Estimating Feasible Nodal Power Injections in Distribution Networks

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***Abstract*—This paper proposes a set-theoretic method to estimate feasible nodal injections (generation or load) in a distribution network, while respecting power system performance requirements. In the setting that we study, performance requirements are constraints in the form of interval ranges on the values that line flows can take, which are modelled by a convex poloytope. At its core, the proposed method relies on the solution of a linearly constrained least-squares optimization problem, which is formulated using sensitivity factors computed by linearizing the power flow equations around the operating point. Since this optimization problem admits a linear closed-form solution, the polytope of permissible line flows is propagated through the solution, via set operations, to obtain the set of feasible nodal injections.**

I. Introduction

This paper proposes an analytically tractable and computationally efficient set-theoretic method to estimate feasible active-power generation and load in a distribution network, while respecting power system performance requirements. (We refer to generation and load at a node generically as a *nodal injection*, which is positive for generation and negative for load.) These performance requirements include constraints in the form of interval ranges on the values that line flows and nodal voltage magnitudes, can take. Guaranteeing such constraints in real time poses significant challenges for existing power system operational tools due to increased penetration of renewable electricity sources, such as wind and solar photovoltaic generation. In particular, since these resources may vary rapidly due to volatile weather conditions, there exists an impetus to develop real-time tools to determine acceptable levels of variability in renewable generation given a set of performance requirements. Furthermore, since renewable generation varies in rated power output, they affect the power system at different voltage levels. For example, wind farms are usually connected at the transmission level, whereas small-scale solar installations are usually connected at the distribution level. Our proposed method is general enough to be applied to both transmission- and distribution-level systems. In this paper, without loss of generality, we focus on distribution network applications.

The problem of determining feasible injections has long been recognized as a challenging problem in the literature [1], [2]. In recent years, convexification of the AC optimal power flow problem has spurred renewed interest in necessary and sufficient conditions for convexity of the feasibility region of the power-flow equations [3], [4]. Distinct from these theoretical contributions, this paper proposes a computationally efficient method to identify feasible injections that satisfy certain system performance requirements. Admittedly, the proposed method does not identify the *entire* set of feasible injections, but by leveraging a linearized power-flow model, the method identifies feasible injections *near* the current operating point. With respect to renewable resource integration, numerous methods have been proposed to assess whether or not power system performance requirements would be violated given certain levels of renewable generation variability [5], [6]. In contrast with these contributions, this paper tackles the inverse problem of determining allowable variations in renewable generation given a set of performance requirements. Although the proposed method is general enough to include limits on, e.g., nodal voltage magnitudes and active-power generation limits, here we focus on estimating feasible injections that respect line-flow limits.

By linearizing the power flow equations around the operating point, the proposed method obtains a linear relationship that maps active-power nodal injections to active-power line flows. Using this relationship, we then formulate a linearly constrained least-squares optimization problem that solves for active-power nodal injections given desired reference active-power line flows. Line-flow limits are modelled with a zonotope (i.e., a special class of convex polytopes represented as the Minkowski sum of a finite set of line segments) with its centre at the current operating point. Using set operations, this zonotope is propagated through the closed-form solution of the linearly constrained least-squares optimization problem. The result is a zonotope that bounds the feasible active-power nodal injections. We illustrate the proposed method on case studies involving a 4-node distribution network.

II. Preliminaries

In this section, we introduce zonotopes—the geometric objects that are key to our approach—and describe the linearized power-system model utilized for the distribution network.

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A. Zonotopes

Zonotopes are a special class of convex polytopes that are represented as a Minkowski sum of a finite set of line segments. Formally, a zonotope, denoted by $\mathcal{X}$, is defined as

$$\mathcal{X} = \left\{ x : x = x_0 + \sum_{k=1}^{s} \alpha_k g_k,\ -c_k \leq \alpha_k \leq c_k \right\}, \quad (1)$$

where $x_0 \in \mathbb{R}^n$ is the centre of the zonotope, and the collection of vectors $g_1, g_2, \ldots, g_s \in \mathbb{R}^n$ is the set of linearly independent *generators* of $\mathcal{X}$ [7].

