# Equilibrium Uniqueness in Entry Games with Private Information*

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**Abstract**

We study equilibria in static entry games with single-dimensional private information. Our framework embeds many models commonly used in applied work, allowing for firm heterogeneity and selective entry. We introduce the notion of strength, which summarizes a firm's ability to endure competition. In environments of applied interest, an equilibrium in which entry strategies are ordered according to the firms' strengths always exists. We call this equilibrium herculean. We derive simple and testable sufficient conditions guaranteeing equilibrium uniqueness and, consequently, a unique counterfactual prediction.

**JEL:** D21, D43, L11, L13

**Keywords:** Entry, Oligopolistic markets, Private information, Equilibrium Uniqueness

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# 1 Introduction

Understanding firms' market entry decisions is a key element of economic policy and regulation. Predicting whether there will be timely entry after a merger or regulatory change requires a framework that determines the number and types of competitors. More broadly, a model with endogenous entry, prices, product characteristics, and welfare outcomes can be used to evaluate policies prospectively. When performing such analyses, researchers use the counterfactual equilibrium of an estimated model to assess the impact of the policy under consideration. A common challenge is the existence of multiple equilibria. Under multiplicity, the model may not yield a unique prediction to the applied question, hindering policy analysis (Berry and Tamer, 2006; Borkovsky *et al.*, 2015). Computing multiple equilibria may also prove challenging when using numerical methods, which may limit the researcher's ability to gain a complete understanding of the impacts of a policy of interest (Iskhakov *et al.*, 2016).

We study equilibrium uniqueness in static binary-action entry games with single-dimensional private information. Our framework accommodates a large variety of entry games, allowing for rich forms of firm *heterogeneity* and *selective* entry. Our main contribution is to provide a sufficient condition that guarantees equilibrium uniqueness. The condition is solely based on the model's fundamentals and verifying it does not require equilibrium computation. In many common applications, we can check the condition by performing a simple calculation. For example, Roberts and Sweeting (2013) and Grieco (2014) use numerical methods to show that their fitted models have a unique equilibrium. Using their estimates and our sufficient condition, we can confirm equilibrium uniqueness in their fitted models, highlighting the usefulness of our results. Our findings provide new tools for applied researchers studying entry.

We characterize firms' equilibrium behavior using a simple index, called *strength*, summarizing a firm's ability to endure competition. The strength of a firm is the unique *symmetric* cutoff strategy that makes the firm indifferent between entering and not entering the market. Facing equal competition, a stronger firm is more willing to enter the market than a weaker one. For the class of models studied, we show that there always exists an equilibrium in which entry strategies are ordered according to strength. We call this a *herculean* equilibrium. When our sufficient condition for equilibrium uniqueness holds, only one herculean equilibrium exists, and no non-herculean equilibrium is possible.

Our proposed framework encompasses static entry models commonly used in applied work. The approach accommodates a large variety of post-entry models, including auctions and competitions in price or quantity; it also allows for rich forms of firm heterogeneity, as firms are allowed to differ in their payoff functions or their distribution of types, capturing that firms might be heterogeneous in their public characteristics (e.g., firms might vary in their product characteristics, geographic locations, or levels of vertical integration). Payoffs might depend on the entry decisions and realized types of competitors, allowing a level of strategic interaction often ignored by the entry literature (auctions being an exception). For example, if firms are privately informed about their marginal costs of production, facing a competitor with a lower marginal cost will lead to a lower post-entry profit. The magnitude of this decrease depends on the firms' realized marginal costs, their degree of product substitutability, and the number of entrants. We enrich the set of models available to applied researchers by including these environments.

In the theoretical literature on market entry, Mankiw and Whinston (1986) study welfare in a symmetric model under complete information. Brock and Durlauf (2001) examine a symmetric coordination game with privately-informed agents. Our modeling shares the idea that both the action and type of an agent affects the payoffs of other agents but differs in that entry decisions are strategic *substitutes* and in that we allow for asymmetric agents. Our article generalizes the existing literature on costly entry into second-price auctions. Samuelson (1985) studies ex-ante symmetric bidders. Tan and Yilankaya (2006) study two groups of asymmetric bidders ordered by first-order stochastic dominance, whereas in Cao and Tian (2013) the two groups are ordered by entry costs. In Ye (2007), bidders are partially informed at entry and fully learn their valuations after entry occurs. Our framework allows for more general forms of bidder heterogeneity and, at the same time, embeds both informational environments. A firm's private information might correspond to its type or a signal about its type.

In the empirical literature, Bresnahan and Reiss (1990, 1991) and Berry (1992) developed the first empirical models of market entry that explicitly account for the strategic interaction between post-entry market competition and firms' entry decisions.[1] Under complete information, the entry game often contains multiple equilibria. Tamer (2003) shows that, without further assumptions, multiple equi-

[1] See also Ciliberto *et al.* (2020) in the context of entry and Bresnahan (1987), Berry (1994), and Berry *et al.* (1995) when the number of competitors is exogenous

libria can lead to set, rather than point, identification.[2] Using numerical methods, Seim (2006) shows that firms having private information may solve the problem of equilibrium multiplicity. Berry and Tamer (2006), however, construct examples of multiple equilibria under private information, raising the question of when uniqueness can be achieved. Glaeser and Scheinkman (2003) show that games of strategic complements in which competitors' types do not directly affect payoffs have a unique equilibrium when they satisfy a Moderate Social Influence (MSI) condition.[3] We contribute to this discussion by identifying a testable condition guaranteeing equilibrium uniqueness in the context of games of strategic substitutes and general payoffs structures.

The importance of allowing for private information in entry models lies beyond the possibility of solving the multiple equilibria problem. Using complementary methodologies, Grieco (2014) and Magnolfi and Roncoroni (2021) test and reject the hypothesis that firms possess complete information at the moment of entry. Furthermore, compared to models that allow for private information, they show that assuming complete information delivers estimates that can lead to qualitatively different predictions. Roberts and Sweeting (2013, 2016) provide evidence of selection at entry, which cannot be accounted for by complete information models.

The article is organized as follows. For illustrative purposes, Section 2 presents our results in the context of a second-price auction. The section introduces and discusses the notions of strength and herculean equilibrium, developing key intuitions. Section 3 introduces the general model and extends the results showing that the existence of a herculean equilibrium is guaranteed and provides a sufficient condition for when the herculean equilibrium is the unique equilibrium of the game. Finally, Section 4 concludes. All the proofs are relegated to the Appendix.

# 2 An Illustrative Example

We begin by illustrating our results in the context of entry into an asymmetric second-price auction (SPA) with independent private values. We generalize our results to a richer set of entry models in Section 3.

[2] Sweeting (2009) shows that multiplicity can help with the model's identification in the context of coordination games. De Paula and Tang (2012) show that multiplicity can be used to infer the signs of strategic interactions. Marcoux (2020) provides a statistical test for whether firms play the same equilibrium across a sample of entry decisions.

[3] The MSI condition has been used to establish uniqueness in the context of linear-payoffs models by Lee *et al.* (2014), Lin and Xu (2017), Xu (2018), and Lin *et al.* (2021).

## 2.1 Second-Price Auction with Entry Costs

**Set up.** Consider a SPA with reservation price $r \geq 0$. The auction consists of one seller, $n$ potential bidders, and one indivisible good. Before making any entry decision, each bidder $i \in \{1, 2, \ldots, n\}$ observes her valuation for the object, $v_i$, which is drawn from an atomless distribution function $F_i$ with full support on $[0, \infty)$.[4] Each $F_i$ is continuously differentiable and has a finite expectation. After observing $v_i$, each bidder, independently and simultaneously, decides whether to enter the auction. If bidder $i$ decides to enter, she incurs in a entry cost $K_i > 0$. The tuple $(F_i, K_i)_{i=1}^n$, which includes the number of potential bidders $n$, is commonly known to all the bidders. Observe that bidders may differ in their distribution of valuations and entry costs. After bidders make their entry decisions, a participating bidder bids their valuation (i.e., its weakly dominant strategy).

**Strategies, payoffs, and equilibrium.** An entry strategy for bidder $i$ is called *cutoff* if there is a threshold $x_i$ such that bidder $i$ enters the auction when its valuation is higher than $x_i$ ($v_i \geq x_i$) and stays out otherwise. Online Appendix C shows that focusing on cutoff strategies is without loss of generality.

To ease the notation, we order bidders' identities according to their cutoffs, with $x_1$ being the bidder with the lowest cutoff and $x_n$ the highest. For a given vector of cutoff strategies $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ define: i) $A_i^k = \prod_{j>i}^k F_j(x_j)$, the probability that bidders playing cutoffs greater than (or *above*) bidder $i$, up to bidder $k$, do not enter the auction; and, ii) $B_i(v) = \prod_{j<i} F_j(v)$, the probability that bidders playing cutoffs lower than (*below*) bidder $i$ obtain valuations lower than $v$. Let $\mathbf{x}_i = (x_1, x_2, \ldots, x_i)$ be a vector containing the cutoff strategies up to bidder $i$. Bidder $i$'s expected *revenue* of entering with a valuation $v_i = x_i$, when there are only $i$ potential bidders, and the other $i-1$ bidders play cutoffs lower than $x_i$ is: [5]

$$R_i(x_i; \mathbf{x}_{i-1}) = x_i B_i(x_i) - r A_0^{i-1} - \sum_{j=1}^{i-1} \left( A_j^{i-1} \int_{x_j}^{x_{j+1}} \max\{r, s\} dB_{j+1}(s) \right).$$

This revenue consists of bidder $i$'s value, $x_i$, times the probability of being the highest valuation bidder, $B_i(x_i)$, minus the expected price paid. The expected

[4] Our results still apply if the support of $F_i$ were an interval $[0, b]$ with $b > 0$. We chose the current formulation to avoid the existence of corner solutions in which a bidder never enters.

[5] The following notation is being used throughout the article: $\sum_\emptyset = 0$ and $\prod_\emptyset = 1$.

price consists of the reserve price, $r$, when no competitor enters, which occurs with probability $A_0^{i-1}$, and the maximum between the reserve price and the highest competitors' bid when entry occurs. A price in the interval $[x_j, x_{j+1})$ is observed only if opponents playing cutoffs higher or equal to $x_{j+1}$ stay out of the auction, which occurs with probability $A_j^{i-1}$. Thus, the price in such interval distributes according to $B_{j+1}$.

Given the opponents' entry cutoff $\mathbf{x}_{-i}$, bidder $i$'s expected *profit* of entering the auction with a valuation $x_i$ is equal to

$$\Pi_i(x_i; \mathbf{x}_{-i}) = A_i^n R_i(x_i; \mathbf{x}_{i-1}) - K_i. \quad (1)$$

The expected profit consists on the expected revenue minus the entry costs $K_i$. Bidder $i$ loses the auction whenever an opponent with a higher valuation than $x_i$ enters the auction. Thus, bidder $i$ obtains a positive payoff only when opponents playing cutoffs larger than $x_i$ stay out, which happens with probability $A_i^n$. In this event, bidder $i$ competes in an auction with $i$ potential bidders, all of which play cutoffs lower than $x_i$, thus receiving the expected revenue $R_i(x_i; \mathbf{x}_{i-1})$.

The function $\Pi_i(x_i; \mathbf{x}_{-i})$ is strictly increasing in each argument. A bidder's expected profit increases in its valuation $x_i$ and in the opponents' entry cutoff $x_j$ (i.e., when opponents enter less often). Because of this monotonicity, we can define bidder $i$'s *best response* to $\mathbf{x}_{-i}$, a cutoff strategy, to be the unique valuation $\chi_i(\mathbf{x}_{-i})$ that solves $\Pi_i(\chi_i(\mathbf{x}_{-i}); \mathbf{x}_{-i}) = 0$. The best response function $\chi_i(\mathbf{x}_{-i})$ is continuous and satisfies $\chi_i(\mathbf{x}_{-i}) \geq r + K_i$, i.e., bidders do not enter the auction if their valuation cannot cover the reservation price plus their entry cost, regardless of what their opponents are playing. Using implicit differentiation, we can show that bidder $i$'s best response is monotonically decreasing in the opponents' cutoffs, $\partial \chi_i(\mathbf{x}_{-i})/\partial x_j < 0$; i.e., when an opponent enters less often (higher $x_j$) a bidder is more willing to enter the auction (lower $x_i$).

A Bayesian equilibrium is a vector of cutoff strategies $\mathbf{x}$ such that every bidder $i$ is indifferent to enter the auction when it draws a valuation equal to its cutoff strategy. That is, $\mathbf{x}$ is an equilibrium vector if and only if $\Pi_i(\mathbf{x}) \equiv \Pi_i(x_i; \mathbf{x}_{-i}) = 0$ or, equivalently, $\chi_i(\mathbf{x}_{-i}) = x_i$, for every bidder $i$. Online Appendix C shows that a Bayesian equilibrium always exists.

## 2.2 Strength and Herculean Equilibrium

We now introduce our two main definitions: bidder *strength* and *herculean* equilibrium. Strength uses the game fundamentals, $(F_i, K_i)_{i=1}^n$, to rank bidders' relative competitiveness. We use strength to identify an equilibrium that exists in every entry game—the herculean equilibrium. This equilibrium is the starting point to find conditions for equilibrium uniqueness.

**Definition** (Strength)**.** The strength of bidder $i$, is the unique number $s_i$ that solves $\Pi_i(s_i; s_i, \dots, s_i) = 0$; that is, the unique $s_i$ satisfying:

$$(s_i - r) \prod_{j \neq i} F_j(s_i) = K_i. \tag{2}$$

We say that bidder $i$ is stronger than bidder $j$ if $s_i < s_j$.

Strength is well defined. It assigns a unique scalar $s_i$ to each bidder $i$, delivering a complete ranking of the bidders.[6] The strength of bidder $i$ is the unique cutoff $s_i$ that is a best response to the other bidders playing the same cutoff strategy $s_i$, i.e., the unique value $s_i$ satisfying $\chi_i(s_i, \dots, s_i) = s_i$. Strength ranks bidders by using the unique symmetric strategy that makes a given bidder indifferent to entering the auction. When bidders are asymmetric, the strategy $s_i$ might differ across bidders. The importance and usefulness of strength relies on summarizing the multidimensional characteristics of bidders, $(F_i, K_i)_{i=1}^n$, into a single scalar.[7]

In intuitive terms, strength ranks firms according to their ability to endure competition. The strength of bidder $i$ encompasses information about a bidder's willingness to enter the auction, relative to that of its competitors. A lower cutoff strategy for bidder $i$ means that bidder $i$ is more willing to enter the auction, as it enters for lower valuations. A lower entry cutoff by competitors, on the other hand, implies that bidder $i$ faces more competition, as competitors are entering more often. Thus, bidder $i$ being stronger than $j$ ($s_i < s_j$) indicates that $i$, despite facing more competition than $j$, is more willing to enter the auction. The next lemma shows that strength generalizes common notions of relative competitiveness used in the entry literature.

**Lemma 1.** *1) If bidders have identical entry costs but the bidders' values are*

[6] The function $(s_i - r)\prod_{j \neq i} F_j(s_i)$ is increasing, unbounded, and equal to 0 when $s_i = r$.

[7] Strength has advantages over other candidates to rank firms, such as expected payoff or entry probability. Online Appendix D presents examples illustrating the advantages of strength over these measures in obtaining information about the bidders' equilibrium behavior.

*ordered by first-order stochastic dominance, the dominating bidder is stronger.*
*2) If bidders have identical distributions of valuations but different entry costs, the bidder with the lowest entry cost is the strongest.*

The ranking provided by strength coincides with that provided by common heuristics used to determine the relative competitiveness of bidders, such as first-order stochastic dominance (FOSD) or entry-cost order. Strength, however, extends the ranking to scenarios in which relative competitiveness is not self-evident. Take, for example, a bidder whose distribution of valuations first-order stochastically dominates the other bidder but has a higher entry cost. This scenario may arise when 'smaller' firms have subsidized entry (Marion, 2007). In this case, the former bidder might be stronger, as it is likely to draw a higher valuation, but it might also be weaker given its higher entry cost. Strength not only ranks bidders in this (or any other) scenario but also, as shown below, provides meaningful information about equilibrium behavior.

**Definition** (Herculean Equilibrium)**.** An equilibrium is called herculean if the equilibrium cutoffs are ordered by strength, with the stronger bidder playing the lower cutoff. That is, $x_i < x_j$ if and only if $s_i < s_j$.

Because stronger bidders are more able to endure competition, they should be more inclined to enter the auction. In terms of equilibrium behavior, the previous intuition translates to stronger bidders playing lower entry cutoffs. In symmetric games, on the other hand, every bidder is equally strong. The herculean equilibrium consists of symmetric strategies in which each bidder plays a cutoff equal to its strength. Thus, in symmetric games, the herculean and symmetric equilibrium coincide.

Herculean equilibrium and strength are incomplete information analogs to risk-dominant equilibrium and risk factor in complete-information games (Harsanyi and Selten, 1988). Both scalars, risk factor and strength, are found by computing an 'indifferent entry' condition. In the context of complete information, a bidder's risk factor is the opponent's highest entry probability for which the bidder is willing to enter. On the other hand, a bidder's strength is the opponent's highest entry probability (lowest entry cutoff) for which the bidder enters if it obtains a valuation equal to said cutoff. Whereas in a herculean equilibrium, a stronger bidder is more likely to enter, the bidder with the lower risk factor enters in a risk-dominant equilibrium. For details, see Espín-Sánchez and Parra (2022).

## 2.3 Auctions with two Potential Bidders

We now illustrate our main results in the context of two potential bidders: a herculean equilibrium always exists, and, under a weak CDF-concavity condition, it is the only equilibrium of the game. From now on, unless otherwise noted, we order bidders' identities by their strength, with bidder 1 being the strongest bidder.

**Proposition 1.** *There always exists a herculean equilibrium. Moreover, the entry game has a unique equilibrium if, for each bidder i, the following condition holds:*[8]

$$\frac{vf_i(v)}{F_i(v)} < 1 \quad \text{for all } v \in [\underline{v}_i, \overline{v}_i], \tag{3}$$

*where* $\underline{v}_i = K_i + r$ *is bidder i's smallest entry cutoff that may lead to positive profits and* $\overline{v}_i = \chi_i(\underline{v}_j)$ *is bidder i's best response to* $\underline{v}_j$ *(i.e., bidder i's highest entry cutoff that she may play in an equilibrium.)*

Proposition 1 provides two results. First, it establishes the existence of a herculean equilibrium, confirming the intuition that an equilibrium in which the strong bidder plays a lower entry cutoff should exist. To see the intuition, consider bidder 1's best response to the opponent's strength relative to its own strength. That is, $\chi_1(s_2)$ relative to $\chi_1(s_1) = s_1$, see Figure 1. Because bidder 1 is stronger, $s_1 < s_2$, bidder 1 faces less competition when bidder 2 plays $s_2$ instead of $s_1$. Consequently, bidder 1 enters more often, $\chi_1(s_2) < s_1 = \chi_1(s_1)$. Similarly, relative to $s_2$, bidder 2 faces more competition when bidder 1 plays $\chi_1(s_2) < s_2$. Thus, bidder 2 needs a higher valuation than its own strength to enter the auction, best responding with an entry cutoff that is higher than $s_2$, $\chi_2(\chi_1(s_2)) > \chi_2(s_2) = s_2$.

These incentives reinforce each other. Iterating mutual best responses starting from the bidders' strength generate two monotonic sequences of cutoffs that are, in each iteration, further apart. Because bidder 1's best response is bounded below by $\underline{v}_1$ and bidder 2's best response is bounded above by $\overline{v}_2$, this process converges to cutoffs $x_1 < x_2$ that are mutual best responses, i.e., a herculean equilibrium. This iteration process can be used in applied research to find a herculean equilibrium when multiple equilibria exist.

