Power Engineering Letters

Power System Loss Divider

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***Abstract*—This letter presents an analytical closed-form expression that quantifies the contributions of nodal active- and reactive-power injections to total loss in a power system operating at sinusoidal steady state. We term this as the *loss divider*, since it is derived by leveraging the ubiquitous current divider law. The proposed loss divider innately embeds the dependence of system loss on both the network topology and the operating point, i.e., voltage profile. The derivation does not rely on any simplifying assumptions, and so the resulting expression delineates the *exact* quadratic relationship between the system loss and nodal active- and reactive-power injections.**

***Index Terms*—Ancillary services, current injection sensitivities, loss coefficients, transmission loss allocation.**

I. Introduction

MODERN electric power systems are undergoing dramatic changes due to deregulation and increased penetration of distributed and renewable generation [1]. Consequently, it is necessary to quantify and allocate the cost associated with system loss among market participants in a fair manner [2], [3]. In this paper, we explicitly demonstrate how system loss can be attributed to nodal active- and reactive-power injections *exactly* (i.e., without resorting to approximations) as a quadratic function, which we term *loss divider*. Then, as an application, we leverage the loss divider for transmission-loss allocation.

There are conceivably many ways to allocate loss amongst producers and consumers of electricity in a power network. Existing methods for loss allocation can be categorized into (i) pro rata, (ii) incremental transmission loss, (iii) proportional sharing, and (iv) loss allocation formulas. Pro rata methods do not consider the network topology [4]. Incremental transmission loss methods depend on the slack bus location [5]. Proportional sharing methods assume that inflows are proportional to the outflows at each bus [6]. Due to the nonlinearity of the problem, simplifying assumptions that may lead to errors are generally utilized in loss formula methods, such as Taylor series expansion of power flow equations [1], quadratic loss expressions [7], and B-coefficients [8]. On the other hand, exact loss formulas are either based on individual power transactions [3] or do not differentiate between loss contributions arising from active- and reactive-power injections [2].

The proposed loss divider outlines the nonlinear dependence of system loss to nodal active- and reactive-power injections while accounting for the electrical distance between participants, and it does not depend on the location of the slack bus or any simplifying assumptions. The key benefit of this characteristic is that we can penalize (or reward) reactive-power demand (or support) from, e.g., loads and distributed energy resources, which may be responsible for total system loss increase (or decrease). Moreover, the loss divider is applicable to both mesh and radial network configurations, as well as those where transmission lines are modelled without ground connecting shunt elements. We demonstrate these aspects via numerical simulations involving standard distribution- and transmission-level test systems.

II. System Model

Consider an electric power network with N buses collected in the set $\mathcal{N}$. Transmission lines are modelled using the lumped-element equivalent Π-circuit model and collected in the set of E edges $\mathcal{E} := \{(m,n)\} \subset \mathcal{N} \times \mathcal{N}$. Collect nodal voltages and current injections in vectors $V \in \mathbb{C}^N$ and $I \in \mathbb{C}^N$ respectively.[1] The branch and shunt currents, respectively, in line $(m,n) \in \mathcal{E}$ can be expressed as

$$I_{(m,n)} = y_{mn}(V_m - V_n) = y_{mn} e_{mn}^{\mathrm{T}} V, \qquad (1)$$

$$I_{(m,n)}^{\mathrm{sh}} = y_{mn}^{\mathrm{sh}} V_m = y_{mn}^{\mathrm{sh}} e_m^{\mathrm{T}} V, \qquad (2)$$

[1]Notation: The matrix transpose is denoted by $(\cdot)^{\mathrm{T}}$, magnitude of a complex number by $|\cdot|$; complex conjugate by $(\cdot)^*$, complex-conjugate transposition by $(\cdot)^{\mathrm{H}}$, real and imaginary parts of a complex number or vector by $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$, respectively, and $\mathrm{j} := \sqrt{-1}$. A diagonal matrix formed with entries of the vector x is denoted by $\mathrm{diag}(x)$; and $\mathrm{diag}(x/y)$ forms a diagonal matrix with the m-th entry given by x_m/y_m, where x_m and y_m are the m-th entries of vectors x and y, respectively. For column vectors $x = [x_1, \ldots, x_M]^{\mathrm{T}}$ and $y = [y_1, \ldots, y_M]^{\mathrm{T}}$, $x \circ y$ denotes the entry-wise product of vectors x and y. The spaces of $N \times 1$ real- and complex-valued vectors are denoted by $\mathbb{R}^N$ and $\mathbb{C}^N$, respectively. The spaces of $M \times N$ real- and complex-valued matrices are denoted by $\mathbb{R}^{M\times N}$ and $\mathbb{C}^{M\times N}$, respectively. The entry in the m-th row and n-th column of the matrix X is denoted by $[X]_{mn}$. The $N \times 1$ vectors with all ones and all zeros are denoted by $\mathbb{1}_N$ and $\mathbb{0}_N$, respectively; e_m denotes a column vector of all zeros except with the m-th entry equal to 1; and $e_{mn} := e_m - e_n$.

