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Reexamining the Distributed Slack Bus

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***Abstract*—Power flow formulated with a distributed slack bus involves modeling the active-power output of each generator with three elements: a *nominal injection* modulated by a fraction of the *net-load imbalance* allocated via a *participation factor*. This setup acknowledges generator dynamics and system operations, but it has long been plagued by ambiguous and inconsistent interpretations of its constituent elemental quantities. In this paper, we establish that, with the: i) nominal active-power injections set to be the economic dispatch setpoints, ii) participation factors fixed to be the ones used in automatic generation control, and iii) net-load imbalance considered to be the total load and loss unaccounted in economic dispatch, the power flow solution best matches results from a simulation of the system differential algebraic equation (DAE) model. Numerical case studies tailored to the New England test system validate the analysis by demonstrating that solutions obtained from a distributed slack formulation offer lower errors (with respect to DAE-model simulations) compared to the exhaustive set of all single slack bus choices.**

***Index Terms*—Automatic generation control, distributed slack bus, economic dispatch, participation factors, power flow.**

I. INTRODUCTION

To the best of our knowledge, the first documented version of the standard textbook power flow problem with a single slack bus is presented in a 1956 paper by J. B. Ward and H. W. Hale [1]. The notion of a slack bus appears in earlier works [2], [3], and its use seems to have been standard in network analyzer studies (a form of analog simulation for power networks) as implied in [1]. A common justification for requiring a slack bus in the power flow problem is that, since the network losses are unknown prior to solving for system states, one cannot specify the power injected at all generator buses. Thus, the power output at one generator is left unspecified and determined as part of the power flow solution. This rationale was perfectly aligned with load frequency control schemes used in the first half of the twentieth century, when it was standard practice for a single generator per area to provide frequency regulation [4]. In support of this hypothesis, we quote the following remark by E. E. George from the Discussion section of [1]:

> *"The slack machine is the regulating generator which controls frequency or tie-line loading, and which cannot be scheduled in megawatt output until the difference between total generation and total load plus loss is calculated, or measured by telemeters, or balanced by a frequency controller."*

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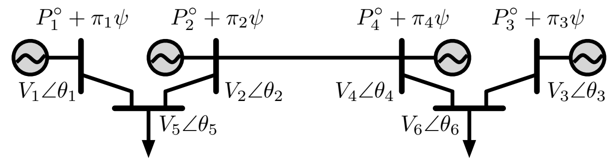

Fig. 1: Generator active-power injections are modeled as $P_g^\circ + \pi_g\psi$ in the distributed slack bus formulation. This paper uncovers appropriate values for P_g°, π_g, and ψ, so that bus voltage magnitudes and phase angles solved from the power flow match steady-state results from a dynamic simulation.

With the introduction of tie-line bias control circa 1948 through the efforts of N. Cohn and R. Brandt [4], [5], the industry shifted to using multiple generators to regulate frequency in a given area. Consequently, the power flow problem formulated with a single slack bus no longer aligned with generator dynamics and system operations. This motivated the conceptualization of the distributed slack bus. The first publication on the topic appears to be a 1971 M.S. thesis advised by F. C. Schweppe [6]. In it, a heuristic strategy is proposed to allocate system loss to generators through successive iterations of power flow solution algorithms.

The underlying premise of formulating the power flow problem with a distributed slack bus is to model the active-power output of each generator g, as follows:

$$P_g^\circ + \pi_g\psi,$$

where P_g° is a nominal active-power injection, π_g is a participation factor, and ψ is the net-load imbalance. (See Fig. 1 for an illustration.) While P_g° and π_g are assumed to be known for all generators, ψ is an unknown *slack* variable that is solved as part of the power flow problem. Power flow solved with the above formulation can yield system states, i.e., bus voltage magnitudes and phase angles, that are in agreement with the simulation of a nonlinear differential algebraic equation (DAE) model simulated unto steady state. This is contingent on choosing correct values for the elemental quantities P_g°, π_g, and ψ—a task that is challenged by the complexity of present-day power system operations, system controllers, generator dynamics, and nonlinearities rife in the constituent models.

We examine a collection of synchronous generators and constant-power loads dispersed over multiple control areas with a standard automatic generation control (AGC) system for frequency regulation and tie-line bias control. Generator references are determined by allocating the net-load imbalance (beyond that acknowledged by economic dispatch) in proportion to AGC participation factors to eliminate the area control error. The main result of the paper is summarized below:

For the most accurate estimate of system states in steady state from the power flow equations, i) P_g° should correspond to the setpoint from economic dispatch, ii) π_g should be set to the AGC participation factor, and iii) ψ should be interpreted as the sum of net load and losses unaccounted in economic dispatch for the corresponding control area.

The statement above invokes variables and control signals across a wide range of temporal and spatial scales: the nominal setpoint derives from slow timescale optimization (economic dispatch), the participation factor is native to fast(er) timescale control (AGC), and the slack variable is specified for a potentially vast geographical expanse (a control area). Our analysis demonstrates that such a cohesive and unified examination across spatio-temporal scales is key to precisely formulating the distributed slack bus. Undeniably, this work has significant pedagogical relevance. Topics pertinent to the main result of this paper (i.e., generator models, AGC, and economic dispatch) are individually covered in detail in popular undergraduate- and graduate-level textbooks (e.g., [7]–[9]). With minimal incremental effort, they can be weaved together to construct the distributed slack version of the power flow problem so that curriculum on this fundamental topic is standardized and aligned with prevailing operational practices.

We provide next, a brief overview of pertinent literature. In [10], participation factors for the distributed slack bus are chosen based on a perturbation analysis of a classical economic dispatch problem. Our analysis reveals that such a choice would yield accurate results for system states only if the AGC participation factors were also set the same way.[1] In [13], the distributed slack bus participation factors are determined from combined cost and reliability criteria. While passing remarks reference selecting participation factors based on AGC, there is no technical discussion on (or justification for) the choice. By and large, the lack of consensus on problem formulation is starkly obvious in the literature. Consider, for instance, that the gamut of participation factors referenced in [14]–[19] includes those attributable to economic dispatch, AGC action, governor control, and inertial response. An analytical treatment that justifies the choice of all elemental components of the distributed slack bus (i.e., the nominal active-power injections, the participation factors, and the slack variable) is notably absent. While the potential of obtaining accurate estimates of system states agnostic to the choice of a single slack bus has driven the incorporation of the distributed slack bus in optimal power flow and economic dispatch [20]–[25], inconsistent interpretations and choices of variables that are key to a precise problem formulation are recurring concerns. We also bring to attention a wide body of literature on the solvability of power flow equations, including [26]–[33], which largely disregards the impact of system operations and generator dynamics that can be readily addressed with a distributed slack bus. Several forays in distribution networks have been made [34]–[36], but we omit a detailed review of these works since our focus is on transmission networks. Finally, we note that software packages like Powerworld [37] and PSAT [38] support distributed slack formulations with customizable participation factors. (Based on the analysis in this paper, we would advocate for the default choice to be the AGC participation factors.)