[Figure 1 around here]

Perhaps more importantly, Proposition 1 provides a sufficient condition on the CDFs' shape for the game to have a unique equilibrium. The uniqueness result is

[8] Condition (3) can hold with equality if one firm is strictly stronger than the other.

significant for applied work, as it provides a testable condition that guarantees that counterfactual equilibria will also be unique.[9] In intuitive terms, condition (3) is an equilibrium-stability condition. It guarantees that bidders do not overreact to a small change in the opponent's cutoff. We show that this lack of overreaction implies that a bidder's expected profit is monotonically increasing in its entry cutoff, even after considering the opponent's best response. In turn, this monotonicity implies that only one valuation makes a bidder indifferent to enter the auction, leading to a unique equilibrium.

To show that condition (3) implies equilibrium stability, let $x_i < x_j$. Using equation (1) when $n = 2$, bidder $i$'s best response to $x_j$ is $\chi_i(x_j) = r + K_i/F_j(x_j)$. Differentiating $\chi_i(x_j)$ with respect $x_j$, substituting for $K_i$, and using $x_i = \chi_i(x_j)$, we find

$$-\chi_i'(x_j) = (x_i - r)\frac{f_j(x_j)}{F_j(x_j)} < x_j \frac{f_j(x_j)}{F_j(x_j)} \leq 1,$$

where the first inequality follows from $x_i < x_j$ (and $r \geq 0$), and the last inequality from sufficient condition (3). That is, when bidder $j$ increases its cutoff, bidder $i$ best responds by decreasing its cutoff less than proportionally. Similarly, using implicit differentiation and analogous arguments, we can also show $-\chi_j'(x_i) = (x_i - r)f_i(x_i)/F_i(x_j) < 1$. Sufficient condition (3), then, guarantees that every pair of cutoff strategies satisfies the local stability condition $\chi_1'(x_2)\chi_2'(x_1) < 1$ (see Fudenberg and Tirole, 1991, p. 24). This implies equilibrium uniqueness. Graphically, an equilibrium is defined by a point at which best response functions cross. Best response functions are continuous and monotone. Thus, stable and unstable equilibria must alternate along the best-response function. By condition (3), however, every equilibrium must be locally stable and, consequently, at most one equilibrium exists. As an equilibrium always exists, the game has a unique equilibrium.

Although the stability argument provides good intuition, we prove uniqueness directly by showing that a bidder's payoff is monotone in its own strategy even after considering the opponent's best response. Later, we scale this method to prove uniqueness under a larger set of players. For any strength order among bidders, define $\hat{\Pi}_i(x) = \Pi_i(x; \chi_j(x))$ to be bidder $i$'s expected profit when their valuation is

[9] In applied work, the distribution of values might have the structure $F_i(v) = F(v|X_i'\beta)$ where $X_i'\beta$ is a vector of bidder and auction characteristics (which may include auctions fixed effects). If $X_i'\beta$ is observed by an econometrician, condition (3) needs to apply conditional on $X_i'\beta$. If some of the elements in $X_i'\beta$ are unobserved, condition (3) delivers a set of unobserved values for which the game would have a unique equilibrium.

$x$, and the opponent best responds to the cutoff strategy $x$, $\chi_j(x)$. By definition, $x$ is an equilibrium strategy when $\hat{\Pi}_i(x) = 0$.[10] We show that $\partial\hat{\Pi}_i(x)/\partial x > 0$ for every $x$, implying that $\hat{\Pi}_i(x)$ crosses zero only once, i.e., a unique equilibrium exists. Let $\underline{x} = \min\{x, \chi_j(x)\}$ and $\overline{x} = \max\{x, \chi_j(x)\}$, differentiating $\hat{\Pi}_i(x)$ with respect to $x$ we obtain:

$$\frac{\partial\hat{\Pi}_i(x)}{\partial x} = F_j(\overline{x}) + (\underline{x} - r) f_j(\chi_j(x))\chi_j'(x) = F_j(\overline{x})\left(1 - \chi_i'(\chi_j(x))\chi_j'(x)\right) > 0.$$

where we used the values $\chi_k'(x)$ derived above. The inequality follows from noting (as shown above) that condition (3) implies $-\chi_k'(x_{-k}) \in (0, 1)$, proving the result.

Equation (3), however, is not a necessary condition. It captures one of the possible mechanisms inducing a unique equilibrium. In particular, the condition ensures that the *shape* of best responses is such that they only cross once. Example 1(c), below, illustrates another mechanism, not captured by (3), that will generate uniqueness: the degree of bidder asymmetry. Asymmetric bidders might have best responses at a different *scale*, ensuring that best responses cross once. For instance, if a bidder has a significantly lower expected valuation (or high entry cost), it will require an extreme (unlikely high) valuation for entry, and the game might have a unique equilibrium despite violating condition (3).

**Lemma 2.** *1) If $(F_1, F_2)$ are concave, then (3) is satisfied and the equilibrium is unique. 2) If the distributions $(F_1, F_2)$ become concave for high valuations, there exists a pair $(\kappa_1, \kappa_2)$ such that, whenever $\underline{v}_i = r + K_i \geq \kappa_i$ for both bidders, the game has a unique equilibrium.*[11]

Lemma 2 further characterizes sufficient condition (3). Lemma 2.1 shows that condition (3) is a weak form of CDF-concavity. In particular, auctions with concave distributions of valuations (e.g., exponential or generalized Pareto) always have a unique equilibrium. Other distributions, such as beta, gamma, or Weibull, are concave for certain parameters, making condition (3) testable. Many distributions used in applications (such as the log-normal distribution) are concave for sufficiently high valuations. Lemma 2.2 shows that, for these eventually-concave distributions, there exist sufficiently high entry costs, or reservation price, guaranteeing equilibrium uniqueness. This last result stands in contrast with tradi-

[10] A concise, albeit less intuitive, proof of existence of herculean equilibrium can be constructed using the function $\hat{\Pi}_i(x)$. Observe: i) $\hat{\Pi}_1(x) < 0$ for $x \leq \underline{v}_1$; and, ii) because $s_1 < s_2$ and $\chi_2(x)$ is decreasing, $0 = \Pi_1(s_1; s_1) < \Pi_1(s_1; \chi_2(s_1)) = \hat{\Pi}_1(s_1)$. By the intermediate value theorem, there exists $x_1 \in (\underline{v}_1, s_1)$ such that $\hat{\Pi}_1(x_1) = 0$, a herculean equilibrium.

[11] The proof of the lemma shows how to find the value $\kappa_i$ for a given $F_i$.

tional complete information intuitions, where large entry costs make entry by both firms unprofitable, leading to coordinated entry and equilibrium multiplicity. With private values, high entry costs (or reservation price) shifts the domain of feasible strategies $[\underline{v}_i, \overline{v}_i]$ to the concave segment of the CDFs, inducing equilibrium uniqueness.

**Example 1** (Log-normal valuations)**.** To illustrate the intuition behind strength, herculean equilibrium, and sufficient condition for uniqueness (3), consider a scenario with no reservation price, $r = 0$, two entrants with identical entry cost, $K$, and valuations that are distributed log-normal with parameters $(\mu_i, \sigma)$. As illustrated by Figure 2a, this distribution family is not concave. Depending on its parameters, the entry game might have multiple or a unique equilibrium.

[Figure 2 around here]

(a) **Uniqueness under sufficiently high entry costs:** Suppose symmetric bidders with $\mu_i = 1$. Because the log-normal distribution becomes concave for high values, by Lemma 2.2, for each value of $\sigma$ we can find a threshold $\kappa$ such that, for every $K \geq \kappa$, sufficient condition (3) holds. Figure 2b depicts the threshold $\kappa$ and the mass of valuations above $\kappa$, as a function of $\sigma$. The shaded area represents the set of entry costs $K$ under which the sufficient condition for uniqueness (3) holds. The relation between $\kappa$ and $\sigma$ is non-monotonic, with $\kappa$ converging to zero when $\sigma$ is high enough. The proportion of valuations above $\kappa$, $\Pr[v \geq \kappa]$, monotonically increases in $\sigma$. That is, the larger the dispersion of the distribution, the less demanding the condition for uniqueness becomes. When $\sigma \to 0$, the mass of valuations above $\kappa$ converges to zero. That is, as the game converges to a complete information game—where equilibrium multiplicity is known to exist—the sufficient condition for uniqueness is never met.

(b) **Multiplicity and uniqueness under symmetry:** We now illustrate the differences between multiple equilibria versus a unique equilibrium. Assume symmetric bidders, with $K = 1$ and $\mu_i = 1$. Figures 3a and 3b illustrate bidders' best response functions and equilibria when $\sigma \in \{1/2, 3/2\}$ (see Figure 2a for the CDFs). When $\sigma = 1/2$ (Figure 3a), the auction has three equilibria, as the best responses cross at three different points. The segment between the points $A$ and $B$ highlights bidder 2's violation of the sufficient condition for uniqueness (3), as $-\chi_2'(x_1) > 1$. Because bidders are symmetric, the herculean equilibrium, denoted by $H$, is symmetric and equal to the bidders' strength.

[Figure 3 around here]

In contrast, when $\sigma = 3/2$ (Figure 3b), sufficient condition (3) holds—the tuple $(K, \sigma)$ is in the shaded area of Figure 2b. Best responses are flatter, satisfying $\chi_i'(x_j) < 1$ throughout. The game has a unique equilibrium (the herculean), which is also stable.

(c) **Asymmetric auctions:** We now illustrate strength and the herculean equilibrium in an asymmetric context. We repeat the previous analysis but now allow bidders to differ in $\mu$. Bidder 1 is stronger, as it has higher expected valuations ($\mu_1 = 1.1 > 1 = \mu_2$). Figures 3c and 3d depict the bidders' best response functions and the strength of each bidder. Strength is computed where a bidder's best response crosses the 45° line; i.e., when $\chi_i(s_i) = s_i$. Because bidder 1 is stronger, a herculean equilibrium must lie above the 45° line. Figure 3c shows that when $\sigma = 1/2$, only one equilibrium is herculean, which is stable. The middle equilibrium is non-herculean and unstable. The other non-herculean equilibrium is stable. Figure 2a shows that as $\sigma$ increases, the CDF becomes more concave. This flattens best responses and the sufficient condition for equilibrium uniqueness holds (see Figure 3d).

To conclude this example, it is interesting to observe what happens when $\mu_1$ increases. Comparing Figures 3a and 3c, we can see that increasing the mean of bidder 1's distribution shifts bidder 2's best response upwards (same shift can be observed comparing Figures 3b and 3d). This shift implies that the non-herculean equilibria get closer to each other. When $\mu_1$ is sufficiently high, the upward shift of bidder 2's best response leads best responses to no longer cross to the right of the 45° line, inducing a unique equilibrium. As explained above, sufficient condition (3) fails to capture this mechanism for equilibrium uniqueness. Condition (3) is about the *shape* of best responses, whereas $\mu_i$ affect their *scale*.

## 2.4 Auctions with $n$ Potential Bidders

We now extend Proposition 1 to auctions with $n$ potential bidders. First, we illustrate that the result generalizes to environments in which bidders can be divided into two asymmetric groups. We then explain why our methods do not generally extend to an arbitrary number of groups. We extend the result to environments with an arbitrary number of groups by imposing further structure to the model.

**Lemma 3.** *In an auction with $n$ potential entrants, if two symmetric firms meet sufficient condition* (3)*, they must play the same cutoff strategy in any equilibrium.*

Two firms are called symmetric if they have identical entry cost $K$ and distribution of valuations $F$. In a herculean equilibrium, symmetric firms must play symmetric strategies. Lemma 3 says that, under condition (3), restricting the uniqueness analysis to strategies in which symmetric firms play symmetric strategies is without loss of generality. A corollary of Lemma 3 is that condition (3) guarantees uniqueness in symmetric games with an arbitrary number of bidders.

A sketch of the proof of the Lemma is as follows. Let $\mathbf{x}$ be an equilibrium vector of cutoffs in which symmetric bidders $i$ and $j$ play $x_i < x_j$. Because of symmetry, bidders $i$ and $j$ have identical best response functions, $\chi(\mathbf{x}_{-k})$. Fix the competitors' strategies $\mathbf{x}_{-i,j}$ and, using symmetry, define $\hat{\Pi}(x) = \Pi_k(x; \chi(x, \mathbf{x}_{-i,j}), \mathbf{x}_{-i,j})$ for $k \in \{i, j\}$. A necessary condition for a cutoff $x$ to be an equilibrium is $\hat{\Pi}(x) = 0$. Using the same steps as in the proof of Proposition 1, we can show that condition (3) implies $\partial\hat{\Pi}(x)/\partial x > 0$ for all $x$. If $\hat{\Pi}(x_i) = 0$, then $\hat{\Pi}(x_j) > 0$, contradicting that $\mathbf{x}$ is an equilibrium.

**Two groups of bidders.** Consider $n$ bidders divided into two groups $g \in \{1, 2\}$. Each group $g$ consists of $n_g$ bidders, $n_1 + n_2 = n$, characterized by two tuples $(F_g, K_g)$. Although bidders are symmetric *within* groups, bidders can be asymmetric *across* groups. The two-group model has been used in applied work when bidders can be divided by exogenous factors. Examples include mills and loggers in the timberwood industry (e.g., Athey *et al.*, 2011), and favored and non-favored bidders in highway procurement auctions (Krasnokutskaya and Seim, 2011).

**Proposition 2.** *In the two-group model, there always exists a herculean equilibrium. If sufficient condition* (3) *holds for each group of bidders $g$, the herculean equilibrium is the unique equilibrium of the game.*

Proposition 2 extends Proposition 1 to the two-group scenario. Because a herculean equilibrium prescribes symmetric firms to play symmetric strategies (i.e., group-symmetric strategies) the proof of existence mimics the two-bidder scenario. We define the group-symmetric best response as the best response of a bidder when every bidder in its group plays the same best-response strategy.[12] This

[12]Formally, let $\tilde{\mathbf{x}} = (x_1, \ldots, x_1, x_2, \ldots, x_2)$ be a vector of group-symmetric strategies. Pick any bidder in group $g \in \{1, 2\}$ and let $\tilde{\Pi}_g(x_1, x_2) = \Pi_g(\tilde{\mathbf{x}})$ represent the expected profit of a bidder in group $g$, entering under valuation $x_g$, when group-symmetric strategies $x_1$ and $x_2$ are played. Define group $g$'s group-symmetric best response $\chi_g(x)$ to be the function that solves $\tilde{\Pi}_g(\chi_g(x), x) = 0$.

definition generates one best-response function per group. As before, iterating mutual (group-symmetric) best responses, starting from the bidders' strengths, pulls cutoffs further apart, converging to a herculean equilibrium.

By Lemma 3, restricting the analysis of uniqueness to group-symmetric strategies is without loss of generality. Following the uniqueness proof when $n = 2$, we use sufficient condition (3) to show that the expected profit of a bidder is strictly increasing in its group-symmetric strategy, even after taking into account the opponents' group-symmetric best response. Consequently, the expected profit of a bidder can cross zero once, inducing a unique equilibrium.

**Asymmetric bidders.** A herculean equilibrium might not exist in environments with $n \geq 3$ asymmetric bidders. We provide an example of non-existence in Online Appendix D. This lack of existence precludes us from obtaining a general result about equilibrium uniqueness.

Our method of showing that iterated best responses are further apart than the bidders' strength does not extend to environments with $n \geq 3$ asymmetric bidders. The strength order between two bidders might reverse with the *behavior* of a third bidder. This reversal implies that, when iterating best responses of bidders, the best responses are no longer getting further apart, and the process might converge to a non-herculean equilibrium or not converge at all.

Consider an auction with no reservation price, $r = 0$, and three asymmetric bidders satisfying $s_1 < s_2 < s_3$. The bidders differ in their distribution of valuations but have identical entry costs, $K$. Using equation (2), bidder $i \in \{1, 2\}$ strength is determined by the solution to $s_i F_j(s_i) = K/F_3(s_i)$, see Figure 4. Iterating best responses between bidder 1 and 2 fixing $x_3 = s_3$, will produce best responses that are further apart as in the previous scenarios. Bidder 1's best response decreases in each iteration, and bidder 2's best response increases (as in Figure 1).

Consider now starting the iteration with bidder 3. Its best response to $(s_1, s_2)$ is $\chi_3(s_1, s_2) > s_3$. Recompute the strength of bidder 1 and 2, but assuming that bidder 3 plays $\chi_3(s_1, s_2)$, i.e., $\overline{s}_i F_j(\overline{s}_i) = K/F_3(\chi_3(s_1, s_2))$, see Figure 4 . Iterating best responses between bidder 1 and 2, fixing the behavior of bidder 3 at $\chi_3(s_1, s_2)$, might have different outcomes depending on the shape of $F_i(v)$. Panel (a) depicts a situation in which bidders are ordered by FOSD. In this example, the relative strength of bidders 1 and 2 remains invariant. As before, the iteration will mimic the process depicted in Figure 1. Panel (b) shows a scenario where the CDFs of bidders 1 and 2 cross. In contrast to the previous situation, iterating best responses will lead to a sequence of best responses in which bidder 2 will decrease in each iteration and bidder 1 will increase. That is, the process of iterating mutual best responses might not converge or converge to a non-herculean equilibrium. To re-establish our results, we need to impose further structure to the model.

[Figure 4 around here]

**Ordered bidders.** We now show that a herculean equilibrium exists in scenarios in which the ranking provided by strength is robust to the opponents' behavior. We call these environments *ordered*.

**Definition** (Ordered Auction)**.** Let $\underline{v} = \min\{\underline{v}_i\}_{i=1}^n$ and $\overline{v} = \max\{\overline{v}_i\}_{i=1}^n$, where $\underline{v}_i = K_i + r$ and $\overline{v}_i = \chi_i(\underline{v}_{-i})$. An auction is ordered if for any two bidders $i$ and $j$, with $i < j$, the following condition holds:

$$F_i(v)K_i \leq F_j(v)K_j \text{ for all } v \in [\underline{v}, \overline{v}]. \tag{4}$$

**Lemma 4.** *If condition* (4) *holds, bidders are ordered by strength with bidder* 1 *being the strongest bidder.*

Ordered environments include, as particular cases, situations in which bidders have: (i) identical entry costs, but distributions of valuations that are ordered by FOSD; or, (ii) identical distributions of valuations and different entry cost. It also allows, with certain restrictions, for bidders that stochastically dominate others but have higher entry costs, as illustrated in the next example.

**Example 2.** Consider a scenario in which the bidders' distribution of valuations belong to the exponentiated family (see Gupta *et al.*, 1998); i.e., $F_i(x) = F(x)^{\theta_i}$ for any distribution $F$ and $\theta_i > 0$. Observe that bidder $i$ first-order stochastically dominates $j$ if and only if $\theta_i > \theta_j$. Let $\theta_i > \theta_j$, using $\overline{v}$ we find that every entry cost $K_i \leq K_j F(\overline{v})^{\theta_j - \theta_i}$ satisfies condition (4) and bidder $i$ is stronger than $j$, i.e., $s_i < s_j$. In particular, when $K_i \in (K_j, K_j F(\overline{v})^{\theta_j - \theta_i}]$, firm $i$ first-order stochastically dominates $j$ and has a higher entry cost.

