DA1

Dimensional Analysis

The fluid dynamic governing Equations are very complex and their solution, even by numerical methods is difficult for many practical applications. If some of the terms causing this complexity can be neglected in certain regions of the flowfield, while the dominant physical features are still retained, then a set of simplified equations can be obtained

To determine the relative magnitude of the various elements in the governing differential equations, the following dimensional analysis is performed. For simplicity, consider the fluid dynamic equations with constant properties (μ = constant, ρ = constant)

(1) $$\vec{\nabla} \cdot \vec{V} = 0$$

(2) $$\rho\left(\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}\right) = \rho \vec{g} - \vec{\nabla} P + \mu \nabla^2 \vec{V}$$

The first step is to define some characteristic or reference quantities, relvant to the physical problem to be studied:

L – reference length (e.g., Wing's chord)
U – reference speed (e.g., free-stream speed)
T – characteristic time (e.g., one cycle of a periodic process, or L/U)

P_o – reference pressure (e.g., free-stream pressure, P_∞)

– We can define the following nondimensional Variables

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad z^* = \frac{z}{L} \qquad \left(\vec{x}^* = \frac{\vec{x}}{L}\right)$$

(3) $$u^* = \frac{u}{U}, \quad v^* = \frac{v}{U}, \quad w^* = \frac{w}{U} \qquad \left(\vec{V}^* = \frac{\vec{V}}{U}\right)$$

$$t^* = \frac{t}{T} = f t \qquad f = \text{characteristic frequency}$$

$$P^* = \frac{P}{P_o}, \quad \vec{g}^* = \frac{\vec{g}}{g}$$

– Next, the governing equations need to be rewritten using the quantities of Eq. (3)

DA2

The first term of the Continuity becomes

(4) $$\frac{\partial u}{\partial x} = \frac{\partial (U u^*)}{\partial (L x^*)} = \frac{U}{L}\frac{\partial u^*}{\partial x^*}$$

and the transformed Continuity equation is

(5) $$\frac{U}{L}\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*}\right) = \frac{U}{L}\,\vec{\nabla}^*\cdot\vec{V}^* = 0.$$

The advective acceleration term in Eq.(2) becomes

(6) $$\rho(\vec{V}\cdot\vec{\nabla})\vec{V} = \rho\left(U\vec{V}^*\cdot\frac{\vec{\nabla}^*}{L}\right)U\vec{V}^* = \frac{\rho U^2}{L}(\vec{\nabla}^*\cdot\vec{V}^*)\vec{V}^*$$

After a similar treatment with other terms, the momentum equation becomes

(7) $$\left[\frac{fL}{U}\right]\frac{\partial\vec{V}^*}{\partial t^*} + (\vec{V}^*\cdot\vec{\nabla}^*)\vec{V}^* = -\left[\frac{P_0}{\rho U^2}\right]\vec{\nabla}^* P^* + \left[\frac{gL}{U^2}\right]\vec{g}^* + \left[\frac{\mu}{\rho U L}\right]\nabla^{*2}\vec{V}^*$$

Each of the terms in square brackets in Eq.7 is a nondimensional grouping of parameters – a Pi group.

first nondimensional number, St, Strouhal Number

(8) $St = \frac{\omega L}{U}$ (likely to be important in unsteady, oscillating flow problems).

It represents a measure of the ratio of inertial forces due to the unsteadiness of the flow (local acceleration) to the inertial forces due to changes in velocity from point to point in the flow field (convective acceleration).

Remark. If the Strouhal number is very small perhaps due to very low frequencies, then the time-dependent first term in Eq. 7 can be neglected compared to the terms of order one.

The second nondimensional number is the Euler number, which represents the ratio between the pressure and the inertia forces

(9) $$Eu = \frac{P_0}{\rho U^2}$$

(DA3)

Very often the Euler number is written in terms of a pressure difference, ΔP, so that

(10) $$Eu = \frac{\Delta P}{\rho U^2}$$

Some form of the Euler number would normally be used in problems in which pressure or the pressure difference between two points is an important variable.