Manuscript received October 25, 2019; revised February 11, 2020 and April 13, 2020; accepted April 19, 2020. Date of publication April 22, 2020; date of current version June 22, 2020. This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC), funding reference numbers RGPIN-2016-04271 and PGSD3-519078-2018. Paper no. PESL-00258-2019. *(Corresponding author: Christine Chen.)*

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Digital Object Identifier 10.1109/TPWRS.2020.2989632

where $y_{mn}, y_{mn}^{\mathrm{sh}} \in \mathbb{C}$ are, respectively, the series and shunt admittances of line (m,n). Kirchhoff's current law can be compactly represented in matrix-vector form as

$$I = YV, \tag{3}$$

where $Y \in \mathbb{C}^{N\times N}$ is the network admittance matrix. Since the admittance matrix is invertible, bus voltages can be expressed as $V = Y^{-1}I =: ZI$, where Z is the network impedance matrix [9]. Then, (1) and (2) can be written as

$$I_{(m,n)} = y_{mn} e_{mn}^{\mathrm{T}} ZI =: \kappa_{(m,n)}^{\mathrm{T}} I, \tag{4}$$

$$I_{(m,n)}^{\mathrm{sh}} = y_{mn}^{\mathrm{sh}} e_{m}^{\mathrm{T}} ZI =: (\kappa_{(m,n)}^{\mathrm{sh}})^{\mathrm{T}} I, \tag{5}$$

where $\kappa_{(m,n)}, \kappa_{(m,n)}^{\mathrm{sh}} \in \mathbb{C}^N$. The entries of $\kappa_{(m,n)}$ and $\kappa_{(m,n)}^{\mathrm{sh}}$ are referred to as the *current injection sensitivity factors* of line (m,n) with respect to the bus current injections. The current injection sensitivity factors in (4)–(5) uncover the impact of bus current injections on the current in line (m,n); in this sense, they serve as a measure of electrical distance between each bus and line (m,n) [9]. We will also find the decomposition of the network admittance and impedance matrices into real and imaginary components, i.e., $Y = G + \mathrm{j}B$ and $Z = R + \mathrm{j}X$, useful later.

Finally, denote the vector of bus complex-power injections by $S = P + \mathrm{j}Q \in \mathbb{C}^N$. (By convention, P_i and Q_i are positive for generation and negative for load.) Then, bus complex-power injections can be compactly written as

$$S = \mathrm{diag}(V) I^*. \tag{6}$$

Remark 1 (Noninvertible Network Admittance Matrix): Suppose that Y is not invertible, i.e., there are no ground-connecting shunt elements in the transmission-line model so that $I_{(m,n)}^{\mathrm{sh}} = 0, \forall\,(m,n) \in \mathcal{E}$ in (2). Here, we can premultiply (3) by the pseudoinverse of the admittance matrix, $Y^{\dagger}$, to get

$$V = Y^{\dagger} I + \frac{1}{N} \mathbb{1}_N \mathbb{1}_N^{\mathrm{T}} V, \tag{7}$$

which follows by recognizing that

$$Y^{\dagger} Y = \mathrm{diag}(\mathbb{1}_N) - \frac{1}{N} \mathbb{1}_N \mathbb{1}_N^{\mathrm{T}}. \tag{8}$$