**Proposition 3.** *In an auction with $n$ asymmetric bidders. If condition* (4) *holds, a herculean equilibrium exists. In addition, if sufficient condition* (3) *holds for each bidder $i$, the herculean equilibrium is the unique equilibrium of the entry game.*

Proposition 3 is neither a particular case nor a generalization of Proposition 2. Although the proposition extends the existence and uniqueness results to the case with $n$ potential bidders, it also requires condition (4).[13]

We prove existence constructively using induction. To sketch the proof, order bidders by strength, with bidder 1 being the strongest. In each step $i$ we show that, taking the best-response functions of bidders $\{1, \ldots, i-1\}$ and cutoffs $\mathbf{x}^{i+1} \equiv (x_{i+1}, \ldots, x_n)$ as given, bidder $i$ has a best response $\chi_i(\mathbf{x}^{i+1})$ satisfying $\chi_i(\mathbf{x}^{i+1}) > \chi_{i-1}(\chi_i(\mathbf{x}^{i+1}), \mathbf{x}^{i+1})$ for every $\mathbf{x}^{i+1}$. Thus, regardless of the cutoffs $\mathbf{x}^{i+1}$ chosen by weaker bidders in subsequent steps, the order between bidder's $i-1$ and $i$ cutoffs will remain. This construction uses condition (4) to show the order among best-response functions is robust, delivering a herculean equilibrium at the last step.

We use sufficient condition (3) in the previous iteration to show uniqueness. In each induction step, we show that bidder $i$'s expected payoff is strictly increasing in its cutoff valuation, even after considering best responses of stronger bidders. This monotonicity delivers a unique best-response function $\chi_i(\mathbf{x}^{i+1})$ in each iteration step. We use this property to show that no other herculean equilibrium exists, and that no non-herculean equilibrium is possible.

Sufficient condition (3) has to be checked for each potential bidder, translating into $n$ conditions that need to be satisfied. In ordered environments, however, there are cases in which condition (3) only needs to be checked for a single bidder. First, consider a scenario in which bidders are ordered by entry costs, i.e., the distribution of valuations, $F$, is symmetric among bidders. In this situation, because entry costs do not directly enter condition (3), if the condition holds on $[\underline{v}, \overline{v}]$, the condition would hold for every bidder in the entry game. Consider, also, the scenario in which firms are ordered by FOSD and belong to the exponentiated family $F_i(v) = F(v)^{\theta_i}$ (see Example 2). In this scenario, sufficient condition (3) for bidder $i$ becomes

$$\frac{v f_i(v)}{F_i(v)} = \theta_i \frac{v f(v)}{F(v)} < 1.$$

The condition, thus, only needs to hold for the strongest bidder (highest $\theta_i$).

# 3 A Model of Market Entry

We now generalize the previous framework to include entry games used in the applied literature. We extend the previous results to environments with two groups

[13] Proposition 3 is a generalization of the existence of an ordered equilibrium result in Miralles (2008), who studied a scenario with $n$-bidders ordered by FOSD and symmetric entry costs.

of firms, with no restrictions on the degree of asymmetry across groups. In Online Appendix G, we extend the results to environments with an arbitrary number of ordered groups.

## 3.1 The Baseline Model

**Set up.** Consider $n$ firms simultaneously deciding on whether to enter a market. Firms are privately informed about their *type* $v_i$ (a scalar), summarizing the firm's information about its profitability upon entering the market. Firm $i$'s *post-entry* profit depends on: (i) the entry decision of every firm; (ii) firm $i$'s type; and (iii) the types of other entrants. We assume that the type of firms not entering the market is payoff irrelevant. The type $v_i$ is drawn according to a cumulative distribution function $F_i$, a continuously differentiable atomless distribution, with full support on $[a, b]$ where $a, b \in \overline{\mathbb{R}}$ (the extended reals). The distributions of types, $F_i$, are independent across firms but not (necessarily) identically distributed.

Let $E = \{1, 2, \ldots, n\}$ be the set of all potential entrants and $\mathcal{E}$ its power set. The set $\mathcal{E}$ contains every potential market structure that we can observe after entry decisions are made. We denote a (realized) market structure by $e \in \mathcal{E}$. The set $e$ lists all the firms participating in a given market structure, whereas the set $e^c = E \backslash e$ lists all the firms that stay out. Similarly, for any firm $j \in e$, we use $e \backslash j$ to denote the market structure without firm $j$. Let $\mathcal{E}_i = \{e \in \mathcal{E} : i \in e\}$ be the set of market structures in which firm $i$ enters. Denote by $v_e = (v_j)_{j \in e}$ the vector of realized types for every firm participating in market structure $e$. For example, $v_E = (v_1, v_2, \ldots, v_n)$ denotes the vector with the realized types of every firm. As a shortcut, we denote by $v_{-i}$ the realized types of every firm except firm $i$ and we write $v_i$ instead of $v_{\{i\}}$ when $i$ is the sole entrant.

Let $\pi_i(v_e)$ be a real valued function representing firm $i$'s *post-entry* profit when the realized market structure is $e$ and the realized types of the participating firms are $v_e$. To illustrate the workings of the notation, observe that $\pi_i(v_i)$ represents firm $i$'s post-entry profit when $i$ is the sole entrant and its type is $v_i$. Similarly, $\pi_i(v_E) = \pi_i(v_i, v_{-i})$ represents firm $i$'s profit when every firm enters the market and the vector of realized types is given by $v_E$. In the SPA example $\pi_i(v_e) = \max\{0, v_i - \max\{r, v_{e\backslash i}\}\} - K_i$. We normalize the payoff of a non-entrant to zero. Finally, we assume that $\pi_i(v_e)$ is continuous, integrable (with finite expectation) in each dimension of $v_e$, and differentiable almost everywhere with respect to its first argument $v_i$. We denote such derivative by $\pi_i'(v_e)$.

The timing of the game is as follows. Before making any entry decision, each firm privately observes $v_i$. After observing $v_i$, each firm independently and simultaneously decides whether to enter the market. After entry decisions are made, market structure $e$ is realized and each firm entering the market gets a payoff $\pi_i(v_e)$. The tuple $(F_i, \pi_i)_{i=1}^n$—which includes the number of potential entrants $n$—is commonly known to every potential entrant.

**Main assumptions.** For a given market structure $e$ in which firm $i$ enters the market ($e \in \mathcal{E}_i$), firm $i$'s profit function satisfies the following three properties.

**A1 (Monotonicity):** The profit function $\pi_i(v_e)$ is (i) weakly increasing in $v_i$; and (ii) strictly increasing in $v_i$ when firm $i$ is the sole entrant.

Assumption A1 gives economic meaning to the firms' type. Upon entering the market, firm $i$'s profit (weakly) increases in $v_i$ in any market structure $e$. A higher $v_i$ can represent a lower marginal cost of production, a lower entry cost, a higher product quality, a better managerial ability, or a higher valuation for a good in an auction. In the SPA example, payoffs are monotone; they increase in $v_i$ when bidder $i$ is the entrant with the highest valuation and are constant in $v_i$ otherwise.

For any market structure $e \in \mathcal{E}_i$, types $v_e$, and competitor $j \in e$, define firm $i$'s *profit gain* induced by $j$'s exit to be

$$\delta_{i,j}(v_e) = \pi_i(v_{e \setminus j}) - \pi_i(v_e). \tag{5}$$

The function $\delta_{i,j}(v_e)$ represents the increase in profit that firm $i$ attains if firm $j$ exits market structure $e$ under types $v_e$. In two-player games, $\delta_{i,j}(v_e)$ represents the difference between monopoly and duopoly profits. In a SPA, with two potential bidders and valuations over the reserve price, $\delta_{i,j}(v_i, v_j) = \min\{v_i, v_j\} - r$.

**A2 (Substitutes):** For each market structure $e$ and competitor $j \in e$:

(i) $\pi_i(v_e)$ is weakly decreasing in $v_j$.

(ii) $\delta_{i,j}(v_e) \geq 0$.

(iii) There exists $\hat{v}_j$ such that $v_j \geq \hat{v}_j$ implies $\delta_{i,j}(v_e) > 0$.

Assumption A2 concerns the impact of competition on profits. It states that firms' entry actions are strategic substitutes, as competition decreases profits. In particular, the assumption states that $\pi_i(v_e)$ decreases when bidder $i$ faces: (i) a more productive competitor (higher type $v_j$), or; (ii) entry ($\delta_{i,j}(v_e) \geq 0$). Condition (iii) is a strengthening of (ii). It indicates that, for every competitor $j$ and market structure $e$ in which $j$ participates, when $j$ exits with a sufficiently high type, $v_j \geq \hat{v}_j$, firm $i$'s payoffs are strictly larger.[14] A SPA satisfies (i), (ii), and (iii).

Let $\phi(v_e) = \prod_{j \in e} f_j(v_j)$ be the joint density of types of every firm participating in market structure $e$.

**A3 (Costly and Interior Entry):** There exist values $\underline{v}_i < \overline{v}_i$ in the interior of the support of $F_i(v_i)$—i.e., $\underline{v}_i, \overline{v}_i \in (a, b)$—such that:

(i) $\pi_i(\underline{v}_i) = 0$.

(ii)

$$\int_{(\underline{v}_j)_{j \in E \setminus i}}^{b} \pi_i(\overline{v}_i, v_{-i}) \phi(v_{-i}) d^{n-1} v_{-i} = 0,$$

where the multiple integral is over each of the $n-1$ dimensions of $v_{-i}$.

Assumption A3 concerns the nature of the entry problem. Condition (i) simply states that entry is costly. Firms need a sufficiently good type, $\underline{v}_i > a$, to enter the market as the sole entrant. In a SPA, $\underline{v}_i = r + K_i$, the reserve price plus the bidder's entry cost. Jointly with assumption A2, A3 implies that, when $v_i < \underline{v}_i$, firm $i$ would never choose to enter the market under any market structure. That is, the value $\underline{v}_i$ represents the minimal type required to enter the market.

Condition (ii) states that any firm will enter the market if its type is sufficiently high. There exists a value $\overline{v}_i < b$ such that drawing $v_i > \overline{v}_i$ ensures entry, even if every potential competitor enters the market whenever $v_j \geq \underline{v}_j$. The assumption that $[\underline{v}_i, \overline{v}_i] \subset (a, b)$ guarantees that every equilibrium is interior; i.e., no firm chooses to either never enter or always enter the market.

**Partial revelation of information.** Reinterpreting $v_i$ as a signal and adding an affiliation assumption between the signal and the firms' type, the previous framework also accommodates models in which, before entry, private information is partially revealed to firms. The partial information framework allows for outcomes commonly observed in applied research but precluded in a full information model, such as *ex-post* regret.[15]

Let $F_i(v_i, t_i)$ be firm $i$'s joint cumulative distribution of *signals* $v_i$ and *types* $t_i$ with support on $[a, b] \times [c, d]$ with $c, d \in \overline{\mathbb{R}}$. The distributions $F_i$ are independent across firms. Before making their costly entry decisions, a firm privately observes

[14] We could dispense of A2(iii) for our results, but we adopt it for brevity in the proofs.

[15] An example of ex-post regret is bidders that pay the entry cost and submit bids below the reserve price (or do not submit bids), after updating their beliefs downward upon learning their true type.

its signal $v_i$, allowing the firm to make inferences about its true type, $t_i$. Firms learn their type after entering the market. Let $F_i(v_i) = \int_c^d F_i(v_i, s)ds$ and let $F_i(t_i|v_i) = F_i(v_i, t_i)/F_i(v_i)$ be the CDF of $t_i$ conditional on $v_i$.

**A4 (Affiliated Signals):** For $v_i' > v_i$, $F_i(t_i|v_i') < F_i(t_i|v_i)$ for all $t_i$.

Assumption A4 states that higher signals lead to a higher expected type in terms of first order stochastic dominance (FOSD) (c.f. Marmer *et al.*, 2013; Gentry and Li, 2014). Let $\tilde{\pi}_i(t_e)$ be firm $i$'s profit under market structure $e$ and vector of types for participating firms $t_e = (t_j)_{j\in e}$. Let $n_e$ be the number of entrants in market structure $e$. Then, we re-interpret $\pi_i(v_e)$ as

$$\pi_i(v_e) = \int_c^d \tilde{\pi}_i(t_e) \prod_{k\in e} f_k(t_k|v_k) d^{n_e} t_e,$$

where the multidimensional integral is across each of the $n_e$ dimensions of $t_e$. Given the properties of FOSD, if the profit function $\tilde{\pi}_i(t_e)$ satisfies analogous conditions to A1-A3, then $\pi_i(v_e)$ satisfies A1-A3 and the results below will hold.

**Example 3.** To illustrate the breadth of models captured by assumptions A1-A4, we show it embeds two frameworks commonly used in applied work.

(a) **Linear model:** We say that the profit function is linear when

$$\pi_i(v_e) = \eta_i - \delta_i(n_e - 1) + v_i,$$

where $\eta_i$ is a scalar summarizing both market and firm characteristics.[16] In this model, only firm $j$'s entry decision, not its type, affects firm $i$'s payoff. A common interpretation of the private information in the linear model is that $-v_i$ represents firm $i$'s entry cost.[17]

(b) **SPA with partial information:** Consider a SPA in which bidders are partially informed about their valuations before entry.[18] Bidder $i$'s valuation (or type) is given by $t_i = v_i\varepsilon_i$, where $v_i \sim F_i$ is the *signal* observed before the entry

[16]Although the term $\eta_i$ is commonly known by the firms', an econometrician may not observe some elements in $\eta_i$. Typically, $\eta_i = X_i\beta_i + \zeta_i$ where $X_i$ is a vector of observed firm and market characteristics and $\zeta_i$ is unobserved by the econometrician.

[17]Examples linear entry models with private information include: Seim (2006); Aguirregabiria and Mira (2007); Bajari *et al.* (2007); Pakes *et al.* (2007); Pesendorfer and Schmidt-Dengler (2008); Sweeting (2009); Aradillas-Lopez (2010); Bajari *et al.* (2010); Krasnokutskaya and Seim (2011); De Paula and Tang (2012); Vitorino (2012); Mazzeo *et al.* (2016).

[18]In the context of auctions, the partial information model been studied by Roberts and Sweeting (2013, 2016), Gentry and Li (2014), and Sweeting and Bhattacharya (2015).

decision is made and $\varepsilon_i$ is the *noise* observed after entry but before submitting a bid. We assume $\varepsilon_i \sim G$ with support in $[0,\infty)$ and is independent from $v_i$.

The expected payoff of a bidder entering with a signal $v_i$, when participating competitors observe signals $v_{e\setminus i}$ is:

$$\pi_i(v_e) = \int_{r/v_i}^{\infty} \left( \int_0^{v_i\varepsilon_i} (v_i\varepsilon_i - \max\{r,s\}) d\Psi_i\left(s, v_e\right) \right) dG(\varepsilon_i) - K_i.$$

Given the signal $v_i$, bidder $i$ values the good in $t_i = v_i\varepsilon_i$, where $\varepsilon_i$ has a cumulative distribution function of $G(\varepsilon_i)$. Entrants submit a bid equal to their valuation. Bidder $i$ obtains the good when it is the highest valuation firm and pays the maximum between the opponents' valuation, $s$, and the reserve price, $r$. Conditional on $v_{e\setminus i}$, the maximal valuation among $i$'s opponents has a CDF equal to $\Psi_i(s, v_e) = \prod_{j\in e\setminus i} G(s/v_j)$.

## 3.2 Strategies, Payoffs, and Equilibrium

**Payoffs and strategies.** A *cutoff* strategy for firm $i$ is a threshold $x_i$ such that firm $i$ enters the market whenever $v_i \geq x_i$ and stays out otherwise. Firm $i$'s expected profit of entering the market with type $v_i$ when facing opponents playing cutoffs $\mathbf{x}_{-i}$ is

$$\begin{aligned}\Pi_i(v_i;\mathbf{x}_{-i}) &= \mathbb{E}_{\mathcal{E}_i}\left[\mathbb{E}_{v_{-i}}\left[\pi_i\left(v_e\right)|v_{-i} \geq \mathbf{x}_{-i}\right]|\mathbf{x}_{-i}\right] \\ &= \sum_{e\in\mathcal{E}_i}\left\{\left(\prod_{j\in e^c} F_j(x_j)\right)\int_{x_{e\setminus i}}^{b} \pi_i\left(v_i, v_{e\setminus i}\right)\phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i}\right\}. \qquad (6)\end{aligned}$$

where $n_e$ is the number of entrants in market structure $e$.

Firm $i$'s expected profit consists of an iterated expectation. First, given the opponents' strategy $\mathbf{x}_{-i}$, the outer expectation is over each market structure in which firm $i$ participates, $e \in \mathcal{E}_i$. Then, for a given market structure $e$, the inner expectation is over the realization of types for every competitor $v_{-i}$, conditional on their type being above their entry cutoff. Expression (6) is the general analog of equation (1). Appendix B shows that (6) is strictly increasing in firm $i$'s type $v_i$ and in an opponent's cutoff, $x_j$. A higher entry cutoff $x_j$ lowers the competitor's probability of entry, inducing firm $i$ to face less competition.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector with cutoff strategies for every firm. A Bayesian *equilibrium* is a vector $\mathbf{x}$ such that $\Pi_i(\mathbf{x}) \equiv \Pi_i(x_i;\mathbf{x}_{-i}) = 0$ for every firm $i$. Online Appendix C shows that an equilibrium always exists and that *every* equilibrium is in cutoff strategies; i.e., focusing on cutoff strategies is without loss of generality. We denote the partial derivative of $\Pi_i(\mathbf{x})$ with respect to $x_i$ by $\Pi_i'(\mathbf{x})$.

**Strength and herculean equilibrium.** We now extend the notion of *strength* to the general framework. Strength uses the game fundamentals, $(F_i, \pi_i)_{i=1}^n$, to rank firms according to their ability to endure competition. As before, we use the firms' strength to identify the equilibrium that remains when the game has a unique equilibrium, the herculean equilibrium.

**Definition** (Strength)**.** The *strength* of firm $i$ is the unique number $s_i \in \mathbb{R}$ that solves $\Pi_i(s_i; s_i, \ldots, s_i) = 0$, where $\Pi_i(\mathbf{x})$ is given by (6). We say that firm $i$ is *stronger* than firm $j$ if $s_i < s_j$.

**Lemma 5.** $\Pi_i(s_i; s_i, \ldots, s_i)$ *is strictly increasing in* $s_i$*, crossing zero once.*

The strength of firm $i$ is the unique cutoff $s_i$ that best responds to every competitor playing the same cutoff strategy $s_i$. A lower value of strength for firm $i$ ($s_i < s_j$) indicates that firm $i$, despite facing more competition than $j$ ($i$ faces competitors with lower entry cutoffs), is more likely than $j$ to enter the market ($i$ plays a lower entry cutoff). Lemma 5 shows that strength is well defined, as it assigns a unique scalar $s_i$ to each firm $i$, delivering a complete ranking of the firms. We call an equilibrium *herculean* if equilibrium cutoffs are ordered by strength, with stronger firms playing *lower* cutoffs.

The next definition is instrumental to characterize the sufficient conditions for equilibrium uniqueness.

**Definition** (Expected Profit Gain)**.** For any vector of cutoff strategies $\mathbf{x}$ define firm $i$'s *expected profit gain* induced by firm $j$'s exit to be

$$\hat{\Delta}_{i,j}(\mathbf{x}) = \sum_{e \in \mathcal{E}_i \setminus \mathcal{E}_j} \left\{ \left( \prod_{k \in e^c \setminus j} F_k(x_k) \right) \int_{(x_k)_{k \in e \setminus i}}^{b} \delta_{i,j}\left(x_i, x_j, v_{e\setminus i}\right) \phi(v_{e\setminus i}) d^{n_e - 1} v_{e\setminus i} \right\}, \qquad (7)$$

where $\delta_{i,j}(v_e) \geq 0$ is firm $i$'s *profit gain* induced by firm $j$'s exit in market structure $e$ with realized types $v_e$, as defined in (5).