A useful property of the zonotope is that it is closed under linear transformations. In other words, given the zonotope defined in (1) and a linear transformation matrix $K \in \mathbb{R}^{m \times n}$, we obtain a corresponding post-transformation zonotope $\mathcal{Y}$ as $\mathcal{Y} = K\mathcal{X}$, or more specifically,

$$\mathcal{Y} = \left\{ y : y = Kx_0 + K \sum_{k=1}^{s} \alpha_k g_k,\ -c_k \leq \alpha_k \leq c_k \right\}. \quad (2)$$

In the remainder of the paper, we will use the property described in (2) to propagate the set of permissible flows, defined by a zonotope, to obtain the corresponding set of feasible nodal injections.

B. Distribution Network Model

Consider a distribution network with N nodes collected in the set $\mathcal{N} = \{1, \ldots, N\}$. Without loss of generality, node 1 denotes the point of common coupling (PCC) at the distribution feeder substation. Assume the voltage at node 1 sets the reference voltage. Let $V_i = |V_i| \angle \theta_i \in \mathbb{C}$ represent the voltage phasor at node i; similarly, let $I_i \in \mathbb{C}$ denote the current injected into node i. Further, collect voltage phasors into the vector $V = [V_1, \ldots, V_N]^{\mathrm{T}}$ and current injections into $I = [I_1, \ldots, I_N]^{\mathrm{T}}$.

The set of E lines is represented by $\mathcal{E} := \{(m,n)\} \subseteq \mathcal{N} \times \mathcal{N}$. Each line is modelled using the lumped-parameter Π-model with series admittance $y_{mn} \in \mathbb{C}$ and shunt admittance $y_{mn}^{\mathrm{sh}} \in \mathbb{C}$. Then, the entries of the network admittance matrix, denoted by Y, are[1]

$$[Y]_{mn} := \begin{cases} y_m + \sum_{(m,k) \in \mathcal{E}} y_{mk}, & \text{if } m = n, \\ -y_{mn}, & \text{if } (m,n) \in \mathcal{E}, \\ 0, & \text{otherwise}, \end{cases} \quad (3)$$

where

$$y_m = g_m + \mathrm{j} b_m := y_{mm} + \sum_{k \in \mathcal{N}_m} y_{mk}^{\mathrm{sh}}, \quad (4)$$

denotes the total shunt admittance connected to node m with $\mathcal{N}_m \subseteq \mathcal{N}$ representing the set of neighbours of node m and $y_m \in \mathbb{C}$ any passive shunt elements connected to node m. Then, applying Kirchhoff's current law at each node and combining them into matrix-vector form, the current balance can be compactly represented as

$$I = YV. \quad (5)$$

[1] $[Y]_{mn}$ denotes the entry in the m-th row and n-th column of matrix Y.

Denote the vector of complex-power nodal injections by $S = [S_1, \ldots, S_N]^{\mathrm{T}} = P + \mathrm{j}Q$, with $P = [P_1, \ldots, P_N]^{\mathrm{T}}$ and $Q = [Q_1, \ldots, Q_N]^{\mathrm{T}}$. (By convention, P_i and Q_i are positive for generation and negative for loads.) Then, complex-power nodal injections can be compactly written as

$$S = \mathrm{diag}\,(V)\, I^*. \quad (6)$$

The above is the complex-valued equivalent of the standard power flow equations, generalized to include active- and reactive-power injections as well as voltage magnitudes and phase angles at all nodes. Separating the real and imaginary components of (6), we recover the ubiquitous power flow equations, which can be compactly written as

$$f(\theta, |V|, P, Q) = 0, \quad (7)$$

where $\theta = [\theta_1, \ldots, \theta_N]^{\mathrm{T}}$ and $|V| = [|V_1|, \ldots, |V_N|]^{\mathrm{T}}$. In (7), the dependence on network parameters, such as line series and shunt impedances, is implicit in the formulation of f.

Next, shifting our focus from nodal injections to line flows, we can express the current flowing in line (m,n) as

$$\begin{aligned} I_{(m,n)} &= y_{mn}(V_m - V_n) + y_{mn}^{\mathrm{sh}} V_m \\ &= \left(y_{mn} e_{mn}^{\mathrm{T}} + y_{mn}^{\mathrm{sh}} e_m^{\mathrm{T}} \right) V, \end{aligned} \quad (8)$$

where $e_m \in \mathbb{R}^N$ denotes a column vector of all zeros except with the m-th entry equal to 1, and $e_{mn} := e_m - e_n$. From (5), the nodal voltages can be expressed as $V = Y^{-1}I$. Subsequently, (8) can be written as

$$I_{(m,n)} = \left(y_{mn} e_{mn}^{\mathrm{T}} + y_{mn}^{\mathrm{sh}} e_m^{\mathrm{T}} \right) Y^{-1} I =: \kappa_{(m,n)}^{\mathrm{T}} I, \quad (9)$$

where $\kappa_{(m,n)} \in \mathbb{C}^N$ are current injection sensitivity factors.