The third group of nondimensional numbers is called the Froude number, which stands for the ratio of inertial force to gravitational force

(11) $$Fr = \frac{U}{\sqrt{gL}}$$

It will generally be important in problems involving flows with free surfaces since gravity principally affects this type of flow. Typical problems would include the study of the flow of water around ships (with the resulting wave action) or flow through rivers or open conduits.

The last nondimensional group in Eq. (7) represents the ratio between the inertial and viscous forces and is called the Reynolds number:

(12) $$Re = \frac{\rho U L}{\mu} = \frac{U L}{\nu}$$

If the Reynolds number is very small ($Re \ll 1$), this is an indication that the viscous forces are dominant in the problem, and it may be possible to neglect the inertia effects; that is the density of the fluid will not be an important variable. Conversely, for large Reynolds number flows, viscous effects are

DA4

Small relative to inertial effects and for these cases it may be possible to neglect the effects of viscosity and consider the problem as one involving a "nonviscous" fluid.

■ **TABLE 7.1**

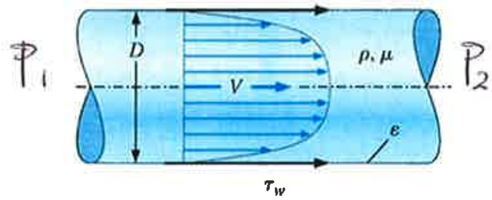
Some Common Variables and Dimensionless Groups in Fluid Mechanics

Variables: Acceleration of gravity, g; Bulk modulus, E_v; Characteristic length, ℓ; Density, ρ; Frequency of oscillating flow, ω; Pressure, p (or Δp); Speed of sound, c; Surface tension, σ; Velocity, V; Viscosity, μ

Dimensionless Groups	Name	Interpretation (Index of Force Ratio Indicated)	Types of Applications
$\frac{\rho V \ell}{\mu}$	Reynolds number, Re	$\frac{\text{inertia force}}{\text{viscous force}}$	Generally of importance in all types of fluid dynamics problems
$\frac{V}{\sqrt{g\ell}}$	Froude number, Fr	$\frac{\text{inertia force}}{\text{gravitational force}}$	Flow with a free surface
$\frac{p}{\rho V^2}$	Euler number, Eu	$\frac{\text{pressure force}}{\text{inertia force}}$	Problems in which pressure, or pressure differences, are of interest
$\frac{\rho V^2}{E_v}$	Cauchy number,[a] Ca	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\frac{V}{c}$	Mach number,[a] Ma	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid in important
$\frac{\omega \ell}{V}$	Strouhal number, St	$\frac{\text{inertia (local) force}}{\text{inertia (convective) force}}$	Unsteady flow with a characteristic frequency of oscillation
$\frac{\rho V^2 \ell}{\sigma}$	Weber number, We	$\frac{\text{inertia force}}{\text{surface tension force}}$	Problems in which surface tension is important

[a]The Cauchy number and the Mach number are related and either can be used as an index of the relative effects of inertia and compressibility. See accompanying discussion.