Given a vector of cutoff strategies $\mathbf{x}$, firm $i$'s expected profit gain induced by firm $j$'s exit, $\hat{\Delta}_{i,j}(\mathbf{x})$, is the probability weighted sum over market structures that firm $j$ can exit of firm $i$'s profit gains due to $j$'s exit, $\delta_{i,j}(v_e)$, integrated over every feasible realization of the competitors' type. The expected profit gain relates to the increase in profit that a firm experiences when a competitor (firm

$j$) marginally increases its entry cutoff.[19] Expression (7) will help us characterize the shape of best responses and, consequently, when the entry game has a unique equilibrium. In an environment with two potential entrants, the expected profit gain equals the profit gain due to firm $j$ not participating. In the context of a SPA, $\hat{\Delta}_{i,j}(\mathbf{x}) = \delta_{i,j}(x_i, x_j) = \min\{x_i, x_j\} - r$. Although assumption A2(ii) only implies that $\delta_{i,j}(v_e) \geq 0$, together with assumption A2(iii) we have that $\hat{\Delta}_{i,j}(\mathbf{x}) > 0$.

### 3.3 Uniqueness with Two Groups of Firms

We now generalize our results to games where entrants can be divided into two groups according to their public characteristics. Firms are symmetric within their group. Across groups, firms can differ in their distribution of types and profit functions. The two-group structure may arise naturally in applications where firms can be divided into incumbents and entrants, high and low-quality firms, local and international producers, discount and traditional retailers, or legacy and low-cost airlines, among other examples.

Firms belong to one of two groups $g \in \{1, 2\}$. Group $g$ consists of $n_g$ potential entrants ($n_1 + n_2 = n$) described by the pair $(\pi_g, F_g)$. For any firm $i$, let $g(i)$ represent firm $i$'s group. We assume that profits are *symmetric* and *anonymous* within a group. That is, for every firm $i$, its profit under market structure $e$ and realized types $v_e$ is equal to $\pi_i(v_e) = \pi_{g(i)}(v_i, \mathbf{v}_r, \mathbf{v}_k)$, where $r$ and $k$ are the number of entrants in $e$, other than $i$, from group $g(i)$ and $-g(i)$, respectively. The vectors $\mathbf{v}_r$ and $\mathbf{v}_k$ represent the types of such entrants. A strategy is called group-symmetric if for each firm $i$, $x_i = x_{g(i)}$. Without loss of generality, let group 1 be the strongest group.

**Proposition 4.** *Let $\Delta_{i,j}(\mathbf{x}) = F_j(x_j)\hat{\Delta}_{i,j}(\mathbf{x})$. A herculean equilibrium always exists. The herculean equilibrium satisfies $x_1 < s_1 < s_2 < x_2$, where $s_g$ and $x_g$ are the strength and the equilibrium cutoff of group $g$. Furthermore, the game has a unique equilibrium if, for every firm $i$ and each opponent $j$, conditions*[20]

$$\frac{f_i(x_i)}{F_i(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1 \qquad \textit{if } g(j) = g(i), \tag{8}$$

$$n_{-g(i)} \frac{f_i(x_i)}{F_i(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1 \qquad \textit{if } g(j) \neq g(i), \tag{9}$$

[19]In particular, Lemma B.2 shows that $\partial\Pi_i(\mathbf{x})/\partial x_j = f_j(x_j)\hat{\Delta}_{i,j}(\mathbf{x})$. An increase in $x_j$ leads firm $j$ to change its entry behavior with probability $f_j(x_j)$, inducing firm $i$ to gain $\hat{\Delta}_{i,j}(\mathbf{x})$.

[20]Conditions (8) and (9) can hold with equality if one group is strictly stronger than the other.

*hold for every vector* $\mathbf{x}$ *such that each dimension* $k$ *satisfies* $x_k \in [\underline{v}_{g(k)}, \overline{v}_{g(k)}]$.

Proposition 4 extends the existence of a herculean equilibrium result to the general framework in the two-group model. The proposition also provides bounds on the herculean equilibrium cutoffs, $x_1 \in (\underline{v}_1, s_1)$ and $x_2 \in (s_2, \overline{v}_2)$. As in the auction example, iterating mutual best responses, starting from the firms' strengths, will lead to a herculean equilibrium.

Proposition 4 also provides four conditions that need to be satisfied for equilibrium uniqueness—two conditions per group. The within-group condition (8) guarantees that, in equilibrium, firms only play group-symmetric strategies. The cross-group condition (9), on the other hand, guarantees that the herculean equilibrium is the only group-symmetric equilibrium of the game. Condition (9) bounds firm $i$'s best response due to a group-symmetric deviation from the opposing group, $-g(i)$. Observe that the left-hand side of condition (9) is multiplied by the number of firms in group $-g(i)$. In group-symmetric strategies, there are $n_{-g(i)}$ opponents deviating simultaneously; thus, the condition needs to bound $n_{-g(i)}$ deviations at the same time. Comparing conditions (8) and (9), we can see that the former condition does not directly depend on $n_{g(i)}$. We can exploit the within-group symmetry among firms to obtain a bound that does not depend on the number of participants in the firm's group.

The following corollary is an immediate implication of Proposition 4 in the context of symmetric entry games.

**Corollary 1.** *If sufficient condition* (8) *holds, a symmetric entry game has a unique equilibrium.*

**Example 4.** Below we illustrate how to apply Proposition 4 in the context of the models introduced in Example 3.

(a) **Linear model:** When studying entry of supercenters into rural grocery markets, Grieco (2014) estimates a symmetric linear model with incomplete information and two potential entrants and $v_i \sim N(0, 1)$ (see Example 3(a)). In the smallest market, where coordination among entrants is more relevant and equilibrium multiplicity is more likely to emerge, the model estimates are given by $\eta = -3.838$ and $\delta = 0.851$.[21] In this context, conditions (8) and (9) collapse into one, becoming

$$\delta F(x_j) \frac{f(x_i)}{F(x_i)} < 1 \text{ for } x_i, x_j \in [\underline{v}, \overline{v}].$$

[21]See Table 7, page 329: $\eta = \mu_0 - \mu_4 = -1.222 - 2.158 = -3.838$. The sufficient also holds for every other specification in the paper.

Because the normal distribution is log-concave, $f(x_i)/F(x_i)$ decreases in $x_i$.[22] Consequently, it is sufficient to check the condition at $x_i$'s lower-bound and $x_j$'s upper-bound. Using the model estimates, we find $\delta F(\overline{v}) f(\underline{v})/F(\underline{v}) = 10^{-4} < 1$. The condition is satisfied, and the equilibrium is unique.

(b) **SPA with partial information:** Roberts and Sweeting (2013, 2016) use a SPA with partial revelation of information model to study the USFS timber auctions (see Example 3(b)). The auction consists of two groups of potential entrants, millers and loggers (groups 1 and 2, respectively). Before entry, each firm observes a signal $v_i$. For the representative (mean) auction, they estimate $\ln v_i \sim N(\mu_{g(i)}, 1.19)$, with $\mu_1 = 3.9607$ and $\mu_2 = 3.5824$. The estimated (symmetric) entry cost is \$2.0543/mfb (dollars per thousand board foot) and the auction's reserve price is \$27.77/mfb.[23] Searching numerically, they found a single equilibrium. We prove that the representative auction indeed has a unique equilibrium when $n_1 = n_2 = 1$. In Online Appendix F, we show that condition (3) implies conditions (8) and (9) in this context. Under log normality $v_i f_i(v_i)/F_i(v_i)$ is decreasing in $v_i$. Thus, condition (3) only needs to hold at $\underline{v}_i$. Then, $\underline{v}_1 f_1(\underline{v}_1)/F_1(\underline{v}_1) = .9436 < 1$ and $\underline{v}_2 f_2(\underline{v}_2)/F_2(\underline{v}_2) = .7568 < 1$, and the game has a unique equilibrium.

**Extensions** The Online Appendix presents two extensions of the previous result.

*A weaker sufficient condition.* Observe that sufficient condition (9) becomes more demanding with an increase in the number of potential entrants—that is, ignoring the effects that the number of entrants has on the expected profit gain, $\hat{\Delta}_{i,j}(\mathbf{x})$. In Online Appendix E, we show that if the expected profit gain satisfies a condition analogous to supermodularity, we can relax Proposition 4 to only require sufficient condition (8) for every competitor, regardless of the group they belong to. In the Appendix, we also show that the supermodularity condition is satisfied in SPAs (Section 2) and in the linear model introduced in Example 3(a). Consequently, in those environments the sufficient condition for uniqueness does not become more demanding as the number of potential entrants increase.

*N groups of ordered entrants.* Online Appendix G also extends our existence of herculean equilibrium and uniqueness results to an arbitrary number of entrants if, similar to the analysis in Section 2.4, the environments is *ordered*. The Appendix

[22] If $G(x) = \ln(F(x))$ is concave, then $G''(x) = \partial(f(x)/F(x))/\partial x < 0$.

[23] In their model, $v_i = \theta_i \varepsilon_i$ where $\ln \theta_i \sim N(\mu_{g(i)}, 0.3321)$ and $\ln \varepsilon_i \sim N(0, 0.8579)$. See Tables 3 or 4 in Roberts and Sweeting (2013, 2016), respectively.

also discusses scenarios in which the set of sufficient conditions can be reduced into a single condition, providing examples.

# 4 Concluding Remarks

This article studies equilibrium uniqueness in static entry games with single-dimensional private information. To this end, we introduce the notions of strength and herculean equilibrium. We show that a herculean equilibrium always exists and develop sufficient conditions guaranteeing equilibrium uniqueness. The proposed framework embeds many models studied in the applied entry literature, accommodating firm heterogeneity and selection. With the aid of strength, we identify the herculean equilibrium. Strength can reduce the computational burden of calculating equilibria with heterogeneous firms, as it provides bounds for the herculean equilibrium. We use our sufficient conditions jointly with the estimates in empirical studies on the literature to illustrate the application of these conditions. We show that their empirical models have a unique equilibrium.

This articled focuses on entry games when firms' entry decisions are strategic substitutes. In games of strategic complements (i.e., when entry becomes more desirable when other firms enter), the restriction to cutoff strategies remains without loss of generality. However, a symmetric entry game under strategic complementarity might have multiple symmetric equilibria (see Brock and Durlauf, 2001; Sweeting, 2009). Consequently, because strength coincides with symmetric equilibrium cutoffs in symmetric games, strength might not be uniquely defined in these types of games. Strategic complementarity also hinders the existence of a unique equilibrium. We can show that a strategic-complement analogous to sufficient condition (8) delivers a unique equilibrium in the context of two firms. However, our methods do not directly extend to those cases when more than two competitors are present.

The focus of this article is on static entry games with private information. We emphasize developing a framework that embeds most of the applied work on endogenous market formation. Beyond the presented results, we see these new developments as the starting point for studying equilibrium uniqueness in dynamic entry games with incomplete information. We hope the tools developed here enable further research in dynamic environments.

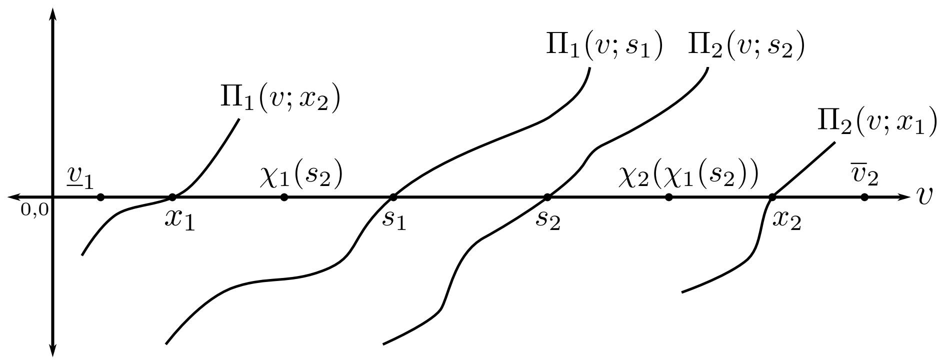
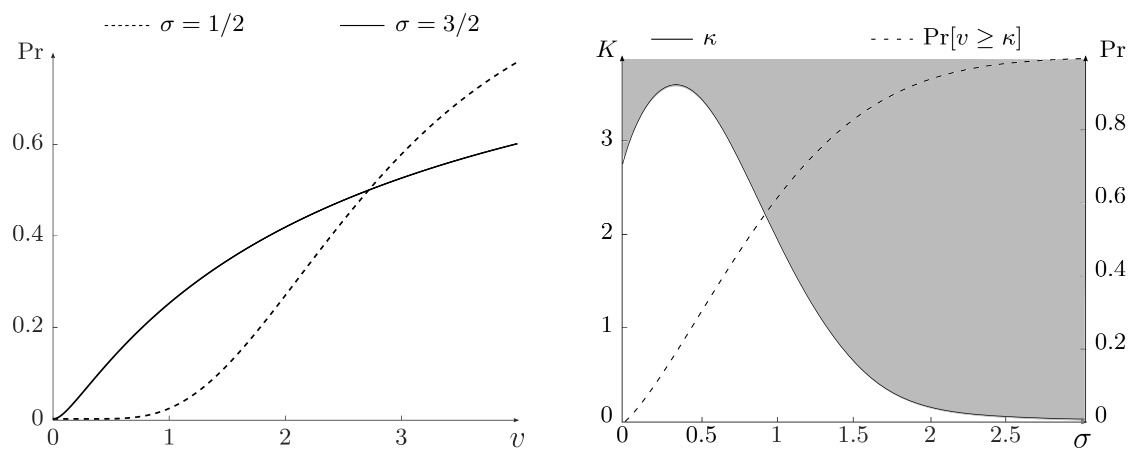


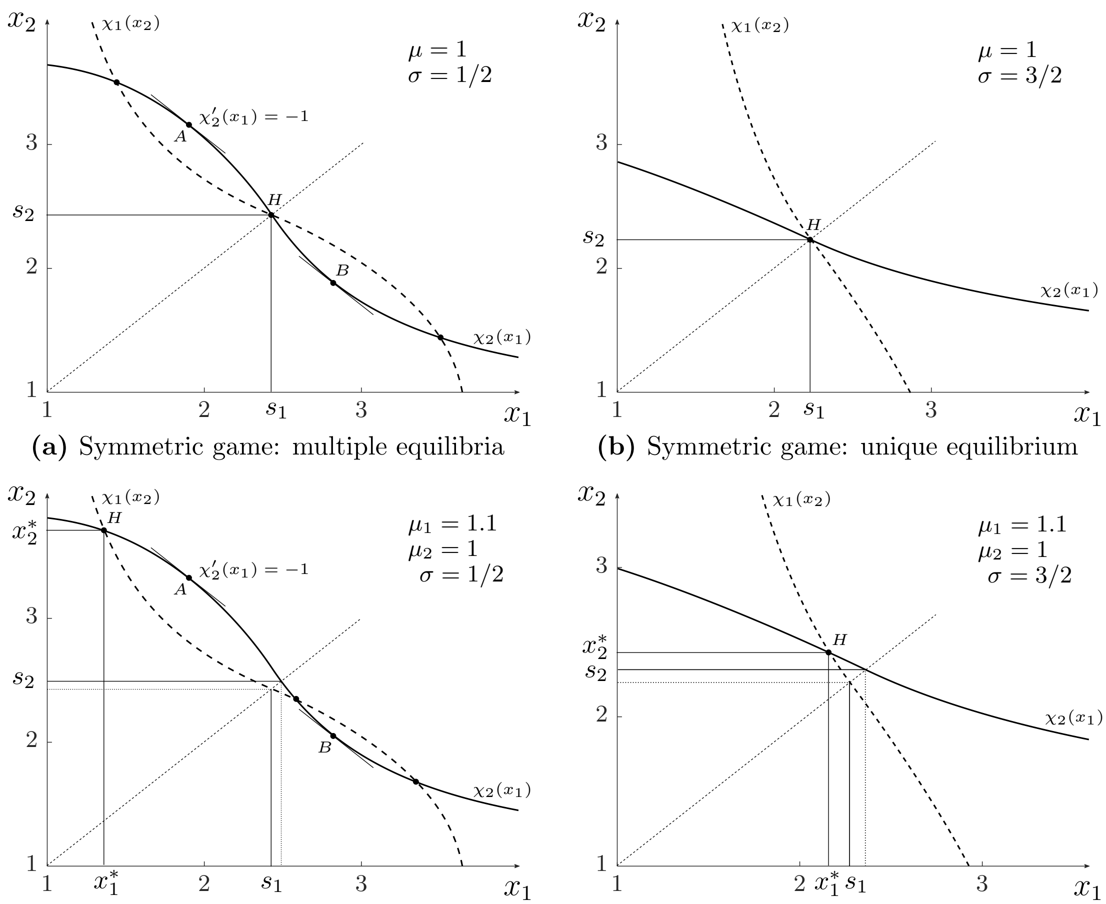

**Figure 1: Construction of a herculean equilibrium from iterated best responses.** Starting from firm 2's strength, $s_2$, firm 1's best response, $\chi_1(s_2)$, is lower than its strength, $s_1 = \chi_1(s_1)$. Similarly, firm 2's best response to $\chi_1(s_2)$ is higher than $s_2 = \chi_2(s_2)$. Iterating these mutual best responses, create bounded monotonic sequences that converge to a herculean equilibrium.




**(a)** Log-normal CDFs ($\mu = 1$)

**(b)** $\kappa$ as a function of $\sigma$

**Figure 2: Sufficiency with log-normal valuations.** Panel (a) shows that log-normal CDFs are not concave. Panel (b) depicts the minimal entry cost $\kappa$ under which uniqueness condition (3) is guaranteed to hold, as a function of $\sigma$. The shaded area represents the set of entry costs under which the entry game has a unique equilibrium. $\Pr[v \geq \kappa]$ represents the proportion of valuations above $\kappa$.



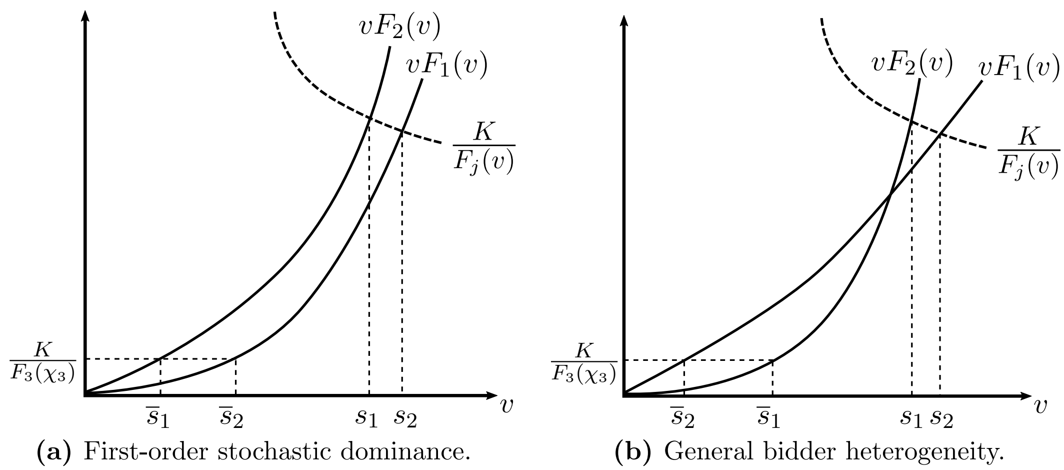

**(a)** Symmetric game: multiple equilibria

**(b)** Symmetric game: unique equilibrium

**(c)** Asymmetric game: multiple equilibria

**(d)** Asymmetric game: unique equilibrium

**Figure 3: Strength and herculean equilibrium under log-normal valuations.** The figure depicts bidders' best response function $\chi_i(x_j)$, their strength $s_i$, and the herculean equilibrium $H$, when valuations distribute log-normal in four different scenarios. Panels (a) and (b) depict symmetric auctions, whereas (c) and (d) asymmetric. Scenario (a) and (c) have multiple equilibria. Sufficient condition for equilibrium uniqueness (3) does not hold between points $A$ and $B$. In scenario (b) and (d), the condition (3) does hold and the game has a unique, the herculean, equilibrium.