Denote, by $S_{(m,n)} = P_{(m,n)} + \mathrm{j}Q_{(m,n)}$, the complex power flowing across line (m,n). We can write

$$S_{(m,n)} = V_m I_{(m,n)}^*. \quad (10)$$

We substitute the current injection sensitivity factors from (9) into (10), and obtain

$$S_{(m,n)} = V_m \left(\kappa_{(m,n)}^{\mathrm{T}} \right)^* I^*. \quad (11)$$

Eliminating I^* from (11) using (6), we get

$$S_{(m,n)} = V_m \left(\kappa_{(m,n)}^{\mathrm{T}} \right)^* \left(\mathrm{diag}\,(V) \right)^{-1} S, \quad (12)$$

which relates complex-power flow in line (m,n) as a function of nodal voltage phasors and complex-power nodal injections.

C. Linearized Model

Below, we sketch out the linearized model that relates small variations in line (m,n) active-power flow to small variations in active-power nodal injections. For the interested reader, details can be found in our recent work [8]. First, separating the real and imaginary components of (12), the active-power flow in line (m,n) is

$$P_{(m,n)} = p_{(m,n)}(\theta, |V|, P, Q). \quad (13)$$

Further denote the solution to (7) by $(\theta_\star, |V_\star|, P_\star, Q_\star)$ and assume $p_{(m,n)}$ is continuously differentiable with respect to

θ, $|V|$, and P at $(\theta_\star, |V_\star|, P_\star, Q_\star)$. Let $\theta = \theta_\star + \Delta\theta$, $|V| = |V_\star| + \Delta|V|$, and $P = P_\star + \Delta P$. Then, by assuming that $\Delta\theta$, $\Delta|V|$, and ΔP are sufficiently small and neglecting any variations in Q, we can express small variations in $P_{(m,n)}$ as

$$\Delta P_{(m,n)} \approx r_1 \Delta\theta + r_2 \Delta|V| + s\Delta P, \tag{14}$$

where, with reference to $p_{(m,n)}$ in (13), we have that[2]

$$r_1 = \nabla_\theta p_{(m,n)},\ r_2 = \nabla_{|V|} p_{(m,n)},\ s = \nabla_P p_{(m,n)},$$

all evaluated at $(\theta_\star, |V_\star|, P_\star, Q_\star)$. By applying a similar small-signal assumption to (7), we can obtain

$$\begin{bmatrix} \Delta\theta \\ \Delta|V| \end{bmatrix} \approx J\Delta P, \tag{15}$$

where J denotes the power-flow Jacobian matrix. Finally, substituting (15) into (14), we get

$$\Delta P_{(m,n)} \approx ([r_1, r_2]J + s)\,\Delta P =: \Gamma_{(m,n)} \Delta P, \tag{16}$$

where $\Gamma_{(m,n)} \in \mathbb{R}^N$ are active-power nodal injection sensitivity factors.

D. Problem Statement

In power systems, performance requirements include constraints in the form of interval ranges on the values that nodal voltage magnitudes and active-power line flows can take. In the remainder of the paper, we focus on active-power line flow limits. With the above in mind, let $P^{\max}_{(m,n)}$ and $P^{\min}_{(m,n)}$ denote the maximum and minimum allowable active-power flow in line (m,n), respectively. Collect the variations in active-power line flows, $\{\Delta P_{(m,n)}\}$, where $(m,n) \in \mathcal{E}$, into the column vector $\Delta f \in \mathbb{R}^E$. Similarly, collect the corresponding active-power line flows for the nominal solution $(\theta_\star, |V_\star|, P_\star, Q_\star)$ into vector $f_\star \in \mathbb{R}^E$. Also, assemble corresponding maximum and minimum allowable active power flows into vectors $f^{\max} \in \mathbb{R}^E$ and $f^{\min} \in \mathbb{R}^E$, respectively. Then, the set of permissible active-power line flows can be described by a (possibly asymmetric) polytope