Similitude, Dimensional Analysis, and Modeling



$\Delta P = [F L^{-3}]$

$D = [L]$

$\rho = [F L^{-4} T^{2}]$

$\mu = [F L^{-2} T]$

$V = [L T^{-1}]$

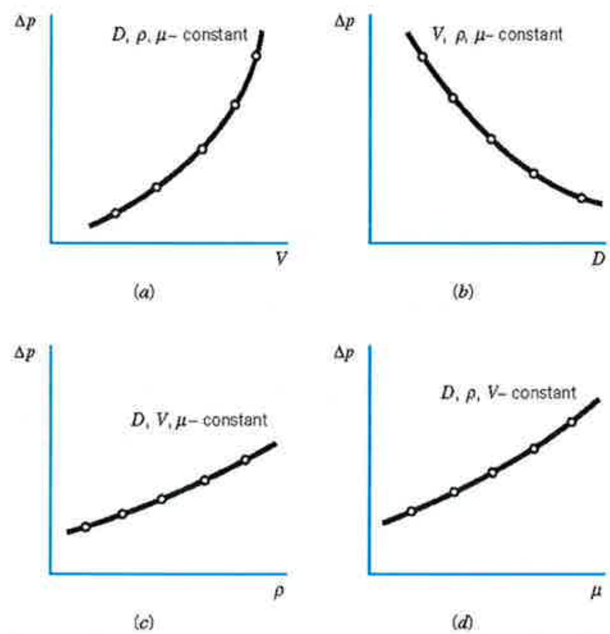


Figure. Illustrative plots showing how the pressure drop in a pipe may be affected by several different factors [1]

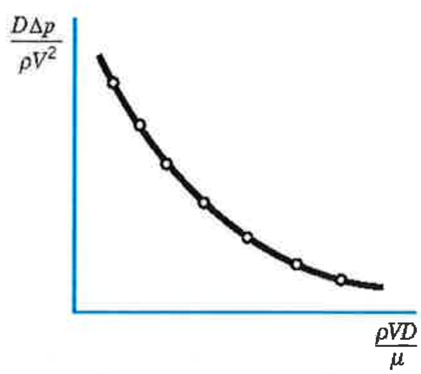


Figure. An illustrative plot of pressure drop data using dimensionless parameters [1]

$$\frac{D\,\Delta P}{\rho V^2} = \frac{L\,(F/L^3)}{(F L^{-4} T^2)(L T^{-1})^2} = F^0 L^0 T^0$$

$$\frac{\rho V D}{\mu} = \frac{(F L^{-4} T^2)(L T^{-1})(L)}{(F L^{-2} T)} = F^0 L^0 T^0$$

DA5

Buckingham Pi Theorem

A fundamental question we must answer is how many dimensionless products are required to replace the original list of variables? The answer to this question is supplied by the basic theorem of dimensional analysis that states the following:

> If an equation involving K variables is dimensionally homogeneous, it can be reduced to a relationship among K−r independent dimensionless products where r is the minimum number of reference dimensions required to describe the variables

Essentially we assume that for any physically meaningful equation involving K variables, such as

$$u_1 = f(u_2, u_3, \ldots, u_K)$$

The dimensions of the variable on the left side of the equal sign must be equal to the dimensions of any term that stands by itself on the right side of the equal sign. It then follows that we can rearrange the equation into a set of dimensionless products (pi terms) so that

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r}).$$

The required number of pi terms is fewer than the number of original variables by r, where r is determined by the minimum number of reference dimensions required to describe the original list of variables.

DA6

Example 1. Consider the force experienced by a body that is in motion through a fluid.

We assume that the force will depend on the following parameters

$$(*) \quad F = f(\rho, V, L, \mu, a)$$

Symbol	Name	Dimensions
F	Force	MLT^{-2}
ρ	Density	ML^{-3}
V	velocity	LT^{-1}
L	characteristic length	L
μ	Coefficient of viscosity	$ML^{-1}T^{-1}$
a	Speed of sound	LT^{-1}

Let us write $(*)$ in the form

$$(**) \quad g(F, \rho, V, L, \mu, a) = 0$$

There are six variables and three fundamental dimensions. Therefore, there are three π

products. If we choose ρ, V, and L as the r-set, the π products are

$$\pi_1 = f_1(F, \rho, V, L)$$

$$\pi_2 = f_2(\mu, \rho, V, L)$$

$$\pi_3 = f_3(a, \rho, V, L)$$

As an example, we find a dimensionless combination of the variables in π_1, in the form $F\rho^a V^b L^c$. Write the quantity in terms of its dimensions

$$\pi_1 = M^0 L^0 T^0 = (MLT^{-2})(ML^{-3})^a(LT^{-1})^b(L)^c$$

The exponents of M, L, and T must be zero, Hence

$$\left.\begin{aligned} 1 + a &= 0 \\ 1 - 3a + b + c &= 0 \\ -2 - b &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} a &= -1 \\ b &= -2 \\ c &= -2 \end{aligned}$$

and π_1 becomes

$$\boxed{\pi_1 = \frac{F}{\rho V^2 L^2}}$$

(DA7)