**(a)** First-order stochastic dominance.



**(b)** General bidder heterogeneity.

**Figure 4: Strength and competition.** Bidder $i \in \{1, 2\}$ strength under symmetric entry costs versus strength when bidder 3's behavior is fixed at $\chi_3(s_1, s_2)$. Panel (a) shows that when the bidders' CDFs are ordered by FOSD, the strength order between bidders 1 and 2 is robust to bidder 3's behavior. Panel (b) shows that when the CDFs cross, the strength order can change with the behavior of the third bidder.

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# Appendix

## A Omitted Proofs

**Proof of Lemma 1.** The result follows from Lemma 4, as both scenarios satisfy condition (4). ■

**Proof of Proposition 1.** It follows from Proposition 2 when $n_1 = n_2 = 1$. ■

**Proof of Lemma 2.** The proof of both statements make use that a concave differentiable function is bounded above by its first-order Taylor approximation; i.e., for every $x$ and $y$ such that $x > y$

$$F(x) - F(y) \geq (x - y)f(x). \tag{A.1}$$

The first claim follows from taking $y = 0$ and using $F(0) = 0$.

For the second statement, let $y$ in equation (A.1) be the inflection point under which $F_i(v)$ becomes concave. Because of concavity, $F_i''(x) \leq 0$, and $f_i(x)$ is non increasing for every $x \geq y$. Because $F_i(v)$ is bounded above by 1, $f_i(x)$ converges to zero as $x$ goes to infinity. If $F_i(y) \leq f_i(y)y$, let $\kappa_i \geq y$ be the valuation that satisfies $F_i(y) = f_i(\kappa_i)y$. Then, for every $x \geq \kappa_i \geq y$ we have:

$$F_i(x) \geq x f_i(x) + F_i(y) - y f_i(x) \geq x f_i(x) + F_i(y) - y f_i(\kappa_i) = x f_i(x),$$

and the inequality hods. If $F_i(y) > f_i(y)y$, let $\kappa_i = y$ and for every $x \geq \kappa_i$ we have: $F_i(x) \geq x f_i(x) + F_i(y) - y f_i(x) \geq x f_i(x)$, proving the result. ■

**Proof of Lemma 3.** It follows from Lemma B.3 in Appendix B. ■

**Proof of Proposition 2.** Using equation (1) and the definition of equilibrium we begin by observing that every herculean equilibrium is characterized by cutoffs $x_1 \leq x_2$ that jointly solve

$$\begin{aligned} (x_1 - r)\, F_1(x_1)^{n_1 - 1} F_2(x_2)^{n_2} &= K_1 \\ F_2(x_2)^{n_2 - 1} \left[ x_2 F_1(x_2)^{n_1} - r F_1(x_1)^{n_1} - \int_{x_1}^{x_2} v d\left(F_1(v)^{n_1}\right) \right] &= K_2. \end{aligned}$$

*Existence.* By construction. If $s_1 = s_2 = s$ there is a herculean equilibrium with cutoffs $x_1 = x_2 = s$. Assume $s_1 < s_2$. By Lemma 3, focusing on group-symmetric strategies is without loss of generality. Let $\chi_1(x)$ be group 1's group-symmetric best response to group 2 playing the group-symmetric strategy $x$. By definition of strength, $\chi_1(s_1) = s_1$. Using implicit differentiation it can be checked that $\chi_1'(x) < 0$ (see uniqueness proof below). Define $\hat{\Pi}_2(x) = \Pi_2(\chi_1(x); x)$ to be the expected profit of a firm in group 2 when it draws valuation $x$, every other firm in group 2 plays the group-symmetric strategy $x$, and group 1 best responds with the group-symmetric cutoff $\chi_1(x)$, that is

$$\hat{\Pi}_2(x) = F_2(x)^{n_2-1}\left[xF_1(\overline{x})^{n_1} - rF_1(\chi_1(x))^{n_1} - \int_{\chi_1(x)}^{\overline{x}} vd\left(F_1(v)^{n_1}\right)\right] - K_2.$$

where $\overline{x} = \max\{\chi_1(x), x\}$. An equilibrium $(x_1, x_2)$ is given when $\hat{\Pi}_2(x_2) = 0$ and $x_1 = \chi_1(x_2)$. Observe that $x_2 \in (s_1, \infty)$ is necessary and sufficient for a herculean equilibrium (i.e., for $x_1 < x_2$). This follows from $\chi_1(x)$ being decreasing in $x$ and $\chi_1(s_1) = s_1$. Then, $x_1 = \chi_1(x_2) < x_2$ if and only if $x_2 \in (s_1, \infty)$.

We prove existence by the intermediate value theorem. By the bounded expectation assumption, $\hat{\Pi}_2(x)$ is unbounded above. Hence, because $\hat{\Pi}_2(x)$ is continuous, it is sufficient to show that $\hat{\Pi}_2(s_1) < 0$. This follows from observing

$$\hat{\Pi}_2(s_1) = \Pi_2(s_1; s_1) < \Pi_2(s_2; s_2) = 0,$$

where the inequality follows from $\Pi_2(s; s)$ being increasing in $s$ (by Lemma B.2) and the definition of strength, $s_2$.

*Uniqueness.* We show that the function $\hat{\Pi}_2(x)$ is strictly increasing, thus it can cross zero only once. We derive the proof in two parts: (i) There exists a unique herculean equilibrium, i.e., $\hat{\Pi}_2(x)$ is strictly increasing in $x$ when $x > s_1$; and, (ii) There is no equilibrium in which $x_2 < x_1$, i.e., $\hat{\Pi}_2(x)$ is strictly increasing in $x$ when $x < s_1$.[24]

To prove part (i) we start differentiating $\hat{\Pi}_2(x)$ in the scenario when $x > s_1$

$$\hat{\Pi}_2'(x) = F_2(x)^{n_2-1}\Bigg\{F_1(x)^{n_1} + n_1\chi_1'(x)\left(\chi_1(x) - r\right) f_1(\chi_1(x))F_1(\chi_1(x))^{n_1-1} \\ + (n_2 - 1)\frac{f_2(x)}{F_2(x)}\left[xF_1(x)^{n_1} - rF_1(\chi_1(x))^{n_1} - \int_{\chi_1(x)}^{x} yd\left(F_1(y)^{n_1}\right)\right]\Bigg\}.$$

Because $F_2(x)^{n_2-1} > 0$, it is sufficient to show that the term in braces is non-negative for all $x \geq s_1$. Implicitly differentiating $\chi_1(x)$ using $\Pi_1(\chi_1(x); x) = 0$

$$\chi_1'(x) = -\frac{n_2\left(\chi_1(x) - r\right) F_1(\chi_1(x))}{F_1(\chi_1(x)) + (n_1 - 1)\left(\chi_1(x) - r\right) f_1(\chi_1(x))}\frac{f_2(x)}{F_2(x)} < 0$$

replacing into the expression in braces delivers

$$(n_2 - 1)\frac{f_2(x)}{F_2(x)}\left[xF_1(x)^{n_1} - rF_1(\chi_1(x))^{n_1} - \int_{\chi_1(x)}^{x} yd\left(F_1(y)^{n_1}\right)\right] \\ + \left[F_1(x)^{n_1} - \frac{n_1 n_2\left(\chi_1(x) - r\right)^2 f_1(\chi_1(x))F_1(\chi_1(x))^{n_1}}{F_1(\chi_1(x)) + (n_1 - 1)\left(\chi_1(x) - r\right) f_1(\chi_1(x))}\frac{f_2(x)}{F_2(x)}\right]. \quad \text{(A.2)}$$

We show that a lower bound for the expression above is always positive. Maximize

[24] By $s_1 < s_2$, we already know that $x_2 = s_1$ it is not an equilibrium.

the subtracting integral term in the first square brackets by taking the upper bound $x\int_{\chi_1(x)}^{x} dF_1(y)^{n_1}$ in the integral to obtain

$$xF_1(x)^{n_1} - rF_1(\chi_1(x))^{n_1} - x\left(F_1(x)^{n_1} - F_1(\chi_1(x))^{n_1}\right) = (x-r)\,F_1(\chi_1(x))^{n_1} > 0.$$

Because $r \geq 0$, sufficient condition (3) implies

$$(x-r)f_i(x) \leq xf_i(x) \leq F_i(x). \tag{A.3}$$

Using this observation, we maximize the subtracting term in the second square brackets by substituting $F_1(\chi_1(x))$ for $(\chi_1(x)-r)f_1(\chi_1(x))$ in the denominator. Then, equation (A.2) becomes

$$F_1(\chi_1(x))^{n_1}\left[\left(\frac{F_1(x)^{n_1}}{F_1(\chi_1(x))^{n_1}} - 1\right) + (n_2-1)\,(x-\chi_1(x))\,\frac{f_2(x)}{F_2(x)}\right] > 0,$$

where $x > \chi_1(x)$ for $x > s_1$ was used to obtain the inequality. Hence the lower bound of (A.2) is positive and $\hat{\Pi}_2(x)$ is increasing in $x$.

To prove part (ii) we differentiate $\hat{\Pi}_2(x)$ when $x < s_1$ (i.e., $x < \chi_1(x)$)

$$\hat{\Pi}_2'(x) = F_1(\chi_1)^{n_1}F_2(x)^{n_2-1}\left[1 + (x-r)\left((n_2-1)\frac{f_2(x)}{F_2(x)} + n_1\chi_1'\frac{f_1(\chi_1)}{F_1(\chi_1)}\right)\right].$$

where we used $\chi_1$ instead of $\chi_1(x)$ to ease the notation. We show that a lower bound of $\hat{\Pi}_2'(x)$ is positive. We start by deriving a lower bound for $\chi_1'(x)$. Implicitly differentiating $\chi_1(x)$ using $\Pi_1(\chi_1(x);x) = 0$ we get that $\chi_1'(x)$ is equal to:

$$\frac{-n_2(x-r)F_2(x)^{n_2-1}f_2(x)}{F_2(\chi_1)^{n_2} + (n_1-1)\left(\chi_1F_2(\chi_1)^{n_2} - rF_2(x)^{n_2} - \int_x^{\chi_1} vd\left(F_2(v)^{n_2}\right)\right)\frac{f_1(\chi_1)}{F_1(\chi_1)}} < 0.$$

First, maximize the subtracting integral term in the denominator by taking the upper bound $\chi_1(x)\int_x^{\chi_1(x)} dF_2(y)^{n_2} = \chi_1(x)\left(F_2(\chi_1(x))^{n_2} - F_2(x)^{n_2}\right)$. Then, rearranging, the denominator becomes

$$F_2(\chi_1)^{n_2}\left(1 + (n_1-1)\,(\chi_1-r)\,\frac{f_1(\chi_1)}{F_1(\chi_1)}\right) \geq n_1\,(\chi_1-r)\,\frac{f_1(\chi_1)}{F_1(\chi_1)}F_2(\chi_1)^{n_2},$$

where in the inequality we used property (A.3), which is implied by sufficient condition (3). Then, substituting in the denominator

$$0 > \chi_1'(x) \geq -\frac{n_2}{n_1}\frac{(x-r)}{(\chi_1(x)-r)}\frac{F_1(\chi_1(x))}{f_1(\chi_1(x))}\frac{f_2(x)}{F_2(x)} > -\frac{n_2}{n_1}\frac{F_1(\chi_1(x))}{f_1(\chi_1(x))}\frac{f_2(x)}{F_2(x)},$$

where in the last inequality we used that, by assumption, $x < \chi_1(x)$. Replacing

$\chi_1'(x)$ into $\hat{\Pi}_2'(x)$, we obtain

$$\hat{\Pi}_2'(x) > F_1(\chi_1)^{n_1} F_2(x)^{n_2-1}\left[1-(x-r)\frac{f_2(x)}{F_2(x)}\right] \geq 0,$$

where in the last inequality we used that sufficient condition (3) implies $1 \geq (x-r)\, f_2(x)/F_2(x)$, proving the result. ■

**Proof of Lemma 4.** By definition of $i$'s strength $(s_i - r)\prod_{j\neq i} F_j(s_i) = K_i$. Equation (4) implies $K_{i+1}F_{i+1}(s_i)/F_i(s_i) \geq K_i$. Substituting for $K_i$ on the RHS of $i$'s strength and rearranging: $(s_i - r)\prod_{j\neq i+1} F_j(s_i) \leq K_{i+1}$. Since the LHS is increasing in $s_i$, $s_{i+1} \geq s_i$. ■

**Proof of Proposition 3.** *Existence.* For a given vector $\mathbf{v} = (v_1, \ldots, v_n)$, let $\mathbf{v}_i = (v_1, \ldots, v_i)$ represent the elements of $\mathbf{v}$ from the 1st to the $i$th dimension and $\mathbf{v}^i = (v_i, \ldots, v_n)$ the elements from the $i$th to the $n$th. Start by ordering bidders by strength, with bidder 1 being the strongest and $n$ the weakest. Recall equation (1), $\Pi_i(x_i;\mathbf{x}_{-i}) = A_i^n R_i(x_i;\mathbf{x}_{i-1}) - K_i$. An equilibrium $\mathbf{x} = (x_1, \ldots, x_n)$ exists if and only if $\Pi_i(\mathbf{x}) \equiv \Pi_i(x_i;\mathbf{x}_{-i}) = 0$ holds for every $i$.

We construct a herculean equilibrium $\mathbf{x}$ recursively. We start by constructing $x_1$ as a function of $\mathbf{x}^2$. For any vector $\mathbf{x}^2$, define $\chi_1(\mathbf{x}^2)$ to be the value of $x_1$ that solves $\Pi_1(x_1;\mathbf{x}^2) = 0$; i.e., $x_1 = r + K_1/A_1^n$.

Construct $x_2$ using $\chi_1(\mathbf{x}^2)$. By substituting $\chi_1(\mathbf{x}^2)$ in for the value of $x_1$ in $\Pi_2(\mathbf{x})$, we can write $\hat{\Pi}_2(\mathbf{x}^2) = \Pi_2(x_2;\chi_1(\mathbf{x}^2),\mathbf{x}^3)$ which is a function of $\mathbf{x}^2$ only. That is, with a slight abuse of notation, $\hat{\Pi}_2(\mathbf{x}^2) = A_2^n R_2(\mathbf{x}^2) - K_2$ where

$$R_2(\mathbf{x}^2) = R_2(\chi_1(\mathbf{x}^2);x_2) = x_2F_1(x_2) - rF_1(\chi_1(\mathbf{x}^2)) - \int_{\chi_1(\mathbf{x}^2)}^{x_2} v\,dF_1(v)$$

is the revenue function $R_2(\mathbf{x}_2)$ after replacing the function $\chi_1(\mathbf{x}^2)$ for the value of $x_1$. The finite expectation assumption implies that $\hat{\Pi}_2(x_2,\mathbf{x}^3)$ is unbounded above in $x_2$. Fix any $\mathbf{x}^3$. Define $\hat{x}_2$ to be the *largest* value of $x_2$ that satisfies $\hat{x}_2 = \chi_1(\hat{x}_2,\mathbf{x}^3)$. Observe that $\hat{x}_2$ always exists, as $x_2 \in \mathbb{R}_+$ and $\chi_1(x_2,\mathbf{x}^3)$ is a continuous function of $x_2$ with range in $(r + K_1, \overline{v}_1)$. Also, for every $x_2 > \hat{x}_2$, $x_2 > \chi_1(x_2,\mathbf{x}^3)$. Otherwise, $x_2$ and $\chi_1(x_2,\mathbf{x}^3)$ would cross again and $\hat{x}_2$ would not be the largest crossing point.

Using $\hat{x}_2 = \chi_1(\hat{x}_2,\mathbf{x}^3) = r + K_1/(F_2(\hat{x}_2)A_2^n)$, we find

$$\hat{\Pi}_2(\hat{x}_2,\mathbf{x}^3) = (\hat{x}_2 - r)A_2^n F_2(\hat{x}_2) = K_1F_1(\hat{x}_2)/F_2(\hat{x}_2) - K_2.$$

If the bidders are equally strong; i.e., condition (4) holds with equality, $\hat{\Pi}_2(\hat{x}_2,\mathbf{x}^3) = 0$. Then, we can define $\chi_2(\mathbf{x}^3) = \hat{x}_2$. If bidder 2 is strictly weaker, condition (4) implies, $\hat{\Pi}_2(\hat{x}_2,\mathbf{x}^3) < 0$. Thus, by the intermediate value theorem, there exists $\chi_2(\mathbf{x}^3) > \hat{x}_2$ such that $\hat{\Pi}_2(\chi_2(\mathbf{x}^3),\mathbf{x}^3) = 0$. Because $\chi_2(\mathbf{x}^3) > \hat{x}_2$, we have $\chi_2(\mathbf{x}^3) > \chi_1(\chi_2(\mathbf{x}^3),\mathbf{x}^3)$ for any $\mathbf{x}^3$, implying that the order will not reverse when constructing cutoffs for weaker firms (though, the actual values of $\chi_1$ and $\chi_2$ do change with $\mathbf{x}^3$). Observe that, by replacing $x_2 = \chi_2(\mathbf{x}^3)$ into $\chi_1(\mathbf{x}^2)$, we can write $\chi_1$ and $\chi_2$ as a function of $\mathbf{x}^3$. That is, $\chi_1(\mathbf{x}^3) = \chi_1(\chi_2(\mathbf{x}^3), \mathbf{x}^3)$.