$$\Delta\overline{\mathcal{H}} = \left\{\Delta f : \eta_i^{\min} \le e_i^{\mathrm{T}} \Delta f \le \eta_i^{\max}, i = 1, \ldots, E\right\}, \tag{17}$$

where $\eta_i^{\max} = e_i^{\mathrm{T}}(f^{\max} - f_\star)$ and $\eta_i^{\min} = e_i^{\mathrm{T}}(f^{\min} - f_\star)$. To ease our exposition, in the remainder of the paper, we consider a symmetric polytope $\Delta\mathcal{H} \subseteq \Delta\overline{\mathcal{H}}$ described by

$$\Delta\mathcal{H} = \left\{\Delta f : -\eta_i \le e_i^{\mathrm{T}} \Delta f \le \eta_i, i = 1, \ldots, E\right\}, \tag{18}$$

where $\eta_i = \min\{|\eta_i^{\min}|, |\eta_i^{\max}|\}$. Equivalently, the subset $\Delta\mathcal{H}$ can also be described by a zonotope using the collection of generators $\{e_1, e_2, \ldots, e_E\}$, where $e_i \in \mathbb{R}^E$, as follows:

$$\Delta\mathcal{H} = \left\{\Delta f : \Delta f = \sum_{i=1}^{E} \pi_i e_i, -\eta_i \le \pi_i \le \eta_i\right\}, \tag{19}$$

Given that the system is operating at the nominal solution $(\theta_\star, |V_\star|, P_\star, Q_\star)$, our goal is to compute the set of feasible variations in active-power nodal injections that satisfy the line-flow constraints described in (18), or equivalently (19).

[2]Given a scalar function $f(x) : \mathbb{R}^n \to \mathbb{R}$, $\nabla_x f(x)$ returns the gradient $[\partial f/\partial x_1, \ldots, \partial f/\partial x_n]$

III. Feasible Active-power Nodal Injection Computation

We formulate a convex optimization problem that utilizes the sensitivities in (16) to compute the set of feasible active-power nodal injections. All active-power nodal injections, except at the PCC, are modelled as independent quantities.

A. Feasible Active-power Nodal Injections

Consider the problem of obtaining the set of feasible active-power nodal injections that best satisfy a set of permissible active-power line flows that lie within $\Delta\mathcal{H}$. In order to solve this problem, recall that the variations in active-power line flows are collected into the vector Δf. Collect the corresponding active-power injection sensitivity factors, i.e., $\Gamma_{(m,n)}$, $(m,n) \in \mathcal{E}$, into the matrix $G \in \mathbb{R}^{E\times N}$. Then, we solve for $\Delta P \in \mathbb{R}^N$ from the following linearly constrained least-squares optimization problem [9]:

$$\begin{aligned} \min_{\Delta P \in \mathbb{R}^N} \quad & ||G\Delta P - \Delta f||^2 \\ \text{s.t.} \quad & \mathbb{1}_N^{\mathrm{T}} \Delta P = \Delta L, \end{aligned} \tag{20}$$

where ΔL denotes the change in total system loss and $\mathbb{1}_N$ denotes the $N \times 1$ vector with all ones.. The unique closed-form solution to (20) is given by [10]

$$\begin{bmatrix} \Delta P \\ \lambda \end{bmatrix} = \begin{bmatrix} 2G^{\mathrm{T}}G & \mathbb{1}_N \\ \mathbb{1}_N^{\mathrm{T}} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2G^{\mathrm{T}}\Delta f \\ \Delta L \end{bmatrix}, \tag{21}$$

where λ is the Lagrange multiplier associated with the equality constraint. Note that the change in system loss ΔL in (20) is not known *a priori*. One option is to set it to zero, which is equivalent to assuming all feasible injections lead to the same system loss as the nominal solution, as would approximately be the case if the variations ΔP are sufficiently small. Then, from (21), the closed-form solution for ΔP is