Proceeding in the same manner with Π_2 and Π_3, we get

$$\Pi_2 = \frac{\rho V L}{\mu}$$

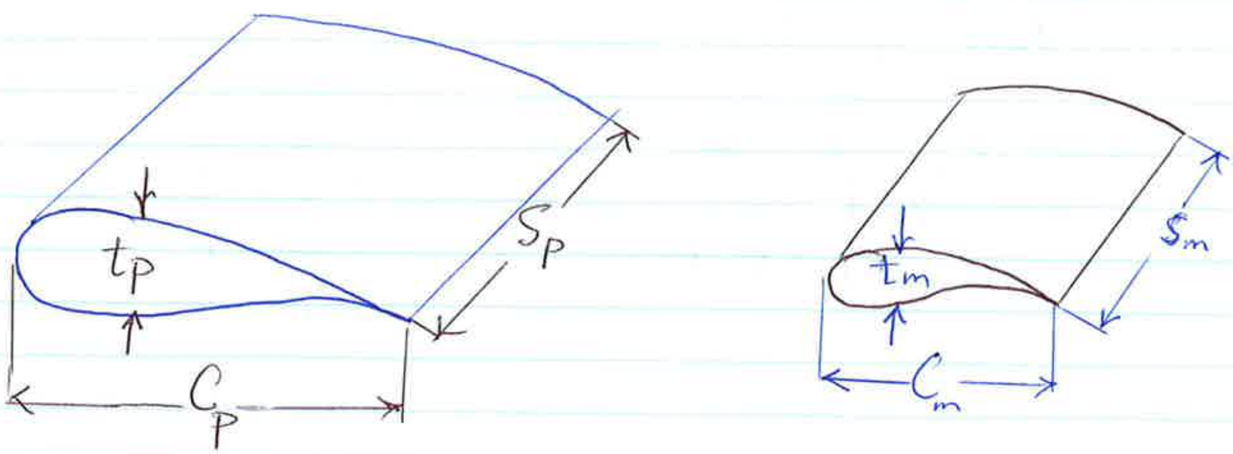
$$\Pi_3 = \frac{V}{a}$$

Then Eq. (*) may be written as

$$\phi\left(\frac{F}{\rho V^2 L^2}, \frac{\rho V L}{\mu}, \frac{V}{a}\right) = 0$$

Flow past an airfoil

Instead of studying flow past an airplane wing (or the entire plane), the same results can be obtained from a scaled down model.



The two are equivalent only when the model and the prototype problems are geometrically, kinematically, and dynamically similar.

- The model is said to be geometrically similar to the prototype when ratios between corresponding geometric parameters are the same. For instance, if t is the thickness and c the

DA8

chord length of the airfoil, the two airfoils are geometrically similar when

$$\frac{t_p}{t_m} = \frac{C_p}{C_m} \equiv L_R .$$

where L_R is the ratio of the linear dimensions. If linear dimensions are similar, corresponding surface areas (A_p, A_m) are then proportional to the square of L_R,

$$\frac{A_p}{A_m} = \frac{C_p^2}{C_m^2} = L_R^2$$

The resulting flow field in the model problem is said to be Kinematically similar when the prototype and model consist of geometrically similar streamlines. Therefore, as some streamlines coincide with geometric surfaces, Kinematic similarity implies also geometric similarity (but not the reverse).

For dynamic similarity, forces at corresponding points must have the same ratio's. Typical forces in a general flow may be the inertia forces F_I, the pressure forces F_p, the viscous forces F_V, the surface tension forces F_T, and the force due to gravity F_G.