Suppose we have shown that, for any vector $\mathbf{x}^i$, $\chi_1(\mathbf{x}^i) \leq \chi_2(\mathbf{x}^i) \leq \cdots \leq \chi_{i-1}(\mathbf{x}^i)$ (strict whenever $s_{k-1} < s_k$). For each step $k \leq i-1$, $\chi_k(\mathbf{x}^{k+1})$ has been recursively constructed by: (i) replacing the previous-step solution $\chi_{k-1}(\mathbf{x}^k)$ into $\chi_j(\mathbf{x}^{k-1})$ for $j \leq k-2$, so that $\chi_j(\mathbf{x}^k) = \chi_j(\chi_{k-1}(\mathbf{x}^k), \mathbf{x}^k)$; (ii) defining

$$\hat{\Pi}_k(\mathbf{x}^k) = \Pi_k(x_k; \chi_1(\mathbf{x}^k), \ldots, \chi_{k-1}(\mathbf{x}^k), \mathbf{x}^{k+1});$$

and, (iii) defining $\chi_k(\mathbf{x}^{k+1})$ to be the highest value $x_k$ that solves $\hat{\Pi}_k(x_k, \mathbf{x}^{k+1}) = 0$. We show that there exists $\chi_i(\mathbf{x}^{i+1}) \geq \chi_{i-1}(\chi_i(\mathbf{x}^{i+1}), \mathbf{x}^{i+1})$ (strict if $s_i > s_{i-1}$) solving $\hat{\Pi}_i(\chi_i(\mathbf{x}^{i+1}), \mathbf{x}^{i+1}) = 0$. By equation (1), $\hat{\Pi}_{i-1}(\mathbf{x}^{i-1}) = 0$ implies $R_{i-1}(\mathbf{x}^{i-1}) = K_{i-1}/A^n_{i-1}$. Substituting the vector of solutions $(\chi_j(\mathbf{x}^i))_{j=1}^{i-1}$ we can write $\Pi_i(\mathbf{x})$ as $\hat{\Pi}_i(\mathbf{x}^i) = A^n_i R_i(\mathbf{x}^i) - K_i$. Because of the finite expectation assumption, $\hat{\Pi}_i(\mathbf{x}^i)$ is unbounded above in $x_i$. Fix any vector $\mathbf{x}^{i+1}$. Take $\hat{x}_i$ to be the largest value of $x_i$ that satisfies $\hat{x}_i = \chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1})$. This value exists by the same argument given to find $\hat{x}_2$ and it also satisfies $x_i > \chi_{i-1}(x_i, \mathbf{x}^{i+1})$ for $x_i > \hat{x}_i$. Using $\hat{x}_i = \chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1})$ and Lemma B.1.2 (see the Auxiliary Result section) we know[25]

$$R_i(\hat{x}_i; \mathbf{x}^{i+1}) = F_{i-1}(\chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1}))R_{i-1}(\chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1}); \hat{x}_i, \mathbf{x}^{i+1}).$$

Then, using the property $R_{i-1}(\chi_{i-1}(\mathbf{x}^i); \mathbf{x}^i) = K_{i-1}/A^n_{i-1}$ and $\hat{x}_i = \chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1})$, we can write $\hat{\Pi}_i(\hat{x}_i, \mathbf{x}^{i+1}) = K_{i-1}F_{i-1}(\hat{x}_i)/F_i(\hat{x}_i) - K_i$. If bidders $i-1$ and $i$ are equally strong, $\hat{\Pi}_i(\hat{x}_i, \mathbf{x}^{i+1}) = 0$ by condition (4) and we define $\chi_i(\mathbf{x}^{i+1}) = \hat{x}_i$. If bidder $i$ is strictly weaker than $i-1$, condition (4) implies $\hat{\Pi}_i(\hat{x}_i, \mathbf{x}^{i+1}) < 0$. Then, by the intermediate value theorem, there exists $\chi_i(\mathbf{x}^{i+1}) > \hat{x}_i$ such that $\hat{\Pi}_i(\chi_i(\mathbf{x}^{i+1}), \mathbf{x}^{i+1}) = 0$. Finally, because $\chi_i(\mathbf{x}^{i+1}) > \hat{x}_i$, we have $\chi_i(\mathbf{x}^{i+1}) > \chi_{i-1}(\chi_i(\mathbf{x}^{i+1}), \mathbf{x}^{i+1})$ for any $\mathbf{x}^{i+1}$, and the order between the cutoffs will be robust to the construction of the equilibrium cutoffs for weaker firms. □

*Uniqueness:* We begin by outlining the induction argument. We order bidders from strongest to weakest. We first show that the strongest bidder has a unique best response to any vector of cutoffs by weaker opponents, $\mathbf{x^2}$. Then, we show that bidder 2 has a unique best response to weaker opponents' cutoffs, $\mathbf{x^3}$, taking bidder 1's unique best response function as given. We also show that these best responses are ordered: bidder 2 always play a higher entry cutoff. Finally, assuming that we have shown that the $k-1$ strongest bidders have a unique best response and that these best responses are ordered, we show that bidder $k$ has a unique best response to any cutoff by weaker bidders, $\mathbf{x}^{k+1}$, and that bidder $k$ always play a higher cutoff than bidder $k-1$. This shows that there is a unique herculean equilibrium. We, then, use the previous argument to show that it also

[25]The equation above uses the recursion notation. The formulation from the lemma is

$$R_i(\hat{x}_i; \mathbf{x}_{i-2}, \chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1})) = F_{i-1}(\chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1}))R_{i-1}(\chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1}); \mathbf{x}_{i-2}).$$

implies that no non-herculean equilibrium exists. Conditions (3) and (4) are used throughout the proof.

*Preliminaries.* Define $\mathbf{\Pi}_k : \mathbb{R}^n \to \mathbb{R}^k$ to be a function equal to $\Pi_i(\mathbf{x})$ (see equation 1) in the $i \leq k$ dimension.[26] Fix $k$, by the existence proof we know that, for every $j \leq k$, there exists recursively defined functions $\chi_j(\mathbf{x}^{k+1})$ satisfying $\mathbf{\Pi}_k(\chi_1(\mathbf{x}^{k+1}), \ldots, \chi_k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}) = 0$. For any $i \leq k$, the total differential of $\Pi_i(\chi_1(\mathbf{x}^{k+1}), \ldots, \chi_k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1})$ with respect to $x_j$, $j > k$, is:

$$A_i^n \left[ \sum_{s=1}^{i-1} A_s^{i-1} R_s(\mathbf{x}_s) f_s(x_s) \frac{d\chi_s}{dx_j} + B_i(x_i) \frac{d\chi_i}{dx_j} + R_i(\mathbf{x}_i) \left( \sum_{s>i}^{k} h_s(x_s) \frac{d\chi_s}{dx_j} + h_j(x_j) \right) \right], \tag{A.4}$$

where $h_i(v) = f_i(v)/F_i(v)$ is the reversed hazard rate of $F_i$ (see Online Appendix D.3 for a step-by-step derivation of A.4). Using (A.4) and the implicit differentiation of $\mathbf{\Pi}_k$, we can write the vector of derivatives $\mathbf{d}_k = (d\chi_1/dx_{k+1}, \ldots, d\chi_k/dx_{k+1})^T$ ($T$ denotes transpose), as the solution to the following system of linear equations:

$$A_i^n \left[ M_k \mathbf{d}_k + \mathbf{R}_k h_{k+1}(x_{k+1}) \right] = 0, \tag{A.5}$$

where $\mathbf{R}_k = (R_1(x_1), R_2(\mathbf{x}_2), \ldots, R_k(\mathbf{x}_k))^T$ and $M_k$ is equal to

$$\begin{pmatrix} B_1(x_1) & R_1(x_1)h_2(x_2) & R_1(x_1)h_3(x_3) & \cdots & R_1(x_1)h_k(x_k) \\ A_1^1 R_1(x_1) f_1(x_1) & B_2(x_2) & R_2(\mathbf{x}_2)h_3(x_3) & \cdots & R_2(\mathbf{x}_2)h_k(x_k) \\ A_1^2 R_1(x_1) f_1(x_1) & A_2^2 R_2(\mathbf{x}_2) f_2(x_2) & B_3(x_3) & \cdots & R_3(\mathbf{x}_3)h_k(x_k) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1^{k-1} R_1(x_1) f_1(x_1) & A_2^{k-1} R_2(\mathbf{x}_2) f_2(x_2) & A_3^{k-1} R_3(\mathbf{x}_3) f_3(x_2) & \cdots & B_k(x_k) \end{pmatrix}.$$

If $M_k$ is invertible, the solution to (A.5) is given by:

$$\mathbf{d}_k = -M_k^{-1} \mathbf{R}_k h_{k+1}(x_{k+1}). \tag{A.6}$$

We will show that $M_k$ is invertible. Then, using the derivatives in (A.6), we show that $d\hat{\Pi}_{k+1}(\mathbf{x}^{k+1})/dx_{k+1} > 0$. This implies that $\hat{\Pi}_{k+1}(\mathbf{x}^{k+1})$ single crosses zero in the $x_{k+1}$ dimension, and $\chi_{k+1}(\mathbf{x}^{k+2})$ is uniquely defined. By the existence proof, we also know that $\chi_k(\chi_{k+1}(\mathbf{x}^{k+2}), \mathbf{x}^{k+2}) < \chi_{k+1}(\mathbf{x}^{k+2})$. Then, by the induction argument, each step of the construction $\chi_{k+1}(\mathbf{x}^{k+2})$ is ordered and uniquely defined. Thus, the herculean equilibrium is unique.

**Claim 1.** There exists a unique herculean equilibrium.

*Proof.* Fix a step $k$ and let $\left(\chi_j(\mathbf{x}^{k+1})\right)_{j=1}^k$ be the vector of functions constructed until step $k$ in the recursion in the existence proof above. For ease in notation, for any $\mathbf{x}^{k+1}$ we write $\left(\chi_j(\mathbf{x}^{k+1})\right)_{j=1}^k = \mathbf{x}_k$. We need to show that there is a unique value of $x_{k+1}$ that solves $\hat{\Pi}_{k+1}(\mathbf{x}^{k+1}) = 0$ for any vector $\mathbf{x}^{k+2}$. In particular, we

[26] For ease in notation, we use $\Pi_i(\mathbf{x})$ and $R_i(\mathbf{x}_i)$ to refer to $\Pi_i(x_i; \mathbf{x}_{-i})$ and $R_i(x_i; \mathbf{x}_{i-1})$.

show $d\hat{\Pi}_{k+1}(\mathbf{x}^{k+1})/dx_{k+1} > 0$, so that $\hat{\Pi}_{k+1}(\mathbf{x}^{k+1})$ single crosses zero from below.

Using (A.4),

$$\frac{d\hat{\Pi}_{k+1}(\mathbf{x}^{k+1})}{dx_{k+1}} = A^n_{k+1}(\mathbf{m}_k\mathbf{d}_k + B_{k+1}(x_{k+1}))$$

where $\mathbf{m}_k = (A^k_1 R_1(x_1)f_1(x_1),\, A^k_2 R_2(\mathbf{x}_2)f_2(x_2),\, \ldots,\, A^k_k R_k(\mathbf{x}_k)f_k(x_k))$. Using (A.6), if $M_k$ is invertible we can write $\mathbf{d}_k = -M_k^{-1}\mathbf{R}_k h_{k+1}(x_{k+1})$ and

$$\frac{d\hat{\Pi}_{k+1}(\mathbf{x}_{k+1})}{dx_{k+1}} = A^n_{k+1}\left(B_{k+1}(x_{k+1}) - q_k h_{k+1}(x_{k+1})\right)$$

where $q_k = \mathbf{m}_k M_k^{-1}\mathbf{R}_k$ is a scalar. Because $A^n_{k+1} > 0$, it is sufficient to show that the term inside the parenthesis is positive for all relevant values of $x_{k+1}$. We prove the previous statement and the invertibility of $M_k$ by induction.

Observe $\Pi_1(\mathbf{x}) = A^n_1(x_1 - r) - K_1$, thus $d\Pi_1(\mathbf{x})/dx_1 > 0$ and bidder 1 has a unique best response (given by $\chi_1(\mathbf{x}^2) = r + K_1/A^n_1$). For bidder 2, observe $M_1 = B_1(x_1) = 1$ is invertible and $q_1 = (x_1 - r)^2 f_1(x_1)$ is well defined. Then, $B_2(x_2) - q_1 h_2(x_2) = F_1(x_2) - (x_1 - r)^2 f_1(x_1)h_2(x_2)$. Using condition (3) twice, $x_1F_1(x_1)/x_2$ is an upper bound for the subtracting term. Since, by construction, we are interested in $x_2 \geq x_1$, $B_2(x_2) - q_1h_2(x_2) > 0$. Then, $d\hat{\Pi}_2(\mathbf{x}^2)/dx_2 > 0$ and $\chi_2(\mathbf{x}^3)$ is uniquely defined.

Suppose we have shown that $M_{j-1}$ is invertible and $B_j(x_j) - q_{j-1}h_j(x_j) > 0$ for all $j \leq k$. Let $l_k = (B_k(x_k) - q_{k-1}h_k(x_k))^{-1}$ and observe that $l_k > 0$ by induction hypothesis; then, by the definition of $M_k$

$$M_k = \begin{pmatrix} M_{k-1} & \mathbf{R}_{k-1}h_k(x_k) \\ \mathbf{m}_{k-1} & B_k(x_k) \end{pmatrix}.$$

Using blockwise inversion,

$$M_k^{-1} = \begin{pmatrix} M_{k-1}^{-1} + h_k(x_k)l_k(M_{k-1}^{-1}\mathbf{R}_{k-1}\mathbf{m}_{k-1}M_{k-1}^{-1}) & -h_k(x_k)l_k(M_{k-1}^{-1}\mathbf{R}_{k-1}) \\ -l_k(\mathbf{m}_{k-1}M_{k-1}^{-1}) & l_k \end{pmatrix}$$

and the inverse of $M_k$ is well defined. We need to show $B_{k+1}(x_{k+1}) - q_k h_{k+1}(x_{k+1}) > 0$. Observing that $\mathbf{R}_k = (\mathbf{R}_{k-1}, R_k(\mathbf{x}_k))^T$, $\mathbf{m}_k = (\mathbf{m}_{k-1}F_k(x_k), R_k(\mathbf{x}_k)f_k(x_k))$, and using the definition of $M_k^{-1}$ and $l_k$ we can write:

$$q_k = F_k(x_k)q_{k-1} + f_k(x_k)\left(R_k(\mathbf{x}_k) - q_{k-1}\right)^2 / \left(B_k(x_k) - q_{k-1}h_k(x_k)\right), \tag{A.7}$$

Thus, $B_{k+1}(x_{k+1}) - q_k h_{k+1}(x_{k+1}) > 0$ is equivalent to show:

$$\left(\frac{B_k(x_{k+1})F_k(x_{k+1})}{f_k(x_k)h_{k+1}(x_{k+1})} - \frac{q_{k-1}}{h_k(x_k)}\right)\left(B_k(x_k) - q_{k-1}h_k(x_k)\right) > \left(R_k(\mathbf{x}_k) - q_{k-1}\right)^2$$

where $B_{k+1}(x_{k+1}) = B_k(x_{k+1})F_k(x_{k+1})$ was used. By the existence proof we are only interested in $x_{k+1} \geq x_k$; using this condition, that $B_k(v)$ is decreasing in $v$, and condition (3) we find that $(B_k(x_k)x_k - q_{k-1})^2$ is a lower bound for the LHS of the expression above. Lemma B.1.1 shows $B_i(x_k)x_k \geq R_k(\mathbf{x}_k)$. Thus we just need to show that $B_k(x_k)x_k - q_{k-1} \geq 0$, which is done by proving $R_k(\mathbf{x}_k) - q_{k-1} \geq 0$. We do this by induction. Since $q_0$ is not defined, we begin with $i = 2$. Integrating by parts $R_2(\mathbf{x}^2)$, $R_2(\mathbf{x}^2) - q_1$ is equal to

$$(x_1 - r)F_1(x_1) + \int_{x_1}^{x_2} F_1(v)dv - (x_1 - r)^2 f_1(x_1) > \int_{x_1}^{x_2} F_1(v)dv \geq 0$$

where $x_1 \geq x_1 - r$ and condition (3) was used in the last step. Suppose we have shown $R_j(\mathbf{x}_j) \geq q_{j-1}$ for $j \leq i$. We show $R_{i+1}(\mathbf{x}_{i+1}) \geq q_i$. Using equation (A.7), this is equivalent to:

$$R_{i+1}(\mathbf{x}_{i+1})/F_i(x_i) - q_{i-1} - (R_i(\mathbf{x}_i) - q_{i-1})^2 \bigg/ \left(\frac{B_i(x_i)}{h_i(x_i)} - q_{i-1}\right) \geq 0.$$

Lemma B.1.2 shows $R_{i+1}(\mathbf{x}_{i+1})/F_i(x_i) \geq R_i(\mathbf{x}_i)$. By the induction hypothesis $R_i(\mathbf{x}_i) \geq q_{i-1}$ and we can rewrite the condition as

$$1 \geq (R_i(\mathbf{x}_i) - q_{i-1}) \bigg/ \left(\frac{B_i(x_i)}{h_i(x_i)} - q_{i-1}\right).$$

The result follows from condition (3) and Lemma B.1.1. Thus $R_{i+1}(\mathbf{x}_{i+1}) \geq q_i$, which proves $d\hat{\Pi}_{k+1}(\mathbf{x}^{k+1})/dx_{k+1} > 0$ for all $x_{k+1} \geq x_k$ and a unique herculean equilibrium exists. □

**Claim 2.** There is no non-herculean equilibria.

*Proof.* Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be an ordered vector of equilibrium cutoffs. beginning from the lower cutoff, let $i$ be the first bidder to play a smaller cutoff than a stronger bidder $i + 1$; i.e., $x_i < x_{i+1}$ but $s_i > s_{i+1}$. In other words, every bidder $k \leq i$ have their cutoffs in the same order as their strength. Because of this, we can use our recursive construction in the existence proof and our induction argument in the uniqueness proof up to bidder $i$, so that best responses are uniquely defined for any vector $\mathbf{x}^{i+1}$ that bidders may play.

Let's analyze $\hat{\Pi}_{i+1}(\mathbf{x}^{i+1})$. Because $\hat{\Pi}_i(\mathbf{x}^i) = 0$ we know $R_i(\mathbf{x}^i) = K_i/A_i^n$. Writing $\hat{\Pi}_{i+1}(\mathbf{x}^{i+1}) = A_{i+1}^n R_{i+1}(\mathbf{x}^{i+1}) - K_{i+1}$. Take $x_{i+1}$ to be value that satisfies $x_{i+1} = \chi_i(x_{i+1}, \mathbf{x}^{i+2}) = x_i$ and notice that Lemma B.1.2 implies $R_{i+1}(x_i, \mathbf{x}^{i+2}) = F_i(x_i)R_i(x_i, x_i, \mathbf{x}^{i+2})$. Then, using $R_i(\mathbf{x}^i) = K_i/A_i^n$, we can write $\hat{\Pi}_{i+1}(x_i, \mathbf{x}^{i+2}) = K_iF_i(x_i)/F_{i+1}(x_i) - K_{i+1} > 0$; which is *positive* under (4) and the condition that bidder $i + 1$ is stronger than bidder $i$. We need to show that there is no $x_{i+1}^* > x_i$ such that $\hat{\Pi}_{i+1}(x_{i+1}^*, \mathbf{x}^{i+2}) = 0$. This follows from the proof of uniqueness, as condition (3) implies $d\hat{\Pi}_{i+1}(\mathbf{x}^{i+1})/dx_{i+1} > 0$ for $x_{i+1}^* > x_i$, implying the result. □ ■

**Proof of Lemma 5.** We show that $s_i$ exists and that $\sigma_i(s) \equiv \Pi_i(s; s, \ldots, s)$ single crosses zero.

*Existence*: Observe that assumptions A3 and A2 jointly imply $\sigma_i(\underline{v}_i) < 0$. Similarly, assumption A3 and Lemma B.2 (see Appendix B) imply, $\sigma_i(\overline{v}_i) \geq \Pi_i(\overline{v}_i; a_{-i})$ $> 0$ (where $a$ is the lower bound of the support of $F_i$). Then, by the intermediate value theorem, there exist $\hat{s}$ such that $\sigma_i(\hat{s}) = 0$.
*Uniqueness*: By Lemma B.2 and the chain rule, we have that $\sigma_i'(s) > 0$. Thus, $\sigma_i(s)$ single crosses zero; i.e., there is a unique value $s_i$ satisfying $\sigma_i(s_i) = 0$. ■

**Proof of Proposition 4.** *Proof preliminaries*: If $s_1 = s_2$ the herculean equilibrium corresponds to the strength of the firms. Assume, without loss of generality, that $s_1 < s_2$. Let $\hat{\mathbf{x}} = (x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2)$ be a vector of group-symmetric cutoff strategies. Pick any firm in group $i \in \{1, 2\}$ and let $\Pi_i^{gs}(x_1, x_2) = \Pi_i(\hat{\mathbf{x}})$ —where $gs$ stands for group-symmetric— represent the expected profit of a firm belonging to group $i$ entering with a valuation $x_i$, when opponents play group-symmetric strategies $x_1$ and $x_2$. Observe that the function $\Pi_i^{gs}(x_1, x_2)$ has a two-dimensional domain, taking as input the group-symmetric strategy of each group.

Lemma B.3 in the Online Appendix implies that, under condition (8), restricting to group-symmetric strategies is without loss. If there is another type of equilibrium, it must be that symmetric firms play asymmetric cutoffs, contradicting the Lemma. Define $\chi_1(x)$ to be the function that solves $\Pi_1^{gs}(\chi_1(x), x) = 0$. Thus, $\chi_1(x)$ corresponds to group 1's symmetric best response to group 2 playing the group-symmetric cutoff $x$. By Lemma B.2, $\Pi_1^{gs}(x_1, x_2)$ is increasing in each argument, and the value $\chi_1(x)$ exists and is unique; i.e., $\chi_1(x)$ is well defined.