Substituting (7) into (1), we find that the current injection sensitivity factors in this case are given by

$$\kappa_{(m,n)}^{\mathrm{T}} = y_{mn} e_{mn}^{\mathrm{T}} Y^{\dagger}, \tag{9}$$

where we have used the fact that $e_{mn}^{\mathrm{T}} \mathbb{1}_N \mathbb{1}_N^{\mathrm{T}} = \mathbb{0}_N^{\mathrm{T}}$. In summary, if Y is not invertible, we define the network impedance matrix $Z := Y^{\dagger}$ for the derivation in Section III. ■

III. Derivation of Exact Quadratic Loss Model

The system loss, denoted by L, can be expressed as the sum of losses on all lines, as follows:

$$L = \sum_{(m,n)\in\mathcal{E}} |I_{(m,n)}|^2 \mathrm{Re}\{y_{mn}^{-1}\} + |I_{(m,n)}^{\mathrm{sh}}|^2 \mathrm{Re}\{(y_{mn}^{\mathrm{sh}})^{-1}\}. \tag{10}$$

Collect all line branch and shunt currents in vectors $I_{\mathrm{line}} \in \mathbb{C}^E$ and $I_{\mathrm{line}}^{\mathrm{sh}} \in \mathbb{C}^E$, respectively. Similarly, collect the line series and shunt admittances into vectors $y_{\mathrm{line}} \in \mathbb{C}^E$ and $y_{\mathrm{line}}^{\mathrm{sh}} \in \mathbb{C}^E$, respectively. Recognizing that $|I_{(m,n)}|^2 = I_{(m,n)} I_{(m,n)}^*$, and $|I_{(m,n)}^{\mathrm{sh}}|^2 = I_{(m,n)}^{\mathrm{sh}} (I_{(m,n)}^{\mathrm{sh}})^*$ we can rewrite (10) as

$$\begin{aligned} L &= I_{\mathrm{line}}^{\mathrm{T}} \mathrm{diag}\left(\mathrm{Re}\{\mathbb{1}_E / y_{\mathrm{line}}\}\right) I_{\mathrm{line}}^* \\ &\quad + \left(I_{\mathrm{line}}^{\mathrm{sh}}\right)^{\mathrm{T}} \mathrm{diag}\left(\mathrm{Re}\{\mathbb{1}_E / y_{\mathrm{line}}^{\mathrm{sh}}\}\right) \left(I_{\mathrm{line}}^{\mathrm{sh}}\right)^*. \end{aligned} \tag{11}$$

A. System Loss and Current Injections

Stack up instances of (4) and (5), respectively, into matrix-vector form as $I_{\mathrm{line}} = K^{\mathrm{T}} I$ and $I_{\mathrm{line}}^{\mathrm{sh}} = K_{\mathrm{sh}}^{\mathrm{T}} I$, and further substitute them into (11) to yield

$$\begin{aligned} L &= I^{\mathrm{T}} K \mathrm{diag}\left(\mathrm{Re}\{\mathbb{1}_E / y_{\mathrm{line}}\}\right) K^{\mathrm{H}} I^* \\ &\quad + I^{\mathrm{T}} K_{\mathrm{sh}} \mathrm{diag}\left(\mathrm{Re}\{\mathbb{1}_E / y_{\mathrm{line}}^{\mathrm{sh}}\}\right) K_{\mathrm{sh}}^{\mathrm{H}} I^* =: I^{\mathrm{T}} \Gamma I^*. \end{aligned} \tag{12}$$

In the above, the current injection sensitivity matrices $K \in \mathbb{C}^{E\times N}$ and $K_{\mathrm{sh}} \in \mathbb{C}^{E\times N}$ are defined as

$$K := (\mathrm{diag}(y_{\mathrm{line}}) A Z)^{\mathrm{T}} = Z^{\mathrm{T}} A^{\mathrm{T}} \mathrm{diag}(y_{\mathrm{line}}), \tag{13}$$

$$K_{\mathrm{sh}} := \left(\mathrm{diag}(y_{\mathrm{line}}^{\mathrm{sh}}) A_{\mathrm{sh}} Z\right)^{\mathrm{T}} = Z^{\mathrm{T}} A_{\mathrm{sh}}^{\mathrm{T}} \mathrm{diag}(y_{\mathrm{line}}^{\mathrm{sh}}), \tag{14}$$

where the network incidence matrices $A \in \mathbb{R}^{E\times N}$, $A_{\mathrm{sh}} \in \mathbb{R}^{E\times N}$ are formed by stacking up row vectors e_{mn}^{T} and e_m^{T}, respectively, analogous to the way I_{line}, $I_{\mathrm{line}}^{\mathrm{sh}}$, y_{line}, and $y_{\mathrm{line}}^{\mathrm{sh}}$ are constructed in (11).