Dynamic similarity is achieved when the significant nondimensional force ratios between the prototype and the model are the same

$$Fr_p = Fr_m \,, \qquad Re_p = Re_m$$

$$Eu_p = Eu_m \,, \qquad We_p = We_m$$

For instance, in flow past an airfoil the relevant forces are those due to inertia F_I, the viscosity F_V, and the drag force F_D. In principle, F_D is a function of the pressure and viscous forces along the body. Here we assume that

DA9

the drag is primarily due to the pressure forces.
In addition to geometric similarity, if we select the flow parameters such that the two of forces, say F_I and F_V are similar,

$$\frac{F_{I_m}}{F_{I_p}} = \frac{F_{V_m}}{F_{V_p}} \Rightarrow Re_p = Re_m$$

we automatically get

$$C_{D_p} = \left(\frac{F_D}{\rho V^2 L^2}\right)_p = \left(\frac{F_D}{\rho V^2 L^2}\right)_m = C_{D_m}.$$

Example 2. A 0.8 m airfoil is designed to be used in flight at operating conditions such that $Re = 10^7$. The performance of the airfoil is evaluated experimentally using a 10:1 scaled-down model. Find the air velocity required in the experiment in order to reproduce the conditions of the prototype.
What is the velocity if the experiment is performed in water? Find the ratio between the drag force

obtained in the experiment and the force expected in the prototype. Consider air conditions at 20 °C.

Soln. For the given $Re = 10^7$, air viscosity $\mu = 1.82 \times 10^{-5}\ \frac{N.S}{m^2}$, and density $\rho = 1.204\ kg/m^3$, the velocity in the prototype problem is

$$Re = \frac{\rho V L}{\mu} \Rightarrow V = Re \frac{\mu}{\rho L} = 10^7 \frac{1.82 \times 10^{-5}}{1.204 \times 0.8} = 188.95\ m/s$$

To maintain dynamic similarity, the Reynolds numbers of the prototype and model must be equal. We can write

$$\left(\frac{\rho V L}{\mu}\right)_p = \left(\frac{\rho V L}{\mu}\right)_m$$

When the working fluid is air for both cases, we have

$$\frac{V_m}{V_p} = \frac{L_p}{L_m} = \frac{1}{10}$$

The experiment in air then must be performed at velocity $V_m = 10\ V_p = 1889.5\ m/s$, which is

(DA 10)

obviously very large (speed of sound at this temperature is only about 350 m/s)!

- If the working fluid is water, the viscosity is $\mu = 10^{-3}\ \frac{N.s}{m^2}$, and the density is $\rho = 998.2\ \frac{kg}{m^3}$

$$\left(\frac{\rho V L}{\mu}\right)_m = \left(\frac{\rho V L}{\mu}\right)_p$$

$$\frac{(988.2)\, V_m L_m}{10^{-3}} = \frac{(1.204)\, V_p L_p}{1.82 \times 10^{-5}}$$

$$\Rightarrow \frac{V_m}{V_p} = 6.64 \times 10^{-2}\ \frac{L_p}{L_m} = 0.664.$$

$$\Rightarrow V_m = 0.664 \times 188.95 = 125.46\ m/s.$$

which is still very large for a water tunnel! Therefore the only way to reduce this velocity further is to use a larger model airfoil.

The drag force obtained from the model air foil is related to the actual forces on the prototype through

$$\left(\frac{F_D}{F_I}\right)_p = \left(\frac{F_D}{F_I}\right)_m \Rightarrow \frac{F_{Dp}}{F_{Dm}} = \frac{F_{Ip}}{F_{Im}} = \frac{(\rho V^2 L^2)_p}{(\rho V^2 L^2)_m}$$

$$\Rightarrow \frac{F_{Dp}}{F_{Dm}} = \frac{1.204}{998.2} \times \left(\frac{188.95}{125.46}\right)^2 \times \left(\frac{10}{1}\right)^2 = 0.274$$

the force on the model is larger than the actual force on the prototype!

References

[1] B.R. Munson, D.F. Young, T.H. Okiishi Fundamentals of Fluid Mechanics, 6th Edition, John Wiley & Sons, 2009, chapter 7

[2] Y.A. Cengel, J.M. Cimbala, Fluid Mechanics, Fundamentals and applications, Mc Graw Hill 2006, chapter 7.

[3] A. Alexandrou. Principles of Fluid Mechanics, Prentice-Hall, 2001, chapter 6.