**Lemma A.1.** *$\chi_1(s_1) = s_1$ and, under condition (9), $0 > \chi_1'(x) > -\frac{f_2(x)}{F_2(x)} \frac{F_1(\chi_1(x))}{f_1(\chi_1(x))}$.*

*Proof.* By definition of strength we know $\Pi_1^{gs}(s_1, s_1) = 0$, therefore $\chi_1(s_1) = s_1$. Let $G_i$ be the set of firms belonging to group $i$. Using implicit differentiation, the chain rule, that groups members are symmetric, and Lemma B.2

$$\chi_1'(x) = -\frac{\frac{\partial \Pi_1^{gs}(\chi_1(x), x)}{\partial x_2}}{\frac{\partial \Pi_1^{gs}(\chi_1(x), x)}{\partial x_1}} = -\frac{\sum_{j \in G_2} \frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_j}}{\sum_{j \in G_1} \frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_j}} = \frac{-n_2 \frac{f_2(x)}{F_2(x)} \Delta_{1,2}(\hat{\mathbf{x}})}{\Pi_1'(\hat{\mathbf{x}}) + (n_1 - 1) \frac{f_1(\chi_1(x))}{F_1(\chi_1(x))} \Delta_{1,1}(\hat{\mathbf{x}})}$$

where $\Delta_{i,j}(\hat{\mathbf{x}}) = F_j(x_j) \hat{\Delta}_{i,j}(\hat{\mathbf{x}})$ is defined by equation (7). Because numerator and denominator are positive, the equation above proves $\chi_1'(x) < 0$ for all $x$. For the lower bound of $\chi_1(x)$ observe that $\Delta_{1,1}(\hat{\mathbf{x}}) > 0$. Take a lower bound for $\chi_1'(x)$ by making $\Delta_{1,1}(\hat{\mathbf{x}})$ zero. The lower bound $\chi_1'(x) > -\frac{f_2(x)}{F_2(x)} \frac{F_1(\chi_1(x))}{f_1(\chi_1(x))}$ follows by using sufficient condition (9). □

*Existence of a herculean equilibrium*: Define $\hat{\Pi}_2(x) = \Pi_2^{gs}(\chi_1(x), x)$. This function is continuous and corresponds to the expected profit of a firm in group 2 when it enters the market under valuation $x$, group 2 plays the group-symmetric cutoff $x$, and group 1 plays their group-symmetric best response $\chi_1(x)$. Define $x_2$ to be the value satisfying $\hat{\Pi}_2(x_2) = 0$ and let $x_1 = \chi_1(x_2)$. Observe that $x_2 \in (s_1, \infty)$ is necessary and sufficient for $x_1 < x_2$. This is so, as $\chi_1(x)$ is decreasing in $x$ and $\chi_1(s_1) = s_1$. The next claim proves that an herculean equilibrium ($x_1 < x_2$) exists, $x_1 < s_1$, and $x_2 > s_2$.

**Claim 3.** $\hat{\Pi}_2(s_2) < 0$ and there exists $\tilde{x} > s_2$ such that $\hat{\Pi}_2(\tilde{x}) > 0$. Thus, by the intermediate value theorem, the herculean equilibrium cutoff $x_2 \in (s_2, \tilde{x})$ exists.

*Proof.* Because group two is weak, and $\chi_1(x)$ is decreasing in $x$, we know that $\chi_1(s_2) < \chi_1(s_1) = s_1 < s_2$ (where Lemma A.1 was used in the equality). Lemma B.2 and the definition of strength implies $\hat{\Pi}_2(s_2) = \Pi_2^{gs}(\chi_1(s_2), s_2) < \Pi_2^{gs}(s_2, s_2) = 0$, proving $\hat{\Pi}_2(s_2) < 0$. For the second part of the claim, observe that, by Lemma B.2, $\Pi_2^{gs}(x_1, x_2)$ is increasing in $x_1$; then, $\Pi_2^{gs}(\chi_1(x), x) \geq \Pi_2^{gs}(a, x)$ for all $x$. Take $\tilde{x} = \overline{v}_2$ and observe that, by assumption A3, $\Pi_2^{gs}(a, \tilde{x}) > 0$, proving the result. ☐

*Uniqueness of equilibrium:* Observing that, under condition (8), Lemma B.3 applies. Therefore, it is without loss to restrict the analysis to group-symmetric strategies. To prove uniqueness, then, we need to show that no other herculean equilibrium exists and that we can not have an equilibrium where $x_2 < x_1$.

**Claim 4.** There exists a unique herculean equilibrium.

*Proof.* To prove uniqueness within the herculean class, we shown $\hat{\Pi}_2'(x) > 0$ so that $\hat{\Pi}_2(x)$ single crosses zero from below. Recall $\hat{\mathbf{x}} = (\chi_1(x), \ldots, \chi_1(x), x, \ldots, x)$. Differentiating $\hat{\Pi}_2(x)$, using the chain rule, and that firms play group-symmetric strategies, we obtain

$$\begin{aligned}
\hat{\Pi}_2'(x) &= \sum_{j\in G_2} \frac{\partial \Pi_2(\hat{\mathbf{x}})}{\partial x_j} + \chi'(x) \sum_{j\in G_1} \frac{\partial \Pi_2(\hat{\mathbf{x}})}{\partial x_j} \\
&> \Pi_2'(\hat{\mathbf{x}}) + (n_2 - 1)\frac{f_2(x)}{F_2(x)}\Delta_{2,2}(\hat{\mathbf{x}}) - n_1 \frac{f_2(x)}{F_2(x)}\Delta_{2,1}(\hat{\mathbf{x}}) \\
&> (n_2 - 1)\frac{f_2(x)}{F_2(x)}\Delta_{2,2}(\hat{\mathbf{x}}) > 0.
\end{aligned}$$

The first inequality follows from using equation (B.2) and the bound for $\chi_1'(x)$ in Lemma A.1. The second inequality follows from sufficient condition (9). Proving that the derivative is positive and uniqueness within the herculean class. ☐

**Claim 5.** There is no group-symmetric equilibrium in which the strong group plays a higher cutoff than the weak group.

*Proof.* We show that no non-herculean equilibrium—i.e., $x_1 > x_2$ but $s_1 < s_2$—can exist. Define $\chi_2(x)$ to be the function that satisfies $\Pi_2^{gs}(x, \chi_2(x)) = 0$; $\chi_2(x)$ corresponds to group two's best response to the cutoff of group one when $x_1 = x$. As before, Lemma H.1 implies that $\chi_2(x)$ is well defined. Similarly, following the steps of Lemma A.1, it can be shown: $\chi_2(s_2) = s_2$, $\chi_2'(x) < 0$ , and, under condition (9), $\chi_2'(x)$ is bounded below by $-\frac{f_1(x)F_2(\chi_2(x))}{F_1(x)f_2(\chi_2(x))}$.

Define the continuous function $\hat{\Pi}_1(x) = \Pi_1^{gs}(x, \chi_2(x))$ which corresponds to the expected profit of a firm in group 1 when entering the market under valuation $x$ and its opponents play the pair of group-symmetric strategies $(x, \chi_2(x))$. We show that there is no $x$ satisfying $x_1 = x > \chi_2(x) = x_2$ and $\hat{\Pi}_1(x) = 0$; i.e., no non-herculean equilibrium exists. Start by observing that $x > \chi_2(x)$ if and only if $x \in (s_2, \infty)$. In Lemma 5 we showed the function $\sigma_1(s) = \Pi_1^{gs}(s, s)$ is strictly increasing in $s$. By the definition of strength and by firm 2 being weak ($s_1 < s_2$),

$$\sigma_1(s_1) = \Pi_1^{gs}(s_1, s_1) = 0 < \sigma_1(s_2) = \Pi_1^{gs}(s_2, s_2) = \Pi_1^{gs}(s_2, \chi_2(s_2)) = \hat{\Pi}_1(s_2),$$

showing that $\hat{\Pi}_1(s_2) > 0$. Following analogous steps to those in Claim 4, which requires the using lower bound for $\chi_2'(x)$ and sufficient condition (9), we can show that $\hat{\Pi}_1'(x) > 0$. Then, because $\hat{\Pi}_1(s_2) > 0$ and $\hat{\Pi}_1'(x) > 0$ for all $x$, $\hat{\Pi}_1(x)$ never crosses zero when $x > s_2$ and the result follows. □ ■

# B Auxiliary Results

**Lemma B.1.** *In a second-price auction, let $(x_1, x_2, \ldots, x_n)$ be an ordered vector of cutoff strategies (i.e., $x_1 \leq x_2 \leq \cdots x_n$). Then, the following properties hold:*

1. $x_i B_i(x_i) \geq R_i(\mathbf{x}_i)$ *and strict if $r > 0$ or if exists $j < i$ such that $x_j < x_i$.*

2. $R_i(\mathbf{x}_i) > F_{i-1}(x_{i-1})R_{i-1}(\mathbf{x}_{i-1})$ *and with equality if $x_i = x_{i-1}$.*

*Proof.* Recall the definition of $R_i(\mathbf{x}_i)$ in equation (1). For the first claim simply observe,

$$x_i B_i(x_i) - R_i(\mathbf{x}_i) = rA_0^{i-1} + \sum_{k=1}^{i-1}\left(A_k^{i-1}\int_{x_k}^{x_{k+1}} s dB_{k+1}(s)\right)$$

which is strictly positive if $r > 0$ or if there exists a bidder $j < i$ such that $x_j < x_i$. For the second claim we show that $R_i(\mathbf{x}_i) = F_{i-1}(x_{i-1})R_{i-1}(\mathbf{x}_{i-1}) + \int_{x_{i-1}}^{x_i} B_i(s)ds$, which proves the claim. Rewriting $R_i(\mathbf{x}_i)$ using definition in (1):

$$R_i(\mathbf{x}_i) = x_i B_i(x_i) - F_{i-1}(x_{i-1})\left[rA_0^{i-2} - \sum_{k=1}^{i-2}\left(A_k^{i-2}\int_{x_k}^{x_{k+1}} s dB_{k+1}(s)\right)\right] - \int_{x_{i-1}}^{x_i} s dB_i(s).$$

Integrating by parts the last term, $R_i(\mathbf{x}_i)$ becomes:

$$x_{i-1}B_i(x_{i-1}) - F_{i-1}(x_{i-1})\left[rA_0^{i-2} - \sum_{k=1}^{i-2}\left(A_k^{i-2}\int_{x_k}^{x_{k+1}} s dB_{k+1}(s)\right)\right] + \int_{x_{i-1}}^{x_i} B_i(s)ds.$$

Since, by definition, $B_i(x_{i-1}) = B_{i-1}(x_{i-1})F_{i-1}(x_{i-1})$, the result follows. □

**Lemma B.2.** *$\Pi_i(\mathbf{x})$ is strictly increasing in every dimension of $\mathbf{x}$.*
**Proof of Lemma B.2.** Start with the derivative of $\Pi_i$ with respect to $x_i$, then

$$\frac{\partial \Pi_i}{\partial x_i} \equiv \Pi_i'(\mathbf{x}) = \sum_{e \in \mathcal{E}_i}\left\{\left(\prod_{j \in e^c} F_j(x_j)\right)\int_{\{x_j\}_{j \in e \setminus i}}^{b} \pi_i'\left(x_i, v_{e\setminus i}\right)\phi(v_{e\setminus i})d^{n_e - 1}v_{e\setminus i}\right\} > 0, \quad \text{(B.1)}$$

which is positive as, by assumption A3, there is a positive probability that firm $i$ is the sole entrant. To compute $\partial\Pi_i/\partial x_j$, pick a market structure $e$ in which $j$ stays out ($j \in e^c$). Conditional on $e$, the derivative of $\Pi_i$ with respect to $x_j$ is equal to

$$f_j(x_j)\left(\prod_{k\in e^c\setminus j} F_k(x_k)\right)\int_{\{x_k\}_{k\in e\setminus i}}^{b} \pi_i\left(x_i, v_{e\setminus i}\right)\phi(v_{e\setminus i})d^{n_e-1}v_{e\setminus i}.$$

Now take market structure $e$, from above, and use Leibnitz differentiation, to compute $\partial\Pi_i/\partial x_j$ conditional on market structure $e\cup j$; i.e., entry decisions by every firm remain the same as in $e$ except firm $j$, which now participates

$$-f_j(x_j)\left(\prod_{k\in e^c\setminus j} F_k(x_k)\right)\int_{\{x_k\}_{k\in e\setminus i}}^{b} \pi_i\left(x_i, x_j, v_{e\setminus i}\right)\phi(v_{e\setminus i})d^{n_e-1}v_{e\setminus i}.$$

Observe that both expressions from above only differ in sign and in the profit function that is integrated over. Summing both equations delivers

$$f_j(x_j)\left(\prod_{k\in e^c\setminus j} F_k(x_k)\right)\int_{\{x_k\}_{k\in e\setminus i}}^{b} \delta_{i,j}\left(x_i, x_j, v_{e\setminus i}\right)\phi(v_{e\setminus i})d^{n_e-1}v_{e\setminus i}.$$

where $\delta_{i,j}(v_e)\geq 0$ is defined in equation (5). Summing across every market structure in which $j$ stays out and using equation (7) we obtain

$$\frac{\partial\Pi_i}{\partial x_j} = f_j(x_j)\hat{\Delta}_{i,j}(\mathbf{x}) = \frac{f_j(x_j)}{F_j(x_j)}\Delta_{i,j}(\mathbf{x}) > 0 \tag{B.2}$$

Thus, the derivative is positive. ■

**Lemma B.3.** *Under condition (8), two symmetric firms that best respond to each other must play the same cutoff strategy.*

**Proof.** Consider two symmetric firms, $p$ and $q$, and fix *any* profile of cutoffs strategies $\mathbf{x}_{E\setminus\{p,q\}}$ for the rest of the firms. The equilibrium condition for firm $p$ holds whenever there exists $x_p$ and $x_q$ such that $\Pi_p(x_p; x_q, \mathbf{x}_{E\setminus\{p,q\}}) = 0$. Define $\chi(x_p)$ to be firm $q$'s best response to $x_p$ (and to $\mathbf{x}_{E\setminus\{p,q\}}$, which is fixed throughout the proof). By Lemma B.2, $\Pi_p(x_p; x_q, \mathbf{x}_{E\setminus\{p,q\}})$ is strictly increasing in both $x_p$ and $x_q$, which implies that $\chi(x_p)$ exists and is uniquely defined for each $x_p$. To prove the Lemma we need to prove three claims.

**Claim 6.** There exists a unique pair of symmetric cutoffs, $x_p = x_q = z$, such that $\Pi_p(z; z, \mathbf{x}_{E\setminus\{p,q\}}) = 0$.

*Proof.* Suppose firms $p$ and $q$ play a symmetric cutoff, $x_p = x_q = z$. Define $\hat{\sigma}(z) = \Pi_p(z; z, \mathbf{x}_{E\setminus\{p,q\}}) = \Pi_q(z; z, \mathbf{x}_{E\setminus\{p,q\}})$, where the last equality follows from symmetry among firms. Thus, if the equilibrium condition is satisfied by firm $p$, it is also satisfied by firm $q$. We want to show there exists a unique value $\hat{z}$ such that $\hat{\sigma}(\hat{z}) = 0$. Following analogous steps to those in Lemma 5, it is easy to show $\hat{\sigma}(\underline{v}_p) < 0$ and $\hat{\sigma}(\overline{v}_p) > 0$; so that, there exists $\hat{z}$ such that $\hat{\sigma}(\hat{z}) = 0$. Similarly, using Lemma B.2 and the chain rule, we can show that $\hat{\sigma}'(z) > 0$. Hence, the value $\hat{z}$ is unique. □

**Claim 7.** Under condition (9):[27] $0 > \chi'(x_p) > -\frac{f(x_p)}{F(x_p)} \frac{F(\chi(x_p))}{f(\chi(x_p))}$.

*Proof.* Let $\mathbf{x} = (x_p, \chi(x_p), \mathbf{x}_{E\setminus\{p,q\}})$. Using implicit differentiation and equations (B.1) and (B.2) from Lemma B.2, we obtain

$$\chi'(x_p) = -\frac{\frac{\partial \Pi_q(\chi(x_p);x_p,\mathbf{x}_{E\setminus\{p,q\}})}{\partial x_p}}{\frac{\partial \Pi_q(\chi(x_p);x_p,\mathbf{x}_{E\setminus\{p,q\}})}{\partial x_q}} = -\frac{f(x_p)}{F(x_p)} \frac{\Delta_{q,p}(\mathbf{x})}{\Pi_q'(\mathbf{x})} < 0$$

which is negative as the denominator and numerator are positive. To obtain the lower bound for $\chi'(x_p)$ simply use condition (8). □

**Claim 8.** An increase in $x_p$, when firm $q$ best responds by playing $\chi(x_p)$, leads firm $p$ to strictly increase its profit; i.e., $\Pi_p(x_p; \chi(x_p), \mathbf{x}_{E\setminus\{p,q\}})$ is increasing in $x_p$.

*Proof.* Differentiating $\Pi_p(x_p; \chi(x_p), \mathbf{x}_{E\setminus\{p,q\}})$ with respect to $x_p$, using the chain rule, and equations (B.1) and (B.2) we obtain

$$\begin{aligned} \frac{d\Pi_p(\mathbf{x})}{dx_p} &= \frac{\partial \Pi_p(\mathbf{x})}{\partial x_p} + \frac{\partial \chi(x_p)}{\partial x_p} \frac{\partial \Pi_p(\mathbf{x})}{\partial x_q} \\ &= \Pi_p'(\mathbf{x}) + \frac{\partial \chi(x_p)}{\partial x_p} \frac{f(\chi(x_p))}{F(\chi(x_p))} \Delta_{p,q}(\mathbf{x}) > \Pi_p'(\mathbf{x}) - \frac{f(x_p)}{F(x_p)} \Delta_{p,q}(\mathbf{x}) > 0, \end{aligned}$$

where $\mathbf{x} = (x, \chi(x), \mathbf{x}_{E\setminus\{p,q\}})$. The first inequality follows from Claim 7, whereas the second from condition (8); which proves the claim. □

We prove Lemma B.3 by contradiction. Recall that $\mathbf{x}_{E\setminus\{p,q\}}$ is fixed throughout the proof. Suppose, without loss of generality, that there exists $x_q < x_p$ constituting an equilibrium. By Claim 6 there exists a unique value $\hat{z}$ such that $\hat{\sigma}(\hat{z}) = 0$.

Suppose first $x_q < \hat{z} < x_p$. Because

$$\hat{\sigma}(\hat{z}) = \Pi_p(\hat{z}; \hat{z}, \mathbf{x}_{E\setminus\{p,q\}}) = \Pi_p(\hat{z}; \chi(\hat{z}), \mathbf{x}_{E\setminus\{p,q\}}) = 0,$$

and $x_p > \hat{z}$, Claim 8 implies that we must have $\Pi_p(x_p; \chi(x_p) = x_q, \mathbf{x}_{E\setminus\{p,q\}}) > 0$; which contradicts $(x_p, x_q)$ being an equilibrium.

Suppose now $x_q < x_p < \hat{z}$. Lemma B.2 and Claim 6 imply

$$0 = \hat{\sigma}(\hat{z}) > \hat{\sigma}(x_p) = \Pi_p(x_p; x_p, \mathbf{x}_{E\setminus\{p,q\}}) > \Pi_p(x_p; \chi(x_p) = x_q, \mathbf{x}_{E\setminus\{p,q\}})$$

which contradicts $(x_p, x_q)$ being an equilibrium. Analogous arguments can be constructed for the case $\hat{z} < x_q < x_p$, proving the Lemma. ■

[27] For ease in notation, we use symmetry, and drop the sub-indexes from $F$ when referring to firms $p$ and $q$.