The expression in (12) uncovers the quadratic relationship between the system loss and current injections. Entries of $\Gamma \in \mathbb{C}^{N\times N}$ can be interpreted as complex-valued second-order sensitivities of loss with respect to bus current injections. Assuming that the network admittance matrix is invertible and transmission-line shunt conductances in the Π-circuit model are negligibly small (which is the case for overhead transmission lines [10]), straightforward algebraic manipulations allow us to simplify Γ defined in (12) as

$$\Gamma = R. \tag{15}$$

See Appendix A for the derivation of (15).

B. System Loss and Complex-Power Injections

We can express the system loss in (12) as the following quadratic function of the complex-power injections:

$$L = S^{\mathrm{H}} \mathrm{diag}\left(\Lambda^*\right) \cdot \Gamma \cdot \mathrm{diag}\left(\Lambda\right) S. \tag{16}$$

The expression in (16) is obtained by rearranging (6) to get

$$I^* = \mathrm{diag}\left(\frac{\mathbb{1}_N}{V}\right) S =: \mathrm{diag}\left(\Lambda\right) S, \tag{17}$$

and further substituting (17) into (12). In order to isolate the real component of (16) define $\mathrm{diag}(\Lambda) := \Xi + \mathrm{j}\Psi$ and recall that $S = P + \mathrm{j}Q$, so that we can rewrite (16) as

$$\begin{aligned} L &= (P - \mathrm{j}Q)^{\mathrm{T}} \left(\Xi - \mathrm{j}\Psi\right) \Gamma \left(\Xi + \mathrm{j}\Psi\right) \left(P + \mathrm{j}Q\right) \\ &=: (P - \mathrm{j}Q)^{\mathrm{T}} \left(U + \mathrm{j}W\right) \left(P + \mathrm{j}Q\right), \end{aligned} \tag{18}$$

where (recognizing that $\Gamma = R$)

$$U = \Xi R \Xi + \Psi R \Psi, \tag{19}$$

$$W = \Xi R \Psi - \Psi R \Xi. \tag{20}$$

Finally, we decompose (18) into its real component

$$L = P^{\mathrm{T}}UP + Q^{\mathrm{T}}UQ + P^{\mathrm{T}}(W^{\mathrm{T}} - W)Q, \tag{21}$$

and imaginary component

$$0 = P^{\mathrm{T}}WP + Q^{\mathrm{T}}WQ + P^{\mathrm{T}}(U - U^{\mathrm{T}})Q. \tag{22}$$

The above hold since system loss L is real valued by definition, and indeed, numerical simulations verify (21) and (22). The expression in (21) delineates the *exact* quadratic relationship for how the nodal active- and reactive-power injections contribute to system loss. We refer to (21) as the *loss divider*, since it specifies how the system loss is *divided* amongst active- and reactive-power injections at each bus. The derivation presented in this section does not rely on any simplifying assumptions. Effects of both the network topology and the operating point are embedded in the sensitivities U and W. In practice, when nodal voltages are near unity, the entries of U are much larger than those in W. Taking the limit where all voltages are precisely unity leads to $U = R$ and $W = \mathbb{0}_{N\times N}$ in (21).

IV. Transmission-Loss Allocation

In this section, we present one particular application for the exact quadratic expression for loss in (21) to decompose system loss into an admittedly arbitrary sum of contributions from nodal active- and reactive-power injections. We describe how the decomposition is related to the transmission-loss allocation method in [2]. From (21), we can express the system loss as the following summation of $2N$ scalar terms:

$$L=\sum_{i=1}^{N}(P^{\mathrm{T}}Ue_i+Q^{\mathrm{T}}We_i)P_i+\sum_{i=1}^{N}(Q^{\mathrm{T}}Ue_i-P^{\mathrm{T}}We_i)Q_i. \tag{23}$$

With (23), for each bus i, we compute the contributions of its active- and reactive-power injection components to the loss respectively as

$$\left(P^{\mathrm{T}}Ue_i + Q^{\mathrm{T}}We_i\right) P_i, \quad \left(Q^{\mathrm{T}}Ue_i - P^{\mathrm{T}}We_i\right) Q_i. \tag{24}$$

Note that, although (21) represents the exact relationship between system loss and nodal active- and reactive-power injections, the decomposition above represents one of many ways that the sum in (23) can be divided into constituent parts, each of which attributable to a particular active- or reactive-power injection.