The literature surveyed above reveals several gaps. First, limited effort has been expended in precisely teasing out the impact of generator dynamics and system operations on the power flow problem in general, and on the distributed slack bus formulation in particular. Second, interpretations of the net-load imbalance, generator setpoints, and participation factors have been shrouded in ambiguity, which has limited their applicability and hindered widespread adoption. Finally, the connections to economic dispatch and AGC have not been clearly explained. This has meant that participation factors for the distributed slack bus—while unambiguously attributable to AGC dynamics—have frequently been interpreted incorrectly in an economic context. With the main result of the paper summarized previously, we address these gaps in the literature. Our conclusions follow from examining the swing, turbine governor, and AGC dynamics in steady state. The analysis is temporally situated between successive executions of economic dispatch, so that the net-load imbalance and generator active-power setpoints assume precise and quantifiable interpretations. While a reduced-order model of machine and controller dynamics for generators is leveraged for analysis, numerical validation is provided with simulations involving detailed two-axis generator, governor and exciter controls, and network models that preserve losses, nonlinear elements, and higher-order behavior [39].

The remainder of this paper is organized as follows. The system DAE model and power flow problem formulation with a single slack bus and the distributed slack bus are overviewed in Section II. Interpretation and justification for the values of the nominal generator setpoints, participation factors, and the slack variable are provided in Section III. In Section IV, we provide a suite of case studies to validate the analysis. Concluding remarks are presented in Section V.

II. Preliminaries

This section first presents the system dynamical model (covering network power balance equations, generator dynamics, AGC, and economic dispatch). The single and distributed slack variants of the power flow problem are then overviewed.

A. Network Power Balance and Generator Dynamical Models

Consider an AC electric power network with buses indexed by elements in set $\mathcal{N}$. Let $\mathcal{G} \subseteq \mathcal{N}$ and $\mathcal{L} \subset \mathcal{N}$ denote the subsets of generator and load buses, respectively. The network is divided into control areas indexed by elements in set $\mathcal{A}$. Buses situated in control area $a \in \mathcal{A}$ are denoted by $\mathcal{N}^a \subseteq \mathcal{N}$, and corresponding generator and load buses by $\mathcal{G}^a \subseteq \mathcal{G}$ and $\mathcal{L}^a \subseteq \mathcal{L}$, respectively. Let V_k [p.u.] and θ_k [rad] denote the bus voltage magnitude and phase angle at bus k, and let P_k [p.u.] and Q_k [p.u.] denote the active- and reactive-power injections

[1] Selecting AGC participation factors based on a perturbation analysis of the companion economic dispatch routine encourages economic optimality between successive dispatch updates [9]. However, there is no indication that this is followed in practice (see, e.g., [11], [12]).

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(originating from a generator, load, or neighboring control area via tie-line flow) at bus k, respectively.[2] The algebraic power flow equations capture the balance of injected active and reactive power. For bus $k \in \mathcal{N}^a$, $a \in \mathcal{A}$, these take the form

$$P_k = V_k \sum_{j \in \mathcal{N}^a} V_j (B_{kj} \sin\theta_{kj} + G_{kj} \cos\theta_{kj}), \tag{1}$$

$$Q_k = V_k \sum_{j \in \mathcal{N}^a} V_j (G_{kj} \sin\theta_{kj} - B_{kj} \cos\theta_{kj}), \tag{2}$$

where $\theta_{kj} := \theta_k - \theta_j$, and G_{kj} [p.u.] and B_{kj} [p.u.] denote the real and imaginary parts of the (k, j) entry in the network admittance matrix, respectively. We adopt the constant power model for loads and the classical model for generators [39]. Then, P_k and Q_k in (1)–(2) are assumed to be known and fixed for all load buses, while for generator buses they are given by $\frac{E_g V_g}{X_g'} \sin(\delta_g - \theta_g)$ and $\frac{E_g V_g}{X_g'} \cos(\delta_g - \theta_g) - \frac{V_g^2}{X_g'}$, respectively, where E_g [p.u.] is the magnitude of the constant voltage source behind transient reactance X_g' [p.u.], and δ_g [rad] denotes the rotor electrical angular position for generator g.

The generator electromechanical dynamics are described by the swing equation augmented with a first-order turbine governor model. Dynamic states for generator g include δ_g [rad], electrical frequency, ω_g [rad/s], and the turbine mechanical power, P_g^{m} [p.u.]. Their evolution is governed by

$$\dot{\delta}_g = \omega_g - \omega_{\mathrm{s}}, \tag{3}$$

$$M_g \dot{\omega}_g = P_g^{\mathrm{m}} - P_g, \tag{4}$$

$$\tau_g \dot{P}_g^{\mathrm{m}} = P_g^{\mathrm{r}} - P_g^{\mathrm{m}} - \frac{1}{R_g \omega_{\mathrm{s}}} (\omega_g - \omega_{\mathrm{s}}), \tag{5}$$

where M_g [$\mathrm{s}^2/\mathrm{rad}$] denotes the inertia constant; τ_g [s], R_g [p.u.], and P_g^{r} [p.u.] are the governor time constant, droop constant, and reference setpoint, respectively; ω_{s} [rad/s] is the synchronous frequency; and P_g [p.u.] is the active power injected at generator bus g as described by (1) [39]. We do not consider voltage dynamics in the analytical developments, rather, we treat the generator terminal voltage, V_g [p.u.], as an input (in practice, it is governed by exciter control and a voltage regulator). The system DAE model comprises appropriate instances of (1)–(2) (for all load buses) and (1), (3)–(5) (for all generator buses). Load active- and reactive-power injections, generator reference setpoints, and generator terminal voltages are inputs to the model.