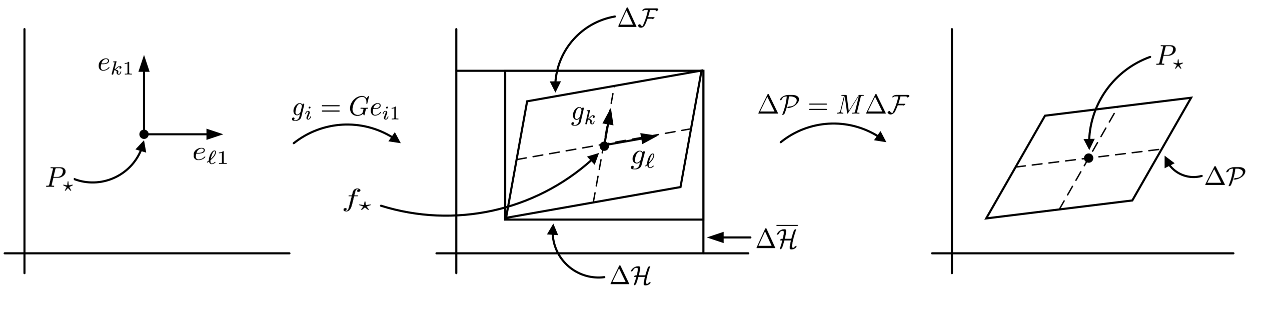
$$\begin{aligned} \Delta P &= \begin{bmatrix} \mathrm{diag}(\mathbb{1}_N) & \mathbb{0}_N \end{bmatrix} \begin{bmatrix} 2G^{\mathrm{T}}G & \mathbb{1}_N \\ \mathbb{1}_N^{\mathrm{T}} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2G^{\mathrm{T}} \\ \mathbb{0}_E^{\mathrm{T}} \end{bmatrix} \Delta f \\ &=: M\Delta f, \end{aligned} \tag{22}$$

where $\mathbb{0}_N$ denotes the $N \times 1$ vector with all zeros; $\mathrm{diag}(\mathbb{1}_N)$ denotes a diagonal matrix formed with entries of $\mathbb{1}_N$.

Denote the set of feasible variations in active-power injections as $\Delta\mathcal{P}$. Since $\Delta f \in \Delta\mathcal{H}$, then from (22), we would like to compute $\Delta\mathcal{P} = M\Delta\mathcal{H}$ using (2). However, we cannot do this directly, due to the following. In (19), the set $\Delta\mathcal{H}$ is expressed as the Minkowski sum of orthogonal basis vectors $e_i \in \mathbb{R}^E$. This suggests that active-power flows on different lines are independent quantities. Under the assumption of independent nodal injections, however, line flows are correlated to each other. Below, we describe a procedure to obtain the set of feasible injections that considers this correlation.

B. Permissible Active-power Line-flow Zonotope

Assume that all active-power nodal injections (except the one at node 1) are independent from each other, i.e., if $m, n \neq 1$, then a change in nodal injection at node m does not affect that at node n. Node 1 absorbs any power



(a) Active-power nodal injection generators. (b) Permissible active-power line flows (c) Feasible active-power nodal injections

Fig. 1: Injection and line-flow zonotope representations.

imbalance in the system. With the above independent injection assumption, the set of active-power nodal injections can be expressed as the Minkowski sum of the collection of generators $\{e_{21}, e_{31}, \ldots, e_{N1}\}$, where $e_{i1} = e_i - e_1$. Then, the corresponding set of active-power line flows can be described using the collection of generators $\{g_2, g_3, \ldots, g_N\}$. Each generator vector g_i is computed as $g_i = Ge_{i1} \in \mathbb{R}^E$ using (2). The set of generators $\{g_2, g_3, \ldots, g_N\}$ are correlated so as to satisfy the assumption that nodal injections are independent at nodes $2, \ldots, N$. These concepts are illustrated in Figs. 1a and 1b.

In (19), the set of permissible active-power line flows, $\Delta\mathcal{H}$, is expressed as the Minkowski sum of orthogonal basis vectors. As stated before, in general the generators that form $\Delta\mathcal{H}$ are at odds with the independent nodal injection assumption. On the other hand, this assumption would be captured if the set of permissible line flows is expressed as a Minkowski sum of generators $\{g_2, g_3, \ldots, g_N\}$. In order to fit such a zonotope to $\Delta\mathcal{H}$, we define a set $\Delta\mathcal{F} \subseteq \Delta\mathcal{H}$ as

$$\Delta\mathcal{F} = \left\{ \Delta f : \Delta f = \sum_{i=2}^{N} \pi_i g_i, -\beta \leq \pi_i \leq \beta \right\}, \tag{23}$$