A. Connection to Z-Bus Loss Allocation [2]

Although many other loss allocation schemes have been proposed in the literature, we expand in particular on the Z-bus method in [2] since it is circuit theoretic and is the most closely related method to the proposed one. The Z-bus allocation method extracts the real part of (12) and expresses the total system loss as the following sum:

$$L = \mathrm{Re}\left\{\sum_{k=1}^{N} I_k^* \left(\sum_{j=1}^{N}[R]_{kj}I_j\right)\right\}, \tag{25}$$

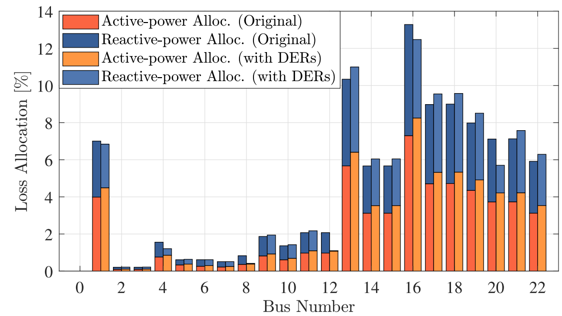

Fig. 1. Indian 22-Bus Power Distribution System: nodal injection contributions to system loss with and without DERs.

where the k-th term in the summation represents the contribution of the current injection at bus k to the total system loss. Our proposed loss-allocation method also begins with (12), but in addition, we explicitly uncover the exact quadratic relationship with respect to nodal active- and reactive-power injections. Thus, the results of the Z-bus allocation method can be recovered as the sum of the two expressions in (24).

B. Numerical Comparisons

We apply the loss allocation scheme in (24) for benchmark distribution and transmission test systems. Starting with a power-flow solution that consists of all nodal voltages, we compute the second-order sensitivities of loss with respect to nodal active- and reactive-power injections, as specified in (19)–(20).

1) Indian 22-Bus Power Distribution System [11]: The contribution of each active- and reactive-power injection to system loss is computed using (24). They are plotted as the darker orange and blue bars, respectively, in Fig. 1. Next, we assume that distributed energy resources (DERs) at buses 4, 8, 12, 16, and 20 inject additional $\Delta Q = 0.002$ p.u. at each bus. The updated active- and reactive-power contributions to loss are plotted in lighter orange and blue colours, respectively, in Fig. 1. For each bus k, adding active- and reactive-power injection contributions in (24) yields the same numerical value as the bus k current-injection contribution in the Z-bus allocation method (i.e., the k-th term in (25)) from [2]. However, by being able to decompose the bus k contribution to loss into components attributable to active- and reactive-power injections, our method can reward a DER for reactive-power support. As shown in Fig. 1, the reactive-power contributions to loss decrease at all buses with additional ΔQ injections.

2) New England 39-Bus System: In Fig. 2, we plot nodal active- and reactive-power injection contributions to system loss as the darker orange and blue bars, respectively. In this case, the nodal injections at buses 20–23 and 25–32 would, in fact, be rewarded for their negative contributions to system loss. Next, we eliminate shunt admittances from the Π-circuit model of all transmission lines, i.e., we set $y_{mn}^{\mathrm{sh}} = 0, \forall\,(m,n) \in \mathcal{E}$, which renders the network admittance matrix singular. Then, as described in Remark 1, we make use of the pseudoinverse of Y for the derivation in Section III. Resulting active- and reactive-power injection contributions to system loss are plotted

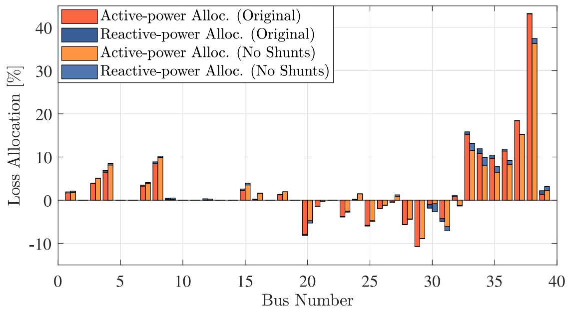

Fig. 2. New England 39-Bus System: nodal injection contributions to system loss with and without line shunt elements.

as the lighter orange and blue bars, respectively, in Fig. 2. Although removing the shunts from the Π-circuit model shifts the system operating point, using the pseudoinverse yields a reasonable outcome for loss allocation as compared with the original case with shunts.