B. AGC and Economic Dispatch

In each control area, AGC action modulates generator active-power outputs to eliminate frequency deviations (that may arise, e.g., from load fluctuations not acknowledged in economic dispatch) and satisfy scheduled net tie-line flow. Next, we describe one particular instance of a practical implementation for the same. For generator $g \in \mathcal{G}^a$, the reference, P_g^{r}, (see (5)) is given by

$$P_g^{\mathrm{r}} = P_g^\star + \alpha_g \Big(\xi^a - \sum_{j \in \mathcal{G}^a} P_j^\star \Big), \tag{6}$$

[2]In the remainder of the paper, quantities related to a generator bus (or a generator) are indexed by subscript g, a load bus by ℓ, and a generic bus by k.

where $P_g^\star$ is the economic dispatch setpoint, α_g is the *AGC participation factor* (chosen such that $\sum_{g \in \mathcal{G}^a} \alpha_g = 1$), and ξ^a [p.u.] is the AGC state. The evolution of ξ^a is governed by

$$\dot{\xi}^a = -\xi^a - \mathrm{ACE}^a + \sum_{g \in \mathcal{G}^a} P_g, \tag{7}$$

where ACE^a is the area control error that accounts for deviations in net tie-line flows with neighboring control areas from their scheduled values as well as frequency deviations from the synchronous value [9], [40]. Typically, the area control error is defined as

$$\mathrm{ACE}^a := (P_{\mathrm{tl}}^a - P_{\mathrm{tl}}^{a\star}) - B^a(\omega^a - \omega_{\mathrm{s}}), \tag{8}$$

where P_{tl}^a and $P_{\mathrm{tl}}^{a\star}$ denote the actual and scheduled net tie-line flows away from control area a, respectively, $B^a < 0$ is the area bias factor, and ω^a denotes the prevailing frequency [9], [41]. While practical setups differ on how ω^a is computed from measurements, we assume it is the average electrical frequency of all generators in control area a, i.e.,

$$\omega^a = \frac{1}{|\mathcal{G}^a|} \sum_{g \in \mathcal{G}^a} \omega_g, \tag{9}$$

where $|\mathcal{G}^a|$ denotes cardinality of the set $\mathcal{G}^a$. As discussed previously, $P_g^\star$, $g \in \mathcal{G}$, are optimizers of a network-wide economic dispatch problem, which we assume takes the form

$$\begin{aligned} \underset{P_g,\, g \in \mathcal{G}^a,\, a \in \mathcal{A}}{\text{minimize}} \quad & \sum_{a \in \mathcal{A}} \sum_{g \in \mathcal{G}^a} C_g(P_g) \\ \text{subject to} \quad & \sum_{a \in \mathcal{A}} \sum_{g \in \mathcal{G}^a} P_g = \sum_{a \in \mathcal{A}} P_{\mathrm{load}}^{a\star}, \end{aligned} \tag{10}$$

where $C_g(\cdot)$ [\$/hr] denotes the cost function for generator g and $P_{\mathrm{load}}^{a\star}$ is the look-ahead net load for control area a (without accounting for system loss or real-time load fluctuations). The precise formulation of the economic dispatch problem is not critical to subsequent developments; the only variables of importance in the context of the distributed slack formulation are the optimizers, $P_g^\star$, $g \in \mathcal{G}$. These may result from a more detailed rendition of the optimization problem in (10).

C. Single and Distributed Slack Variants of Power Flow

The combination of the power balance equations, generator electromechanical dynamics, AGC, and economic dispatch dictates the evolution of system states. If one solely cares about their values in steady state, the general solution strategy is to focus on the algebraic power balance equations since the alternative of performing time-domain simulations of a DAE model (such as the one given by (1)–(8)) unto steady state is not computationally commensurate. A major challenge in this regard is to determine the active-power outputs from the generators in steady state since these depend on dynamic *and* static state variables (δ_g and V_g, θ_g, respectively).

The standard workaround for the above problem is to: i) assume that generator active-power outputs are fixed to some nominal setpoints, and ii) attribute all unforecasted load variations and unmodeled system loss to one generator. This recovers the power flow problem with a single slack bus. The

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distributed slack bus alternative involves modeling the active-power balance at each generator bus g as follows:

$$P_g^\circ + \pi_g \psi = V_g \sum_{j \in \mathcal{N}^a} V_j (B_{gj} \sin\theta_{gj} + G_{gj} \cos\theta_{gj}), \tag{11}$$

where P_g° is a nominal active-power injection, π_g is a participation factor, and ψ denotes the slack that is to be allocated amongst all generators. Similar to the standard power flow formulation with a single slack bus, the nominal active-power injection, P_g°, and the voltage magnitude, V_g, are fixed at generator buses (as are the active- and reactive-power injections at load buses). Unspecified voltage magnitudes and phase angles are solved from power flow.

III. Formalizing the Elements of the Distributed Slack Bus

In Section III-A, we uncover appropriate values for P_g°, π_g, and ψ, so that the power flow solution yields estimates of bus voltage magnitudes and phase angles that best match results from DAE simulations executed through to steady state. First, we show that the nominal active-power injection, P_g°, should correspond to the optimizer from the economic dispatch problem in (10):

$$P_g^\circ = P_g^\star. \tag{12}$$

Second, the appropriate choice for the participation factor, π_g, is the AGC participation factor that appears in (6), i.e.,

$$\pi_g = \alpha_g. \tag{13}$$

Finally, the slack variable, ψ, is shown to be the sum of all active-power load and loss not acknowledged in economic dispatch for the control area to which the generator belongs. In particular,

$$\psi = \Delta P^a := \Delta P^a_{\mathrm{load}} + P^a_{\mathrm{loss}}, \tag{14}$$

where ΔP^a is the net-load imbalance for control area a, given by the sum of unaccounted load $\Delta P^a_{\mathrm{load}}$ and loss P^a_{loss}. These are defined as

$$\Delta P^a_{\mathrm{load}} := \sum_{\ell \in \mathcal{L}^a} P_\ell - P^{a\star}_{\mathrm{load}}, \tag{15}$$

$$P^a_{\mathrm{loss}} := \sum_{k \in \mathcal{N}^a} \big(V_k^2 G_{kk} + 2 \sum_{j \in \mathcal{N}^a} V_k V_j G_{kj} \cos\theta_{kj} \big). \tag{16}$$

Figure 2 illustrates the system architecture and the main result. The optimization, control, and physical layers of the system are shown in Fig. 2(a). During periods of steady-state operation between successive economic dispatch updates (see Fig. 2(b)), specifying the active-power injection for each generator bus as $P_g^\star + \alpha_g \Delta P^a$ (see Fig. 2(c)) in the power flow equations yields the best estimate of bus voltage magnitudes and phase angles. To explain what ΔP^a precisely represents, suppose Fig. 2(b) illustrates system frequency through the course of the following stylized changes. The economic dispatch problem (10) is executed at some initial time with the load perfectly known. The system then settles to steady state following minor dynamic fluctuations during which losses are compensated. At a later time instant, there is a non-trivial step increase in the system net load. Following this, the

