Applications:

① Evaluate $\int e^{-x^2}\,dx$ as an infinite series

__Sln:__ e^{-x^2} has no elementary (closed form) antiderivative.

Will find MacLaurin series for $e^{-x^2} \longrightarrow \int \ldots \, dx$

1st hour

$\rightsquigarrow$ $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$, $R = \infty$ (i.e., series converges for all y)

$\xrightarrow{y=-x^2}$ $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$

$\rightarrow$ will now integrate term-by-term

$$\int e^{-x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}\,dx =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1} + C$$

(new R - from Friday - $= \infty$)

② express $\int \sin(x^3)\,dx$ as a P.S.

<u>Sln:</u>

$$\sin y = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}, \quad R = \infty$$

$y = x^3$ $\longrightarrow$

$$\sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{6n+3}$$

$$\Rightarrow \int \sin(x^3)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{6n+3}\,dx =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int x^{6n+3}\,dx =$$

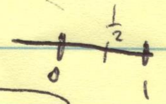
$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{6n+4}}{6n+4} + C}$$

③ Find the sum of the series:

$$\frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \ldots$$

<u>Sln:</u>

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n\cdot 2^n} \overset{\text{rewrite}}{=} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{1}{2}\right)^n}{n}$$



Remember: $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$, $\boxed{R=1}$

because I integrated $\int \sum x^n$

because $1+x = 1-(-x)$

is in there

Series Valid for x: $|x-0| < 1 \ni \frac{1}{2}$

plug in $x = \frac{1}{2}$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n} = \ln\left(1 + \frac{1}{2}\right) = \ln\left(\frac{3}{2}\right)$$

If there is a Q about finding the sum

DON'T PANIC cuz

- is it geometric? (if yes, $\sum ar^n = \frac{a}{1-r}$)
- is it telescopic? (If yes, $S_n = \dots$, $\sum = \lim_{n \to \infty} S_n$)
- if neither, think about one of these Taylor / Maclaurin series when a special x (from the interval of convergence) is plugged in

④ (Another application)

Evaluate $\boxed{\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}}$

<u>Sln:</u> $\boxed{\text{Sol 1}}$: L'Hopital → $\frac{0}{0}$ ... ✓

$\boxed{\text{Sol 2}}$: use a series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\Rightarrow \lim_{x \to 0} \frac{e^x - (1+x)}{x^2} = \lim_{x \to 0} \frac{\left(\cancel{(1+x)} + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) - \cancel{(1+x)}}{x^2}$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{x^2} =$$

$$= \lim_{x \to 0} \left(\frac{1}{2} + \cancelto{0}{\frac{x}{3!}} + \cancelto{0}{\frac{x^2}{4!}} + \cancelto{0}{\frac{x^3}{5!}} + \cancelto{0}{\cdots}\right) = \frac{1}{2}$$

⑤ (Using the series to calculate derivatives)

a) Find $f^{(124)}(0)$ for $f(x) = e^{-2x^2}$

Sln) $f^{(124)}(0) = c_{124}\,(124)!$

↳ c_{124} multiplies x^{124} in the series expansion $e^{-2x^2} = \sum_{m=0}^{\infty} c_m x^m$

$$* \quad e^{-2x^2} = \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^k 2^k}{k!}}_{c_m} \underbrace{x^{2k}}_{x^m}$$

↳ want to see for what k

$x^{2k} = x^{124} \Rightarrow \boxed{k = 62}$

$\Rightarrow \boxed{k=62} \longrightarrow c_{124}\, x^{124}$

$$c_{124} = \frac{(-1)^{62}\, 2^{62}}{62!} = \frac{2^{62}}{62!}$$

$$\Rightarrow \boxed{f^{(124)}(0) = \frac{2^{62}}{62!}(124)!}$$ (don't try to simplify)