V. Concluding Remarks

In this paper, we derive an analytical closed-form expression that attributes nodal active- and reactive-power injection contributions to the system loss. The derived expression recovers well-known loss allocation methods as special cases with certain simplifications. The utility of the derived expression is demonstrated via loss-allocation application studies involving modified Indian 22-bus and New England 39-bus test systems. Future work includes perturbation analysis of (21) to approximate incremental loss given nodal active- and reactive-power variations. Furthermore, incorporation of (21) to model system loss may be beneficial in a variety of problems, such as economic dispatch, optimal power flow, and transactive energy.

Appendix

A. Derivation of (15)

We first note that since $Z = Y^{-1}$, we have that

$$(G + \mathrm{j}B)(R + \mathrm{j}X) = (R + \mathrm{j}X)(G + \mathrm{j}B) = \mathrm{diag}(\mathbb{1}_N), \tag{26}$$

which leads to the following set of relationships:

$$GR - BX = RG - XB = \mathrm{diag}(\mathbb{1}_N), \tag{27}$$

$$GX + BR = XG + RB = \mathbb{0}_{N\times N}. \tag{28}$$

Then, substitute (13) and (14) into (12) to get

$$\begin{aligned}\Gamma &= Z^{\mathrm{T}} A^{\mathrm{T}} \mathrm{diag}(y_{\mathrm{line}} \circ \mathrm{Re}\{\mathbb{1}_E / y_{\mathrm{line}}\} \circ y^*_{\mathrm{line}}) A Z^* \\ &\quad + Z^{\mathrm{T}} A^{\mathrm{T}}_{\mathrm{sh}} \mathrm{diag}(y^{\mathrm{sh}}_{\mathrm{line}} \circ \mathrm{Re}\{\mathbb{1}_E / y^{\mathrm{sh}}_{\mathrm{line}}\} \circ (y^{\mathrm{sh}}_{\mathrm{line}})^*) A_{\mathrm{sh}} Z^*. \end{aligned} \tag{29}$$

Since $y_{\mathrm{line}} \circ y^*_{\mathrm{line}}$ and $y^{\mathrm{sh}}_{\mathrm{line}} \circ (y^{\mathrm{sh}}_{\mathrm{line}})^*$ are real-valued vectors, we can equivalently express (29) as

$$\begin{aligned}\Gamma &= Z^{\mathrm{T}} A^{\mathrm{T}} \mathrm{diag}\left(\mathrm{Re}\{y_{\mathrm{line}} \circ \mathbb{1}_E / y_{\mathrm{line}} \circ y^*_{\mathrm{line}}\}\right) A Z^* \\ &\quad + Z^{\mathrm{T}} A^{\mathrm{T}}_{\mathrm{sh}} \mathrm{diag}\left(\mathrm{Re}\{y^{\mathrm{sh}}_{\mathrm{line}} \circ \mathbb{1}_E / y^{\mathrm{sh}}_{\mathrm{line}} \circ (y^{\mathrm{sh}}_{\mathrm{line}})^*\}\right) A_{\mathrm{sh}} Z^*. \end{aligned} \tag{30}$$

Furthermore, since A and A_{sh} are real-valued matrices, $\mathrm{Re}\{y_{\mathrm{line}}\} = \mathrm{Re}\{y^*_{\mathrm{line}}\}$, and $\mathrm{Re}\{y^{\mathrm{sh}}_{\mathrm{line}}\} = \mathrm{Re}\{(y^{\mathrm{sh}}_{\mathrm{line}})^*\}$, (30) simplifies as

$$\begin{aligned}\Gamma &= Z^{\mathrm{T}} \mathrm{Re}\{A^{\mathrm{T}} \mathrm{diag}\left(y_{\mathrm{line}}\right) A\} Z^* \\ &\quad + Z^{\mathrm{T}} \mathrm{Re}\{A^{\mathrm{T}}_{\mathrm{sh}} \mathrm{diag}\left(y^{\mathrm{sh}}_{\mathrm{line}}\right) A_{\mathrm{sh}}\} Z^* \\ &= Z^{\mathrm{T}} \mathrm{Re}\{A^{\mathrm{T}} \mathrm{diag}\left(y_{\mathrm{line}}\right) A + A^{\mathrm{T}}_{\mathrm{sh}} \mathrm{diag}\left(y^{\mathrm{sh}}_{\mathrm{line}}\right) A_{\mathrm{sh}}\} Z^*. \end{aligned} \tag{31}$$

