① Find the MacLaurin series of $f(x) = e^x$ (and its radius of convergence)

<u>Sln</u>: $f(x) = e^x \rightarrow f'(x) = e^x \rightarrow f''(x) = e^x$

$f^{(n)} = e^x \leftarrow \ldots$

$\rightarrow f^{(n)}(0) = 1$, for all n

$\Rightarrow c_n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{1}{n!}$

therefore, $$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

For R: $$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{x^n} \frac{n!}{(n+1)!} \right| = \frac{|x|}{n+1} \xrightarrow{n \to \infty} 0 < 1$$

for all x

$\Rightarrow$ converges for all $x \rightarrow R = \infty$, nice :)

② MacLaurin series for $\boxed{\sin x}$

$$c_n = \frac{f^{(n)}(0)}{n!}$$

Sln:

$\rightarrow f(x) = \sin x$	$f(0) = 0$	$c_0 = \frac{0}{0! = 1} = 0$
$f'(x) = \cos x$	$f'(0) = 1$	$c_1 = \frac{1}{1!} = 1$
$f''(x) = -\sin x$	$f''(0) = 0$	$c_2 = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$	$c_3 = \frac{-1}{3!}$
$\rightarrow f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$	$c_4 = 0$
$\vdots$	$\vdots$	$\vdots$

derivatives repeat in a cycle of 4

⇝ MacLaurin series:

$$\boxed{\sin x} = f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$$

and ⇝ $\boxed{R = \infty}$

③ MacLaurin series for $\boxed{\cos x}$

<u>Sln:</u> could again find $\boxed{f^{(n)}(x)} \to \boxed{f^{(n)}(0)} \to \boxed{c_n}$

but why bother?

$$\boxed{\cos x} = (\sin x)' \overset{②}{=} \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) =$$

$$\underset{\text{by term}}{\overset{\text{term}}{=}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}\,\cancel{(2n+1)}}{(2n+1)!}$$

$$'' \; (2n)!\,\cancel{(2n+1)}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$$

④ Maclaurin of $x^3\cos x$?

<u>Sln:</u> $x^3 \cos x = x^3 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n)!}$

(5) Represent $f(x) = \sin x$ as a Taylor series centered at $\boxed{a = \pi/3}$

Sln:

$\rightarrow f(x) = \sin x$	$f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$	$c_0 = \frac{\sqrt{3}}{2}$
$f'(x) = \cos x$	$f'\left(\frac{\pi}{3}\right) = \frac{1}{2}$	$c_1 = \frac{\frac{1}{2}}{1!} = \frac{1}{2}$
$f''(x) = -\sin x$	$f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$	$c_2 = \frac{-\frac{\sqrt{3}}{2}}{2!}$
$f'''(x) = -\cos x$	$f'''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$	$c_3 = \frac{-\frac{1}{2}}{3!}$
$\rightarrow f^{(4)}(x) = \sin x$	$f^{(4)}\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$	$c_4 = \frac{\frac{\sqrt{3}}{2}}{4!}$

$$\therefore \sin(x) = \sum_{n=0}^{\infty} \left(c_n \right) \left(x - \frac{\pi}{3} \right)^n$$

$c_n \downarrow$ exercise

$\Rightarrow$ $\boxed{\text{memorize}}$ MacLaurin series for

$\rightarrow e^x$

$\rightarrow \sin x$

$\rightarrow \cos x$

$\rightarrow \ln(1+x)$