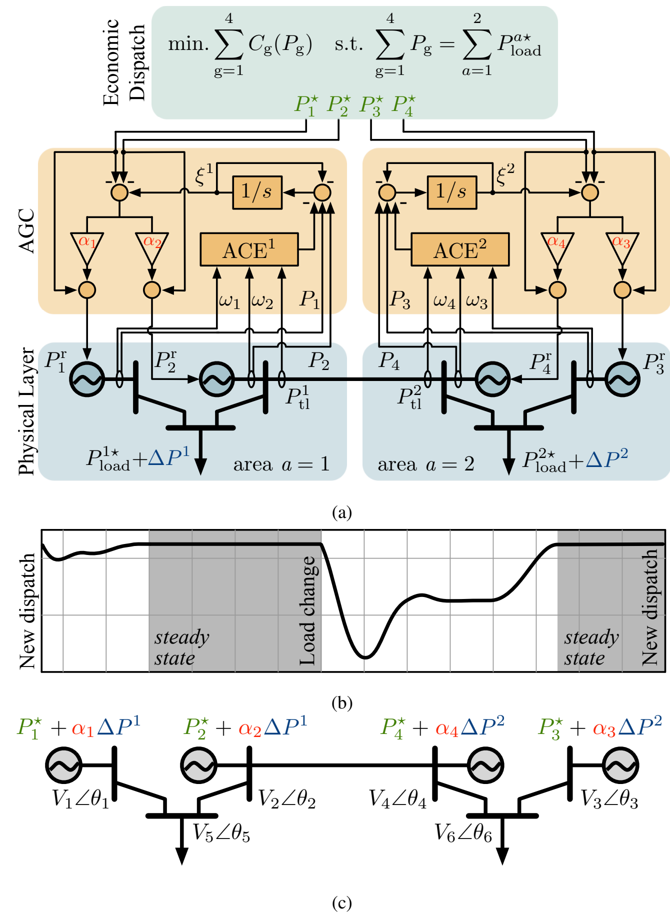

Fig. 2: (a) Illustration of economic dispatch and AGC architecture for a power system with two control areas. (b) [*not to scale*] Sketch of prevailing frequency in a control area. Steady-state operation (shaded gray) is punctuated by dynamic excursions that may arise due to new economic dispatch setpoints or load changes. (c) Power flow formulation with distributed slack bus. Modeling injections of generator buses as $P_g^\star + \alpha_g \Delta P^a$ in the algebraic power flow equations yields the best estimates for bus voltage magnitudes and phase angles during periods of steady state between economic dispatch updates.

system returns to steady state after (sustained and significant) AGC action. With this in mind, consider solving the power flow problem with a distributed slack bus during the two periods of steady-state operation shaded in gray. In the first steady-state period following the initial execution of economic dispatch, ΔP^a *only* represents the unaccounted losses. In the second one, ΔP^a encompasses *both* the load change and accompanying change in loss.

After establishing the main result in Section III-A, in Section III-B we overview the algebraic equations, known variables, and unknown variables in the distributed slack bus power flow problem.

A. *Establishing the Main Result:* (12)–(14)

To show (12)–(14), we have to examine aggregate dynamics in each control area. Summing (4)–(5) over all generators in area a and appending companion AGC dynamics in (7) yields

$$\sum_{g \in \mathcal{G}^a} M_g \dot{\omega}_g = \sum_{g \in \mathcal{G}^a} P_g^{\mathrm{m}} - P^a_{\mathrm{gen}}, \tag{17}$$

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$$\sum_{g\in\mathcal{G}^a} \tau_g \dot{P}_g^{\mathrm{m}} = \xi^a - \sum_{g\in\mathcal{G}^a} \Big(P_g^{\mathrm{m}} + \frac{1}{R_g\omega_{\mathrm{s}}}(\omega_g - \omega_{\mathrm{s}})\Big), \quad (18)$$

$$\dot{\xi}^a = -\xi^a - \mathrm{ACE}^a + P_{\mathrm{gen}}^a, \quad (19)$$

where we have used the fact that $\sum_{g\in\mathcal{G}^a} \alpha_g = 1$, and

$$P_{\mathrm{gen}}^a := \sum_{g\in\mathcal{G}^a} P_g \quad (20)$$

is the total active power generated in control area a. Since the power flow problem is solved to obtain system states in steady state, we consider the aggregate dynamics for each control area a, i.e., (17)–(19), at some time instant $t = t_{\mathrm{ss}}$ when $\dot{\omega}_g(t_{\mathrm{ss}}) = \dot{P}_g^{\mathrm{m}}(t_{\mathrm{ss}}) = 0\ \forall\, g \in \mathcal{G}^a$, and $\dot{\xi}^a(t_{\mathrm{ss}}) = 0$:[3]

$$0 = \sum_{g\in\mathcal{G}^a} P_g^{\mathrm{m}}(t_{\mathrm{ss}}) - P_{\mathrm{gen}}^a, \quad (21)$$

$$0 = \xi^a(t_{\mathrm{ss}}) - \sum_{g\in\mathcal{G}^a}\Big(P_g^{\mathrm{m}}(t_{\mathrm{ss}}) + \frac{1}{R_g\omega_{\mathrm{s}}}(\omega_g(t_{\mathrm{ss}}) - \omega_{\mathrm{s}})\Big), \quad (22)$$

$$0 = -\xi^a(t_{\mathrm{ss}}) - \mathrm{ACE}^a(t_{\mathrm{ss}}) + P_{\mathrm{gen}}^a. \quad (23)$$

Substituting (9) in (8) for time $t = t_{\mathrm{ss}}$ yields

$$\mathrm{ACE}^a(t_{\mathrm{ss}}) = (P_{\mathrm{tl}}^a(t_{\mathrm{ss}}) - P_{\mathrm{tl}}^{a\star}) - \frac{B^a}{|\mathcal{G}^a|}\sum_{g\in\mathcal{G}^a}(\omega_g(t_{\mathrm{ss}}) - \omega_{\mathrm{s}}). \quad (24)$$

Summing (21)–(23) over all control areas and appropriately substituting (24), we get

$$\sum_{a\in\mathcal{A}}\Big(P_{\mathrm{tl}}^a(t_{\mathrm{ss}}) - P_{\mathrm{tl}}^{a\star} + \sum_{g\in\mathcal{G}^a}\Big(\frac{1}{R_g\omega_{\mathrm{s}}} - \frac{B^a}{|\mathcal{G}^a|}\Big)(\omega_g(t_{\mathrm{ss}}) - \omega_{\mathrm{s}})\Big) = 0. \quad (25)$$