Recognizing in the above that, if the real part of $y^{\mathrm{sh}}_{\mathrm{line}}$ is small, we get

$$\mathrm{Re}\{A^{\mathrm{T}} \mathrm{diag}(y_{\mathrm{line}}) A + A^{\mathrm{T}}_{\mathrm{sh}} \mathrm{diag}(y^{\mathrm{sh}}_{\mathrm{line}}) A_{\mathrm{sh}}\} \approx \mathrm{Re}\{Y\}, \tag{32}$$

where the approximation would be exact if shunt admittances were purely imaginary. We substitute (32) into (31) to get

$$\Gamma = Z^{\mathrm{T}} \mathrm{Re}\{Y\} Z^*. \tag{33}$$

Then, we expand (33) to get

$$\Gamma = (R + \mathrm{j}X) G (R - \mathrm{j}X) = (R + \mathrm{j}X)(GR - \mathrm{j}GX).$$

Recognizing that $-GX = BR$ from (28), we get

$$\begin{aligned}\Gamma &= (R + \mathrm{j}X)(GR + \mathrm{j}BR) \\ &= (RG - XB)R + \mathrm{j}(RB + XG)R. \end{aligned} \tag{34}$$

Finally, substitution of (27) and (28) into (34) yields (15) as desired.

References

[1] Q. Ding and A. Abur, "Transmission loss allocation based on a new quadratic loss expression," *IEEE Trans. Power Syst.*, vol. 21, no. 3, pp. 1227–1233, Aug. 2006.

[2] A. J. Conejo, F. D. Galiana, and I. Kockar, "Z-bus loss allocation," *IEEE Trans. Power Syst.*, vol. 16, no. 1, pp. 105–110, Feb. 2001.

[3] Q. Ding and A. Abur, "Transmission loss allocation in a multiple-transaction framework," *IEEE Trans. Power Syst.*, vol. 19, no. 1, pp. 214–220, Feb. 2004.

[4] M. Ilic, F. Galiana, and L. Fink, *Power Systems Restructuring: Engineering and Economics*. Berlin, Germany: Springer Science & Business Media, 2013.

[5] F. D. Galiana, A. J. Conejo, and I. Kockar, "Incremental transmission loss allocation under pool dispatch," *IEEE Trans. Power Syst.*, vol. 17, no. 1, pp. 26–33, Feb. 2002.

[6] D. Kirschen, R. Allan, and G. Strbac, "Contributions of individual generators to loads and flows," *IEEE Trans. Power Syst.*, vol. 12, no. 1, pp. 52–60, Feb. 1997.

[7] G. Gross and S. Tao, "A physical-flow-based approach to allocating transmission losses in a transaction framework," *IEEE Trans. Power Syst.*, vol. 15, no. 2, pp. 631–637, May. 2000.

[8] Y.-C. Chang, W.-T. Yang, and C.-C. Liu, "A new method for calculating loss coefficients of power systems," *IEEE Trans. Power Syst.*, vol. 9, no. 3, pp. 1665–1671, Aug. 1994.

[9] A. J. Conejo, J. Contreras, D. A. Lima, and A. Padilha-Feltrin, "Zbus transmission network cost allocation," *IEEE Trans. Power Syst.*, vol. 22, no. 1, pp. 342–349, Feb. 2007.

[10] H. W. Dommel, *EMTP Theory Book*. Vancouver, Canada: Microtran Power System Analysis Corporation, 1996.

[11] R. D. Zimmerman, C. E. Murillo-Sánchez, and R. J. Thomas, "MATPOWER: Steady-state operations, planning and analysis tools for power systems research and education," *IEEE Trans. Power Syst.*, vol. 26, no. 1, pp. 12–19, Feb. 2011.