where β is a scalar quantity to be determined. According to (23), β serves to expand or shrink $\Delta\mathcal{F}$ uniformly in the directions of all its constituent generators (i.e., g_i's). Computed according to the most limiting line-flow constraint,

$$\beta = \min_k (x[k]) \text{ and } x = \operatorname{diag}\left(\frac{\mathbb{1}_E}{\sum_{i=2}^{N} \operatorname{abs}(g_i)} \right) \left(\sum_{i=1}^{E} e_i \right), \tag{24}$$

where $x[k]$ denotes the kth entry in x, and each entry in $\operatorname{abs}(g_i)$ is the absolute value of the corresponding entry in g_i. The above choice of β ensures that $\Delta\mathcal{F}$ occupies the maximum possible volume within the line-flow constraint set $\Delta\mathcal{H}$. In this way, any further uniform expansion of $\Delta\mathcal{F}$, by increasing the value of β, would cause a subset of $\Delta\mathcal{F}$ to lie outside of $\Delta\mathcal{H}$, thus violating the permissible line-flow constraints. The relationships between sets $\Delta\overline{\mathcal{H}}$, $\Delta\mathcal{H}$, and $\Delta\mathcal{F}$ are illustrated in Fig. 1b.

Now, assuming the permissible variations in active-power line flow lie within the set $\Delta\mathcal{F}$, which accounts for line-flow correlations that result from the independent nodal injection assumption, we return to the closed-form solution (22) of the linearly constrained least-squares problem outlined in Section III-A. Since $\Delta P = M\Delta f$, where $\Delta f \in \Delta\mathcal{F} \subseteq \Delta\mathcal{H}$, then using the set operation in (2), the set of feasible variations in active-power nodal injections $\Delta\mathcal{P} = M\Delta\mathcal{F}$. This is conceptually illustrated in Figs. 1b and 1c.

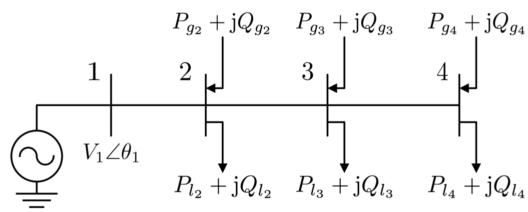


Fig. 2: 4-node distribution network.

IV. Case Studies

In this section, we illustrate the concepts developed in Section III by presenting case study results for a 4-node distribution network. The one-line diagram is reproduced in Fig. 2 and a complete system description can be found in [11]. We validate the framework developed by comparing the estimated set of feasible nodal injections against exact solutions of nonlinear power flow equations.

Using the network parameters, we obtain the power flow solution for the nominal operating point. We then compute active-power nodal injection sensitivity factors using (16), which are used to map active-power nodal injection generators (i.e., e_{i1}'s in Fig. 1a) to generators that describe permissible active-power line-flows (i.e., g_i's in Fig. 1b).

A. Incorporating Active-power Line-flow Limits

Suppose variations in active-power line flows are constrained to lie within lower and upper bounds, i.e., $\Delta P_{(1,2)} \in [-0.5, 0.5]$ p.u., $\Delta P_{(2,3)} \in [-0.3, 0.3]$ p.u., and $\Delta P_{(3,4)} \in [-0.16, 0.16]$ p.u., which form the line-flow constraint set $\Delta\mathcal{H}$. The value of β is computed, via (24), so that any uniform expansion of $\Delta\mathcal{F}$ causes it to exceed $\Delta\mathcal{H}$. The projection of $\Delta\mathcal{H}$ and $\Delta\mathcal{F}$ onto the $P_{(1,2)}$-$P_{(2,3)}$ subspace is shown in