Since losses in transmission lines are negligible, it follows that

$$\sum_{a\in\mathcal{A}} P_{\mathrm{tl}}^a(t_{\mathrm{ss}}) \approx 0. \quad (26)$$

Furthermore, assuming that the scheduled tie-line flows out of a control area match those into the area (with opposite direction), we get:

$$\sum_{a\in\mathcal{A}} P_{\mathrm{tl}}^{a\star} = 0. \quad (27)$$

With (26) and (27), (25) simplifies as

$$\sum_{a\in\mathcal{A}}\sum_{g\in\mathcal{G}^a}\left(\frac{1}{R_g\omega_{\mathrm{s}}} - \frac{B^a}{|\mathcal{G}^a|}\right)(\omega_g(t_{\mathrm{ss}}) - \omega_{\mathrm{s}}) = 0. \quad (28)$$

Since all flows in the network are constant in steady state, it follows that $\dot{\delta}_g(t_{\mathrm{ss}}) = \dot{\theta}_g(t_{\mathrm{ss}})\ \forall\, g \in \mathcal{G}$, and $\dot{\theta}_g(t_{\mathrm{ss}}) = \dot{\theta}_k(t_{\mathrm{ss}})\ \forall\, k \in \mathcal{N}_g$, where $\mathcal{N}_g$ denotes the set of buses electrically connected to bus g. Furthermore, the electric network is connected, which renders $\dot{\delta}_g(t_{\mathrm{ss}}) =: \Delta\omega_{\mathrm{ss}}\ \forall\, g \in \mathcal{G}$. From (3), this implies $\omega_g(t_{\mathrm{ss}}) = \omega_{\mathrm{s}} + \Delta\omega_{\mathrm{ss}}\ \forall\, g \in \mathcal{G}$. Then, (28) becomes

$$\Delta\omega_{\mathrm{ss}}\sum_{a\in\mathcal{A}}\sum_{g\in\mathcal{G}^a}\left(\frac{1}{R_g\omega_{\mathrm{s}}} - \frac{B^a}{|\mathcal{G}^a|}\right) = 0. \quad (29)$$

Since $R_g > 0$ and $B^a < 0$, we see from (29) that $\Delta\omega_{\mathrm{ss}} = 0$. This further implies $\omega_g(t_{\mathrm{ss}}) = \omega_{\mathrm{s}}\ \forall\, g \in \mathcal{G}$. Indeed, AGC action restores the frequency of each generator (and therefore, the prevailing frequency of each area) back to the synchronous value. Summing (21)–(23) with $\omega_g(t_{\mathrm{ss}}) = \omega_{\mathrm{s}}$, we see that $\mathrm{ACE}^a(t_{\mathrm{ss}}) = 0$. Substituting this into (23) yields

$$\xi^a(t_{\mathrm{ss}}) = P_{\mathrm{gen}}^a = \sum_{g\in\mathcal{G}^a} P_g^\star + \Delta P^a, \quad (30)$$

where the total active power generated in control area a, P_{gen}^a, is expressed as the sum of the optimizers from economic dispatch, $P_g^\star$, $g \in \mathcal{G}^a$, (see (10)), and the net-load imbalance, ΔP^a (see (14)). Next, substituting (6) into (5), and recognizing that $\dot{P}_g^{\mathrm{m}}(t_{\mathrm{ss}}) = 0$ and $\omega_g(t_{\mathrm{ss}}) = \omega_{\mathrm{s}}$, we get

$$P_g^{\mathrm{m}}(t_{\mathrm{ss}}) = P_g^\star + \alpha_g\Big(\xi^a(t_{\mathrm{ss}}) - \sum_{j\in\mathcal{G}^a} P_j^\star\Big) = P_g^\star + \alpha_g\Delta P^a, \quad (31)$$

where the second equality above follows by substituting (30). Finally, rearranging terms in (4) for time $t = t_{\mathrm{ss}}$, we get

$$P_g(t_{\mathrm{ss}}) = P_g^{\mathrm{m}}(t_{\mathrm{ss}}) - M_g\dot{\omega}_g(t_{\mathrm{ss}}) = P_g^\star + \alpha_g\Delta P^a, \quad (32)$$

where the second equality follows from (31) and setting $\dot{\omega}_g(t_{\mathrm{ss}}) = 0$. This establishes the main result in (12)–(14).

Remark 1 (Subset of Generators Participate in AGC): Consider the setting where only generators in a subset $\mathcal{G}^{a\prime} \subseteq \mathcal{G}^a$ participate in AGC in control area a. It follows that $\alpha_g = 0\ \forall\, g \in \mathcal{G}^a \setminus \mathcal{G}^{a\prime}$ in (6), with $\sum_{g\in\mathcal{G}^{a\prime}} \alpha_g = 1$. Using the same analytical procedure as above, we recover $\pi_g = \alpha_g\ \forall\, g \in \mathcal{G}^{a\prime}$, and $\pi_g = 0\ \forall\, g \in \mathcal{G}^a \setminus \mathcal{G}^{a\prime}$, as the corresponding participation factors for the distributed slack bus. Only generators that participate in AGC have nonzero participation factors. This is intuitive, since the remainder of the generators have their active-power outputs fixed to economic dispatch setpoints. For ease of exposition, in the remainder of the paper, all generators in a control area are assumed to partake in AGC. ■

Remark 2 (Impact of AGC on Net Tie-line Flow): The analysis in Section III-A demonstrates that AGC action in control area a enforces $\mathrm{ACE}^a(t_{\mathrm{ss}}) = 0$ and $\omega_g(t_{\mathrm{ss}}) = \omega_{\mathrm{s}}\ \forall\, g \in \mathcal{G}^a$. Substituting these into (24), we observe that the steady state net tie-line flow out of the control area matches its scheduled value, i.e., $P_{\mathrm{tl}}^a = P_{\mathrm{tl}}^{a\star}$. This yields the power balance constraint

$$P_{\mathrm{tl}}^{a\star} = \sum_{k\in\mathcal{N}^a}\sum_{j\notin\mathcal{N}^a} -V_k^2 G_{kj} + V_k V_j(B_{kj}\sin\theta_{kj} + G_{kj}\cos\theta_{kj}), \quad (33)$$

which must be satisfied by the power flow solution (obtained from single or distributed slack variants) in all control areas. ■

Remark 3 (No AGC): For completeness, Appendix A examines the special case where generators only participate in primary frequency control. We establish that the correct interpretation for the slack variable, ψ, is the net-load imbalance unaccounted by economic dispatch *over all control areas*, and that the participation factor, π_g, is the *governor-based participation factor*. The nominal active-power injection, P_g°, remains the optimizer from economic dispatch. Governor-based participation factors have been leveraged to form a distributed slack bus (see, e.g., [15], [17], [19]), albeit, with limited analytical justification. Also, interpretations for the nominal active-power injections and slack variable for this special case are not provided in these prior efforts. ■

[3]In what follows, $x(t_{\mathrm{ss}})$ denotes the value that variable x assumes at time instant t_{ss}.

