constraint $g(x,y)=0$)

$\vec{\nabla} f = \langle y e^{xy}, x e^{xy} \rangle$

$\vec{\nabla} g = \langle 2x, 6y \rangle$

- $\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow$ $\left.\begin{array}{l} y e^{xy} = 2\lambda x \\ x e^{xy} = 6\lambda y \end{array}\right\}$ hmmm, harder!

* {multiply 1st by x, second by y}:

$\left.\begin{array}{l} xy e^{xy} = 2\lambda x^2 \\ \quad \| \\ xy e^{xy} = 6\lambda y^2 \end{array}\right\} \Rightarrow \boxed{2\lambda x^2 = 6\lambda y^2}$

$\lambda \neq 0 \to$ divide by it $\longrightarrow$

$\boxed{\lambda = 0}$

1st: $y=0$
2nd: $x=0$ $\} \to (0,0)$ not on $g(x,y)=0$ (x^2+3y^2-16) (getting $-16=0$:))

didn't need to solve for λ

$$\Rightarrow \boxed{x^2 = 3y^2} \quad (A)$$

Remember, these points (x,y) (satisfying (A))

also have to lie on the curve $g(x,y)=0$:

$$x^2+3y^2-16=0$$

$$\Downarrow (A)$$

$$\boxed{3y^2} + 3y^2 - 16 = 0 \Rightarrow y^2 = \frac{16}{6} = \frac{8}{3}$$

$$\Rightarrow \boxed{y = \pm\sqrt{\frac{8}{3}}}$$

$$x^2 = 8 \Rightarrow x = \pm\sqrt{8}$$

$\rightarrow$ 4 points: $\overset{P_1}{\left(\sqrt{8}, \sqrt{\frac{8}{3}}\right)}$, $\overset{P_2}{\left(\sqrt{8}, -\sqrt{\frac{8}{3}}\right)}$, $\overset{P_3}{\left(-\sqrt{8}, \sqrt{\frac{8}{3}}\right)}$, $\overset{P_4}{\left(-\sqrt{8}, -\sqrt{\frac{8}{3}}\right)}$

$\Rightarrow f(P_1) = e^{8/\sqrt{3}}$

$\Rightarrow f(P_2) = e^{-8/\sqrt{3}}$ } global minima

$\Rightarrow f(P_3) = e^{-8/\sqrt{3}}$

$\Rightarrow f(P_4) = e^{8/\sqrt{3}}$ $\rightarrow$ global maxima

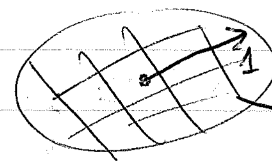
INEQUALITY CONSTRAINT

Example 3: Find max/min of $f(x,y) = 2x^2+y^2$

on $\boxed{x^2+y^2 \le 1}$ → means domain is restricted on

↓

constraint is not a curve anymore

↳  (closed disc of radius 1)

Solution:

STRATEGY ⇝ break it up into 2 subproblems

① $\boxed{x^2+y^2<1}$ (interior) → ⊖ find CPs the usual way

⊖ check if/keep they are in $\{x^2+y^2<1\}$

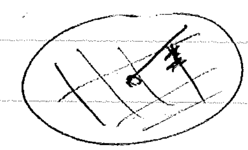
② $\boxed{x^2+y^2=1}$ (boundary) → equality constraint

↓

use Lagrange.

① $f(x,y) = 2x^2 + y^2, \quad x^2 + y^2 < 1$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 4x = 0 \Rightarrow x = 0 \\ \frac{\partial f}{\partial y} &= 2y = 0 \Rightarrow y = 0 \end{aligned}\right\} \Rightarrow \boxed{(0,0)} \text{ inside}$$



CP, don't need to classify it

② • $\vec{\nabla} f = \langle 4x, 2y \rangle$

$g(x,y) = x^2 + y^2 - 1$ (circle; curve)

• $\vec{\nabla} g = \langle 2x, 2y \rangle$

L.M. : $\vec{\nabla} f = \lambda \vec{\nabla} g$

- $4x = \lambda \cdot 2x \rightarrow 2x = \lambda \cdot x$
- $2y = \lambda 2y \rightarrow y = \lambda \cdot y$

$\Rightarrow (2-\lambda)x=0$ $\boxed{\text{and}}$ $(1-\lambda)y=0$

$\downarrow$ $\lambda=2$ or $x=0$ (or both)

$\downarrow$ $\lambda=1$ or $y=0$ (or both)

- $\boxed{\text{If}}$ $\lambda=2$, then $y=0 \Rightarrow \overset{g(x,y)}{x^2+0^2}=1 \Rightarrow \boxed{x=\pm 1}$
- $\boxed{\text{If}}$ $\lambda=1$, then $x=0 \Rightarrow 0^2+y^2=1 \Rightarrow y=\pm 1$

$\therefore$ CPs: $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$

$+$ $(0,0)$ from before

Plug in:

$f(0,0)=0 \leftarrow$ global min

$f(1,0)=2 \leftarrow$ global max

$f(0,1)=1$

$f(-1,0)=2 \leftarrow$ global max

$f(0,-1)=1$