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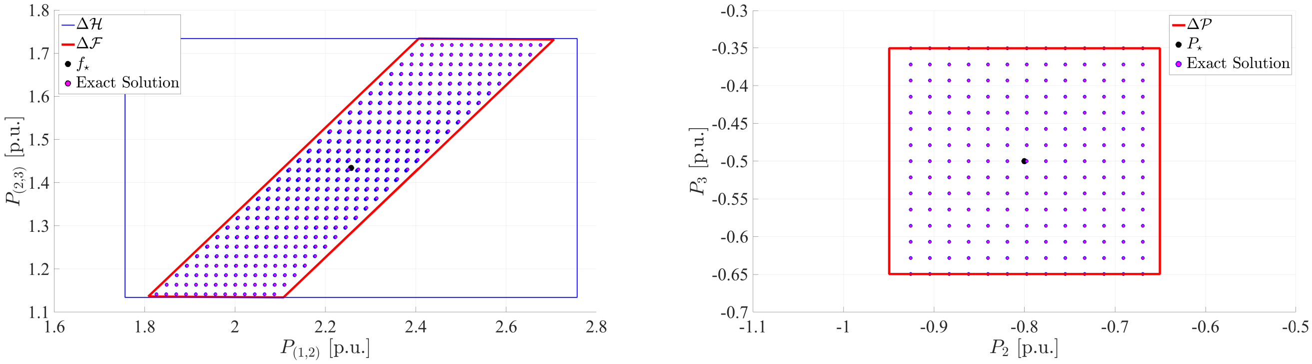
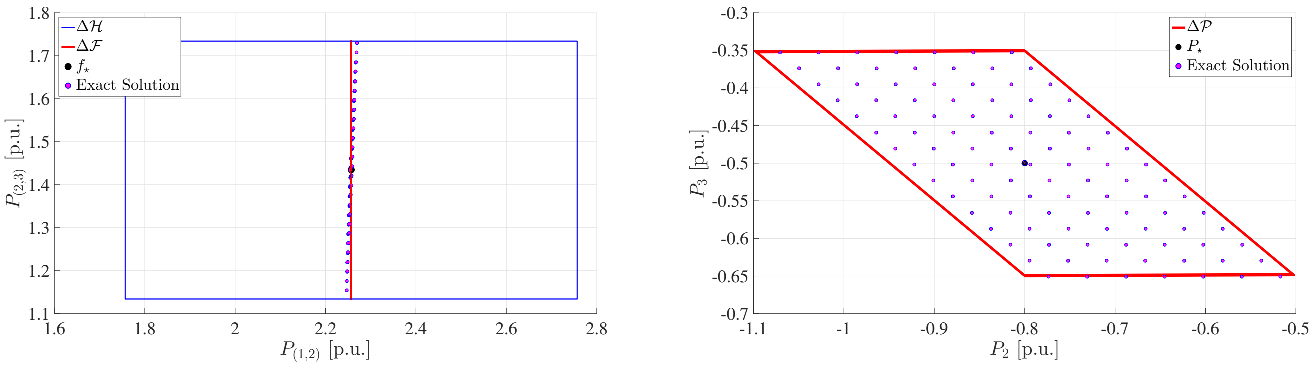


(a) Permissible active-power line flows.

(b) Feasible active-power nodal injections.

Fig. 3: Permissible line flows and feasible nodal injections with active-power line-flow limits.



(a) Permissible active-power line flows.

(b) Feasible active-power nodal injections.

Fig. 4: Permissible line flows and feasible nodal injections with fixed feeder-head active-power injection.

Fig. 3a. Assuming $\Delta f \in \Delta\mathcal{F}$, the set of feasible variations in nodal injections is computed as $\Delta\mathcal{P} = M\Delta\mathcal{F}$, where M is defined in (22). As shown in Fig. 3b, $\Delta\mathcal{P}$ is then sampled to obtain solutions to the nonlinear power flow equations. The set of permissible line flows depicted in Fig. 3a contains all but the extrema of the nonlinear power flow samples. Thus, we can conclude that the linearization is sufficiently accurate.

B. Incorporating Fixed Feeder-head Active-power Injection

Now suppose that the feeder-head active-power injection, and in turn the active-power line flow from node 1 to 2, is also fixed, i.e., $\Delta P_1 = 0$ and $\Delta P_{(1,2)} = 0$. The resulting permissible active-power line-flows and feasible injections are illustrated in Fig. 4a and Fig. 4b, respectively. The set-theoretic method is compared against exact solutions of the power flow equations. Indeed, as shown in Fig. 4a, $P_{(1,2)}$ is fixed at the nominal solution.

V. Concluding Remarks

In this paper, we proposed a set-theoretic method to estimate the feasible nodal power injections of a distribution network, using linear sensitivity factors computed at the operating point. The method was demonstrated via case studies involving a 4-node network. While we focused on distribution networks, the method is general enough to tackle the same problem at the transmission level.

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