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B. Power Flow Formulation with Distributed Slack Bus

We now overview the algebraic equations as well as known and unknown variables in setting up power flow with a distributed slack bus. The problem is formulated and solved *across all* control areas while acknowledging the scheduled tie-line flow out of and the net-load imbalance *within each* control area. As in the single slack formulation, each bus $k \in \mathcal{N}$ is associated with four variables: voltage magnitude and phase angle $V_k,\ \theta_k$, and net active- and reactive-power injections $P_k,\ Q_k$. For load bus $\ell \in \mathcal{L}$, $P_\ell,\ Q_\ell$ are known, $V_\ell,\ \theta_\ell$ are unknown, and we must acknowledge both the active- and reactive-power balance equations (1)–(2). For generator bus $g \in \mathcal{G}$, V_g is fixed while θ_g is unknown, and we only consider the active-power balance equation (11). Note that P_g° and π_g are assumed to be known ($P_g^\circ = P_g^\star$, the economic dispatch setpoint, and $\pi_g = \alpha_g$, the AGC participation factor). On the other hand, $\psi = \Delta P^a$ depends on the control area that the generator belongs to, and, $\Delta P^a,\ a \in \mathcal{A}$ are left as unknowns. Furthermore, we need to impose (33) for each control area to capture AGC action. Recall that scheduled net tie-line flows out of control areas are assumed to satisfy $\sum_{a\in\mathcal{A}} P_{\mathrm{tl}}^{a\star} = 0$; thus, only $|\mathcal{A}| - 1$ instances of (33) are necessary.

Collecting active-power balance equations at all load and generator buses, reactive-power balance equations at load buses, and net tie-line flow equations for $|\mathcal{A}| - 1$ control areas, we obtain a total of $2|\mathcal{L}| + |\mathcal{G}| + |\mathcal{A}| - 1$ algebraic equations. Without loss of generality, we set the voltage phase angle at one generator bus, say $\overline{g} \in \mathcal{G}$, as the system angle reference, i.e., we assume $\theta_{\overline{g}} = 0$. Then, we have a total of $2|\mathcal{L}|+|\mathcal{G}|+|\mathcal{A}|-1$ unknowns: $V_\ell,\ \theta_\ell,\ \ell \in \mathcal{L}, \theta_g,\ g \in \mathcal{G}\setminus\{\overline{g}\}$, and $\Delta P^a,\ a \in \mathcal{A}$. The resultant system of nonlinear algebraic equations can be solved using iterative algorithms like the Newton-Raphson method. To ensure broad applicability of the analytical developments, the DC power flow problem with a distributed slack bus for cases with and without AGC is presented in Appendix B.

IV. Case Studies

We examine the New England 39-bus 10-machine test system. Case studies demonstrate that solutions obtained from a distributed slack AC power flow are in excellent agreement with time-domain simulations of the system DAE model. We also compare solutions with the standard single slack bus option, and find that the distributed slack formulation consistently yields lower errors with no added computation time. The DAE simulations are performed in PSAT [38] using a detailed two-axis model along with governor and exciter controls for each synchronous generator [39]. (Recall that our theoretical developments employed a swing equation model augmented with a turbine governor for analytical convenience.) Custom MATLAB code implements the Newton-Raphson algorithm to solve the distributed slack power flow as described in Section III-B.

A. Simulation Setup

The system one-line diagram is depicted in Fig. 3. The network is split into two control areas belonging to the set $\mathcal{A} = \{1, 2\}$, with area 1 containing generator buses $\mathcal{G}^1 = \{30, 37, 38\}$ and area 2 containing generator buses $\mathcal{G}^2 = \{31, \ldots, 36, 39\}$. Generator parameters and nominal load values are sourced from the PSAT data file in [38], and the generator active-power injections therein (reproduced in Table I) are assumed to be the economic dispatch setpoints $P_g^\star$, $g \in \mathcal{G}$. Without loss of generality, we set AGC participation factors to be proportional to the generator capacities.[4] Numerical values are reported in Table I. The scheduled net tie-line flow between the control areas is taken to be the sum of steady-state flows across lines $(1, 39)$, $(3, 4)$, and $(17, 16)$. For the simulations, we introduce uniform increases/decreases for all loads and compare bus voltage magnitudes and phase angles obtained from the post-disturbance steady state of the DAE simulation with those resulting from the power flow problem formulated with: i) the distributed slack bus, and ii) an exhaustive combination of single slack bus choices.

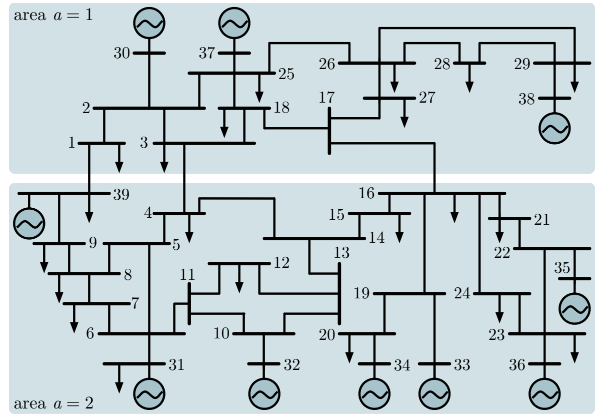

Fig. 3: One-line diagram for the New England test system.

B. Accuracy and Computation Time

Box-and-whisker plots in Figs. 4–5 depict error distributions obtained from the power flow solutions compared to DAE simulations for the cases of $\pm 10\%$ uniform change in load active-power withdrawals. Boxes contain data points between first and third quartiles, and whisker ends indicate minimum and maximum values. Case indices referenced in the x axes in Figs. 4–5 correspond to slack bus choices detailed in Table II.

An inspection of the results in Fig. 4 reveals that the power flow problem formulated with the distributed slack bus yields bus voltage magnitudes, phase angles, and active-power flows that match the DAE simulations very well (hence validating the analytical developments). The distributed slack bus formulation offers significantly greater accuracy compared to all single slack bus alternatives, and this is also the case for voltage phase angles solved from the distributed slack variant of the DC power flow as shown Fig. 5. Note that results from the distributed slack simulations are plotted with reference to the y axes on the left of the plots while the single slack results are with reference to the y axes on the right.

[4]The choice of the AGC participation factors is inconsequential to the analytical developments and case studies as long as the participation factors for the distributed slack bus match them.

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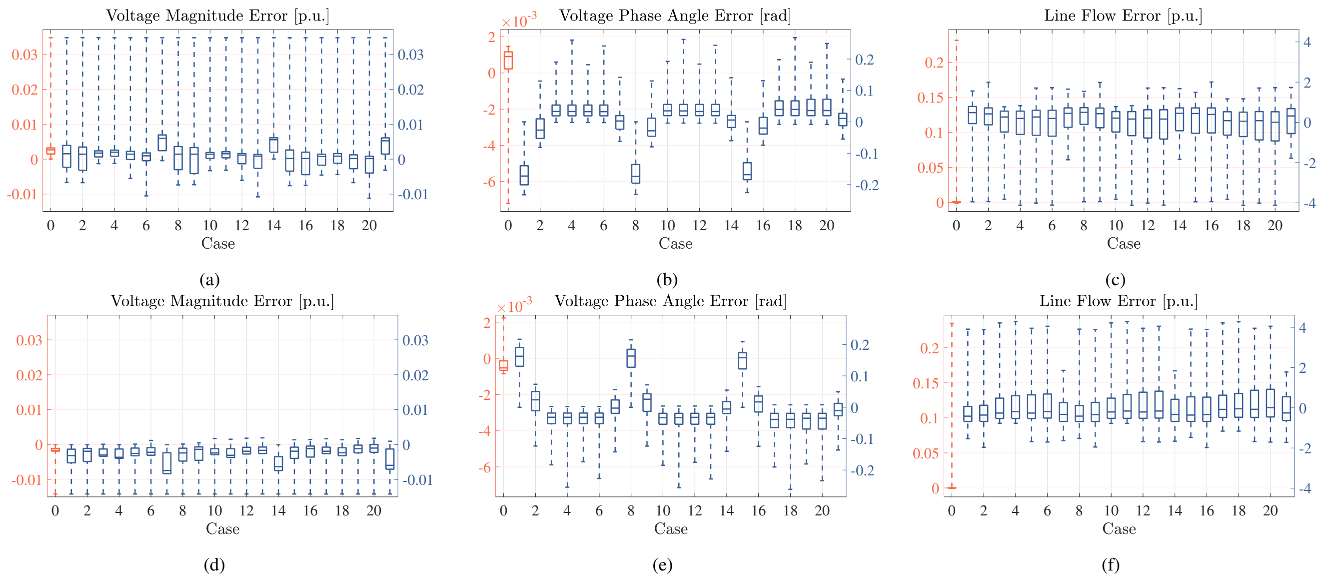

Fig. 4: Distribution of errors in (a)(d) voltage magnitudes, (b)(e) voltage phase angles, and (c)(f) line active-power flows, from AC power flow, for scenarios of uniform load change of $+10\%$ in (a)–(c) and -10% in (d)–(f). Case 0 corresponds to the distributed slack bus formulation, while Cases 1–21 are exhaustive combinations of single slack bus choices (detailed in Table II). The y axes on the left apply to Case 0, while the ones on the right apply to Cases 1-21.

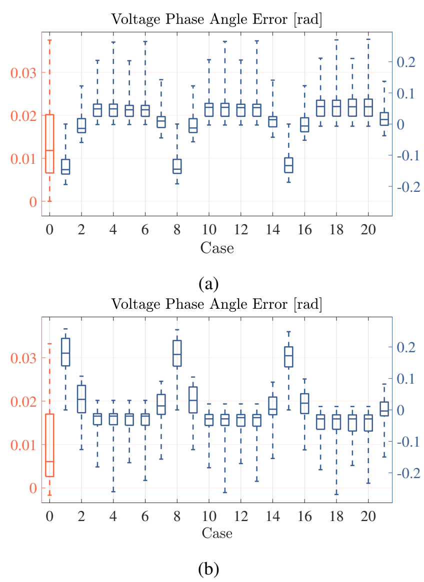

Fig. 5: Distribution of errors in phase angles from DC power flow for scenarios of uniform load change of $+10\%$ in (a) and -10% in (b). Case 0 corresponds to the distributed slack bus formulation, while Cases 1–21 are exhaustive combinations of single slack bus choices (detailed in Table II). The y axes on the left apply to Case 0, while the ones on the right apply to Cases 1-21.

For $+10\%$ and -10% uniform load changes, our custom MATLAB code takes 0.47 s and 0.55 s, respectively, to converge to the solution of the power flow problem with a distributed slack bus. PSAT simulations take 10.34 s and 10.81 s, respectively, to reach steady state. Notably, the power flow problem formulated with the distributed slack bus converges in the same time as taken to solve the single slack version.

V. Concluding Remarks

This paper formalizes choices for the elemental constituents (nominal active-power injections, participation factors, and net-load imbalance) that appear in the distributed slack bus formulation. The main result is tailored to networks with AGC for frequency regulation and tie-line bias control. Corollaries covering the case where generators only participate in primary frequency control and the DC power flow formulation with a distributed slack bus are also provided.

Appendix

A. Primary Frequency Control Only

In the problem formulation outlined in Section III-B, suppose AGC is omitted and tie-line flows are not regulated. Collecting active-power balance equations at generator and

TABLE I: Generator economic dispatch setpoints and AGC participation factors for the New England test system.

	Area 1			Area 2						
Generator at Bus g	30	37	38	31	32	33	34	35	36	39
Economic Dispatch Setpoint, $P_g^{\star}$ [p.u.]	2.50	5.40	8.30	6.78	6.50	6.32	5.08	6.50	5.60	10.0
AGC Participation Factor, α_g	0.4212	0.2284	0.3503	0.1361	0.1459	0.1312	0.1103	0.1383	0.1167	0.2214

TABLE II: Explanation of cases: Case 0 corresponds to the power flow problem formulated with the distributed slack bus and is labeled “Distributed.” The remaining cases correspond to power flow problems formulated with an exhaustive combination of single slack bus options for the two control areas. For instance, in Case 4, the generators at buses 30 and 34 (in control areas 1 and 2, respectively) are chosen as slack buses.

Case	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Area 1	Distributed	30	30	30	30	30	30	30	37	37	37	37	37	37	37	38	38	38	38	38	38	38
Area 2	Distributed	31	32	33	34	35	36	39	31	32	33	34	35	36	39	31	32	33	34	35	36	39

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load buses as well as reactive-power balance equations at load buses yields $2|\mathcal{L}| + |\mathcal{G}|$ algebraic equations, with as many unknowns: V_ℓ, θ_ℓ, $\ell \in \mathcal{L}$, θ_g, $g \in \mathcal{G} \setminus \{\overline{g}\}$, and the net-load imbalance for the entire system, ΔP (defined analogously to (14), except without the notion of control areas). Following a procedure similar to Section III-A, we get that in steady state at time $t = t_{\mathrm{ss}}$,

$$\omega_g(t_{\mathrm{ss}}) = \omega_{\mathrm{s}} \left(1 - \frac{\Delta P}{\sum_{j \in \mathcal{G}} R_j^{-1}}\right), \tag{34}$$

$$P_g(t_{\mathrm{ss}}) = P_g^\star + \frac{R_g^{-1}}{\sum_{j \in \mathcal{G}} R_j^{-1}} \Delta P. \tag{35}$$

Comparing (35) with the left-hand side of (11), we see that the: i) nominal active-power injection, $P_g^\circ = P_g^\star$, ii) appropriate choice for the participation factor,

$$\pi_g = \frac{R_g^{-1}}{\sum_{j \in \mathcal{G}} R_j^{-1}} =: \rho_g, \tag{36}$$

which is referred to in the literature as the governor-based participation factor, and iii) correct interpretation for the slack variable, ψ, is the system-wide net-load imbalance, ΔP.

B. DC Power Flow

Designate bus $\overline{g} \in \mathcal{G}$ as the angle reference, i.e., $\theta_{\overline{g}} = 0$. Collect generator active-power injections at buses $g \in \mathcal{G}$ in vector $P_\mathcal{G}$, and active-power components of loads at buses $\ell \in \mathcal{L}$ by $P_\mathcal{L}$. Let us further denote the vector of unknown voltage phase angles at the generator buses $g \in \mathcal{G} \setminus \{\overline{g}\}$ by $\theta_{\mathcal{G}\setminus\{\overline{g}\}}$, and at load buses by $\theta_\mathcal{L}$. Let $B_{\mathcal{N}\setminus\{\overline{g}\}} \in \mathbb{R}^{|\mathcal{N}| \times |\mathcal{N}|-1}$ denote the matrix obtained by removing all shunt elements and the $\overline{g}$-th column of the imaginary part of the network admittance matrix. The standard DC power flow equations can be expressed as follows:

$$\begin{bmatrix} P_\mathcal{G} \\ P_\mathcal{L} \end{bmatrix} = -B_{\mathcal{N}\setminus\{\overline{g}\}} \begin{bmatrix} \theta_{\mathcal{G}\setminus\{\overline{g}\}} \\ \theta_\mathcal{L} \end{bmatrix}. \tag{37}$$

We now discuss the solution of the above equations with a distributed slack bus for the cases with and without AGC.

1) With AGC: Denote $P_\mathcal{G}^\star$ to be the vector that collects economic dispatch setpoints of all generators, and define $\Delta P_\mathcal{A} := [\Delta P^1, \ldots, \Delta P^{|\mathcal{A}|}]^{\mathrm{T}}$. Following the developments in Section III-A, we substitute $P_g = P_g^\circ + \pi_g \psi = P_g^\star + \alpha_g \Delta P^a$ for entries of $P_\mathcal{G}$ in (37) and rearrange terms to get

$$\begin{bmatrix} P_\mathcal{G}^\star \\ P_\mathcal{L} \end{bmatrix} = -\begin{bmatrix} A & B_{\mathcal{N}\setminus\{\overline{g}\}} \end{bmatrix} \begin{bmatrix} \Delta P_\mathcal{A} \\ \theta_{\mathcal{G}\setminus\{\overline{g}\}} \\ \theta_\mathcal{L} \end{bmatrix}, \tag{38}$$

where

$$A = \begin{bmatrix} \alpha_{\mathcal{G}^1} & \cdots & \alpha_{\mathcal{G}^{|\mathcal{A}|}} \\ 0_{|\mathcal{L}|} & \cdots & 0_{|\mathcal{L}|} \end{bmatrix}, \tag{39}$$

with the g-th entry of $\alpha_{\mathcal{G}^a} \in \mathbb{R}^{|\mathcal{G}|}$ being α_g if $g \in \mathcal{G}^a$ and 0 otherwise, and $0_{|\mathcal{L}|}$ being the $|\mathcal{L}|$-dimensional all-zeros vector. Assuming the system to be lossless, voltage magnitudes to be unity, and angle differences in (33) to be small, we get

$$P_{\mathrm{tl}}^{a\star} = \sum_{k \in \mathcal{N}^a} \sum_{j \notin \mathcal{N}^a} B_{kj} (\theta_k - \theta_j). \tag{40}$$

Collecting (38) and net tie-line flow equations (40) for $|\mathcal{A}| - 1$ control areas, we obtain a system of $|\mathcal{G}| + |\mathcal{L}| + |\mathcal{A}| - 1$ linear equations, from which we solve for as many unknowns: θ_g, $g \in \mathcal{G} \setminus \{\overline{g}\}$, θ_ℓ, $\ell \in \mathcal{L}$, and ΔP^a, $a \in \mathcal{A}$.

2) Primary Frequency Control Only: Following the developments in Appendix A, we substitute $P_g = P_g^\circ + \pi_g \psi = P_g^\star + \rho_g \Delta P$ for entries of $P_\mathcal{G}$ in (37) and rearrange terms appropriately to get

$$\begin{bmatrix} P_\mathcal{G}^\star \\ P_\mathcal{L} \end{bmatrix} = -\begin{bmatrix} \rho & B_{\mathcal{N}\setminus\{\overline{g}\}} \end{bmatrix} \begin{bmatrix} \Delta P \\ \theta_{\mathcal{G}\setminus\{\overline{g}\}} \\ \theta_\mathcal{L} \end{bmatrix}, \tag{41}$$

where the g-th entry of $\rho \in \mathbb{R}^{|\mathcal{N}|}$ is ρ_g if $g \in \mathcal{G}$ and 0 otherwise. The system of $|\mathcal{G}| + |\mathcal{L}|$ equations in (41) can be solved for the unknowns θ_g, $g \in \mathcal{G} \setminus \{\overline{g}\}$, θ_ℓ, $\ell \in \mathcal{L}$, and ΔP.

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