Online Appendix

# Equilibrium Uniqueness in Entry Games with Private Information

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Supplemental Material –Not for Publication

## C Equilibrium Exists and is in Cutoff Strategies

An entry strategy for firm $i$ is a mapping from the firm's type $v_i$ to a probability of entering in the market $\tau_i : [a, b] \to [0, 1]$. We assume that the strategy of firm $i$ is an integrable function with respect to its own type $v_i$. We study the Bayesian Equilibria of the entry game. Denote by $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ the vector of entry strategies. Given a strategy profile $\tau$, the expected profit of firm $i$ *after* drawing the type $v_i$ but *before* entry decisions are realized is

$$\Pi_i(v_i; \tau) = \tau_i(v_i) \left[ \sum_{e \in E_i} \left\{ \int_{[a,b]^{n-1}} \pi_i(v_e) \Pr[e|\tau_{-i}, v_{-i}] \phi(v_{-i}) d^{n-1} v_{-i} \right\} \right] \quad \text{(C.1)}$$

where $\Pr[e|\tau_{-i}, v_{-i}]$ is the probability of observing market structure $e$, given the vector of strategies $\tau_{-i}$ and the realizations of types $v_{-i}$. The integral is over each of the $n-1$ dimensions of firm $i$'s competitors types, $v_{-i}$. Conditional on $i$'s entry, which occurs with probability $\tau_i(v_i)$, the expected profit of firm $i$ consists of the expected sum of profit that firm $i$ would get under each feasible market structure, which is induced by the vector of strategies $\tau$ and the realization of types $v_{-i}$, integrated over all possible realizations of the competitors' types, $\phi(v_{-i})$.

**Definition** (Cutoff Strategy)**.** A strategy $\tau_i(v_i)$ is called *cutoff* if there exists a threshold $x > 0$ such that

$$\tau_i(v_i) = \begin{cases} 1 & \text{if } v_i \geq x \\ 0 & \text{if } v_i < x \end{cases} .$$

A cutoff strategy specifies whether a firm enters a market with certainty depending on whether its type is above or below some given threshold. In any best response, there exists a type, $v_i$, that makes a firm indifferent to enter the market. We break this indifference by assuming that firms enter. For a cutoff strategy, this means that a firm enters when its type is greater or equal to its cutoff. Given a vector $\tau_{-i}$, a best response is given by the strategy $\hat{\tau}_i$ that maximizes (C.1) at every value of $v_i$.

A Bayesian Nash equilibrium is defined by a vector of strategies $\tau$ in which every firm best respond to each other. The next proposition establishes the existence of an equilibrium and that, without loss of generality, we can restrict our analysis to cutoff strategies.

**Lemma C.1.** *For any game $(\pi_i, F_i)_{i=1}^n$ satisfying assumptions A1-A3, there exists an equilibrium. For any vector $\tau_{-i}$, firm i's best response is a cutoff strategy. Therefore, every equilibrium of the game is in cutoff strategies.*

**Proof of Lemma C.1.**

*best responses are cutoff strategies*: Fix any firm $i$ and vector of strategies $\tau$. By assumptions A3 and A2, we know that in equilibrium no firm will enter if they draw $v_j < \underline{v}_j$. For relevance, impose that $\tau$ satisfies the restriction $\tau_j(v_j) = 0$ in that range. Because firm $i$'s profit is linear in $\tau_i$, firm $i$'s best response is to participate with probability one whenever there is a positive payoff of doing so. Suppose firm $i$ enters the market with certainty ($\tau_i(v_i) = 1$). The restriction above implies that there is positive probability that firm $i$ is the sole entrant to the market and, consequently, by A1, profits are strictly increasing in $v_i$. By A3, $\Pi_i(\underline{v}_i; \tau) < 0$, and $\Pi_i(\overline{v}_i; \tau) > 0$. Thus, $\Pi_i(v_i; \tau)$ single crosses zero and $i$'s best response to $\tau_{-i}$ is the cutoff strategy defined by the value $x_i$ that satisfies $\Pi_i(x_i; \tau_i = 1, \tau_{-i}) = 0$.

*Existence*: We check the conditions of Brouwer's fixed-point theorem. Because $F_i$ is atomless and has full support and $\pi_i(v_e)$ being continuous and differentiable in $v_i$, firm $i$'s best response lies in the compact and convex set $[\underline{v}_i, \bar{v}_i]$. Thus the n-dimensional function of best responses is a continuous mapping from $\times_{i=1}^n [\underline{v}_i, \bar{v}_i]$ to itself and the conditions for the proposition are met. ■

Existence follows from Brouwer's fixed-point theorem. The restriction to cutoff strategies is quite intuitive: regardless of which strategy competitors are playing, assumption A1 implies that firm $i$'s expected profit is strictly increasing in its type. Because $i$'s expected profit is linear in its entry probability (see eq. (C.1)), $i$ either prefers to enter with certainty, when it is profitable to do so, or to stay out. The next Lemma characterizes all cutoff equilibria.

**Lemma C.2.** *The vector* $\mathbf{x}$ *of cutoff strategies constitutes an equilibrium if and only if* $\Pi_i(\mathbf{x}) = 0$ *for every firm* $i$.

**Proof of Lemma C.2.** By the previous proof a cutoff strategy is defined as the value $x_i$ satisfying $\Pi_i(x_i; \tau_i = 1, \tau_{-i}) = 0$. Because in a cutoff equilibrium $\Pr[e|\tau, v_i]$ is either zero or one. Integrating (C.1) over payoff-irrelevant firms delivers (6). ■

Lemma C.2 characterizes every equilibrium of the entry game. Firm $i$'s best response to $\mathbf{x}_{-i}$ is defined by a cutoff $x_i$ equal to the value of $v_i$ that satisfies $\Pi_i(v_i; \mathbf{x}_{-i}) = 0$. A profile of equilibrium cutoffs $\mathbf{x}$ is, thus, constructed by the collection of functions $\Pi_i(\mathbf{x})$ evaluated at a point in which every firm $i$ is indifferent between entering the market when drawing type $x_i$.

# D Second Price Auction

## D.1 Alternative notions for Strength

In this section, we explore alternative notions for strength. In particular, we study the relationship between: (i) the cutoff strategies, $x_i$; (ii) the *ex-ante* probability of participating in the auction, $1 - F_i(x_i)$; and (iii) the *ex-ante* expected payoff of each bidder; which, for a given vector of cutoffs strategies $\mathbf{x} = (x_1, x_2)$, is equal to:

$$U_i(\mathbf{x}) = \int_{x_i}^{\infty} \left( vF_j(\max\{v, x_j\}) - \int_{x_j}^{\max\{v,x_j\}} sdF_j(s) - K_i \right) dF_i(v). \tag{D.1}$$

That is, for each valuation $v_i$ under which bidder $i$ participates (i.e., for each $v_i > x_i$), the expected payoff of participating in the auction, weighted by the probability that $v_i$ occurs.

We explore the relation between the previous objects by means of an example. Consider two asymmetric bidders whose distribution of valuations follows a Generalized Pareto distribution (GPD) with shape parameter $\kappa$ and scale parameter $\sigma$.[28] The choice of GPD yields a simple concave distribution with positive support that is flexible enough to change its mean and variance. Suppose both bidders have a symmetric participation cost $K$, but bidder 1 is characterized by $(\kappa_1, \sigma_1) = (0, 1)$ and bidder 2 by $(\kappa_2, \sigma_2) = (0.25, 0.75)$. Both distributions have the same mean but the second distribution has twice the variance. That is, the second distribution is a mean-preserving spread of the first. Because the CDFs cross, distributions are *not* ordered by FOSD. Consequently, the game is not *ordered* and it is not self-evident which bidder is stronger.

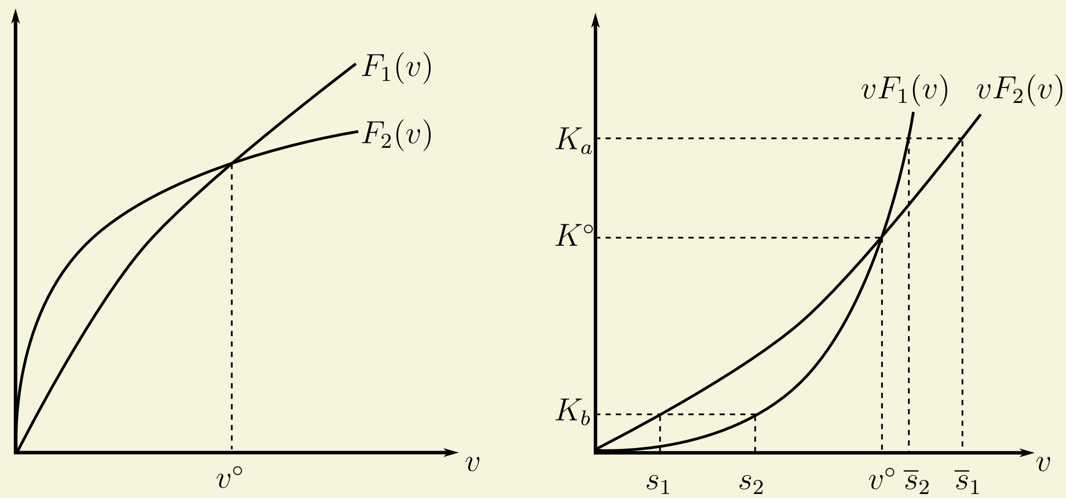
Intuitively, the stronger bidder would be the one whose distribution of valuations has more mass to the right of the equilibrium cutoffs strategies, as this implies the bidder is more likely to obtain higher valuations. If the equilibrium cutoff strategies are high, then bidder 2 would have more mass to the right of the cutoffs, and thus bidder 2 would be the stronger bidder. High equilibrium cutoff strategies are likely to occur when participation costs are high. Conversely, if the cutoff strategies are low, then bidder 1 would have more probability mass to the right of the cutoffs, and thus bidder 1 would be the stronger bidder. Low equilibrium cutoff strategies are likely to occur when participation costs are low.

This situation is illustrated in Figure 5. Panel (a) shows that both distributions are concave, thus Lemma 2 implies that the participation game has a unique equilibrium for any participation costs $K > 0$. Panel (a) also shows that both distributions cross at $v^\circ = 2.2007$. Panel (b) depicts the bidders' strength. It shows that bidders are equally strong when $K^\circ = 1.957$. For participation costs above $K^\circ$, bidder 2 is stronger ($s_2 < s_1$) and, in the unique equilibrium, bidder 2 plays a lower cutoff strategy ($x_2 < x_1$). For instance, if $K_a = 2 > K^\circ$, then the vector of equilibrium cutoffs is $\mathbf{x} = (2.241, 2.238)$. Alternatively, when $K < K^\circ$, bidder 1 is stronger ($s_1 < s_2$) and plays a lower equilibrium cutoff strategy ($x_1 < x_2$). For

[28] For $\kappa \in \mathbb{R}$ and $\sigma \in (0, \infty)$, the Generalized Pareto CDF is defined over $\mathbb{R}_+$ and given by

$$F(x|\kappa, \sigma) = \begin{cases} 1 - \left(1 + \frac{\kappa x}{\sigma}\right)^{-\frac{1}{\kappa}} & \kappa \neq 0 \\ 1 - e^{-\frac{x}{\sigma}} & \kappa = 0 \end{cases}.$$

The CDF is concave whenever $\kappa > -1$, its mean is well defined for $\kappa < 1$ and given by $\sigma/(1-\kappa)$, whereas its variance is defined for $\kappa < 1/2$ and given by $\sigma^2/(1-\kappa)^2(1-2\kappa)$.




**(a)** Distributions are concave **(b)** Strength under different participation costs

**Figure 5: Strength under second-order stochastic dominance.** Distributions are Generalized Pareto with parameters $(\kappa_1, \sigma_1) = (0, 1)$ and $(\kappa_2, \sigma_2) = (0.25, 0.75)$ respectively. Panel (a) shows that distributions cross at $v^\circ = 2.2007$. Panel (b) shows that, depending on the entry cost, either bidder can be stronger.

example, if $K_b = 1 < K^\circ$, then the equilibrium is $\mathbf{x} = (1.281, 1.383)$.

When the participation cost is equal to $K^\circ$, bidders are equally strong ($s_i = v^\circ$). Because the CDFs are concave, the unique equilibrium is given by the symmetric cutoffs equal to the bidders' strength ($x_i = v^\circ$). The expected payoff of bidder 2, however, is greater than the expected payoff of bidder 1. Using equation (D.1), we obtain $(U_1, U_2) = (0.103, 0.185)$. This means that although bidders' cutoffs are not ranked, their expected profits are. The intuition in this scenario follows from $F_2(v) < F_1(v)$ for every $v > v^\circ$. Relative to bidder 1, bidder 2's valuations (distributed according to $F_2(v)$) are skewed to the right tail of the distribution, whereas their expected payment price (distributed according to $F_1(v)$) is skewed towards the left (see Figure 5a). In other words, for valuations greater than $v^\circ$, bidder 2's conditional distribution of valuations FOSD the bidder 1's conditional distribution.

Beginning from the previous example, we construct an equilibrium in which bidder 1 receives a lower expected payoff than bidder 2, despite playing a lower participation cutoff and having a higher participation probability. By decreasing bidder 1's participation cost, bidder 1 becomes stronger than bidder 2 and will play a lower cutoff in the unique equilibrium of the game. By continuity, if the decrease in bidder 1's cost is small, we can construct an equilibrium with said characteristics. Take for example $(K_1, K_2) = (1.9, K^\circ)$, then bidder 1 is stronger and plays a lower cutoff—in this case $\mathbf{x} = (2.1327, 2.2196)$—but also receives lower expected payoffs $(U_1, U_2) = (1.11, 1.83)$. At a cutoff equal to $v^\circ$, both bidders are equally likely to enter. Thus, $x_1 < v^\circ < x_2$ implies that bidder 1 is simultaneously more likely to participate and receive a lower expected payoff.

Finally, to show that cutoff order need not coincide with entry-probability order, modify the participation costs to $(K_1, K_2) = (1.1, 1)$. In this scenario, bidder 1 plays a higher entry cutoff $x_1 = 1.434 > 1.313 = x_2$ while also participating more frequently $1 - F_1(x_1) = .238 > .234 = 1 - F_2(x_2)$.

## D.2 Example of Non-Existence of a Herculean Equilibrium when the Game is not Ordered

We provide an example of a non-ordered game with three entrants which does not have a herculean equilibrium. Suppose the three bidders have identical entry costs, $K = 1$, and the distributions of valuations are given by

$$F_1(v) = 1 - e^{-\frac{v}{2}} \qquad F_2(v) = 1 - (1 + \frac{v}{0.3322})^{-1} \qquad F_3(v) = 1 - e^{-v}.$$

These distributions are concave. In this game, $s_1 = 1.545$ and $s_2 = s_3 = 1.909$. Thus, bidder one is strongest, and bidders 2 and 3 are equally strong. A herculean equilibrium prescribes that bidders 2 and 3 play the same strategy in equilibrium. However, there is no equilibrium with such property. In fact, the unique equilibrium of the game is given by: $x_1 = 1.2938$, $x_2 = 2.1718$, and $x_3 = 2.2180$.

## D.3 Derivation of Equation (A.4)

Recall equation (1)

$$\Pi_i(x_i; \mathbf{x}_{-i}) = A_i^n R_i(x_i; \mathbf{x}_{i-1}) - K_i.$$

where

$$R_i(x_i; \mathbf{x}_{i-1}) = x_i B_i(x_i) - r A_0^{i-1} - \sum_{j=1}^{i-1} \left( A_j^{i-1} \int_{x_j}^{x_{j+1}} \max\{r, s\} dB_{j+1}(s) \right),$$

Before differentiating its is worth noticing that

$$\frac{dB_i(v)}{dv} = B_i(v) \sum_{s=1}^{i-1} h_s(v) \text{ and } \frac{dA_i^n}{dx_j} = A_i^n h_j(x_j) \text{ for } j > i.$$

For a given vector $\mathbf{x} = \left(\chi_1(\mathbf{x}^{k+1}), \ldots, \chi_k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}\right)$, the derivative of $\Pi_i(\mathbf{x})$ with respect to $x_j$ for $j > k$ is

$$\frac{d\Pi_i(\mathbf{x})}{dx_j} = A_i^n R_i(\mathbf{x}_i) \sum_{s=i+1}^{k} h_s(x_s) \frac{d\chi_s}{dx_j} + A_i^n R_i(\mathbf{x}_i) h_j(x_j) + A_i^n \frac{dR_i(\mathbf{x}_i)}{dx_j}$$

where

$$\frac{dR_i(\mathbf{x}_i)}{dx_j} = B_i(x_i)\frac{d\chi_i}{dx_j} + x_iB_i(x_i)\left(\sum_{s=1}^{i-1} h_s\left(x_i\right)\right)\frac{d\chi_i}{dx_j} - rA_0^{i-1}\left(\sum_{\ell=1}^{i-1} h_\ell\left(x_\ell\right)\frac{d\chi_\ell}{dx_j}\right)$$
$$- \sum_{\ell=1}^{i-1}\left(A_\ell^{i-1}\left(\sum_{s=\ell+1}^{i-1} h_s\left(x_s\right)\frac{d\chi_s}{dx_j}\right)\left(\int_{x_\ell}^{x_{\ell+1}} vdB_{\ell+1}\left(v\right)\right)\right.$$
$$+ A_\ell^{i-1}x_{\ell+1}B_{\ell+1}\left(x_{\ell+1}\right)\left(\sum_{s=1}^{\ell} h_s\left(x_{\ell+1}\right)\right)\frac{d\chi_{\ell+1}}{dx_j}$$
$$\left. -A_\ell^{i-1}x_\ell B_{\ell+1}\left(x_\ell\right)\left(\sum_{s=1}^{\ell} h_s\left(x_\ell\right)\right)\frac{d\chi_\ell}{dx_j}\right) \tag{D.2}$$

But observe

$$A_{\ell-1}^{i-1}x_\ell B_\ell\left(x_\ell\right)\left(\sum_{s=1}^{\ell-1} h_s\left(x_\ell\right)\right)\frac{d\chi_\ell}{dx_j} - A_\ell^{i-1}x_\ell B_{\ell+1}\left(x_\ell\right)\left(\sum_{s=1}^{\ell} h_s\left(x_\ell\right)\right)\frac{d\chi_\ell}{dx_j}$$
$$= A_{\ell-1}^{i-1}x_\ell\left(B_\ell\left(x_\ell\right)\sum_{s=1}^{\ell-1} h_s\left(x_\ell\right) - \frac{B_{\ell+1}\left(x_\ell\right)}{F_\ell\left(x_\ell\right)}\sum_{s=1}^{\ell} h_s\left(x_\ell\right)\right)\frac{d\chi_\ell}{dx_j}$$
$$= A_{\ell-1}^{i-1}x_\ell\left(B_\ell\left(x_\ell\right)\sum_{s=1}^{\ell-1} h_s\left(x_\ell\right) - B_\ell\left(x_\ell\right)\sum_{s=1}^{\ell} h_s\left(x_\ell\right)\right)\frac{d\chi_\ell}{dx_j}$$
$$= -A_{\ell-1}^{i-1}x_\ell B_\ell\left(x_\ell\right)h_\ell\left(x_\ell\right)\frac{d\chi_\ell}{dx_j}$$

substituting in, the substracting summation in (D.2) becomes

$$\sum_{\ell=1}^{i-1}\left(A_\ell^{i-1}\left(\int_{x_\ell}^{x_{\ell+1}} vdB_{\ell+1}\left(v\right)\right)\left(\sum_{s=\ell+1}^{i-1} h_s\left(x_s\right)\frac{d\chi_s}{dx_j}\right)\right.$$
$$\left. -A_{\ell-1}^{i-1}x_\ell B_\ell\left(x_\ell\right)h_\ell\left(x_\ell\right)\frac{d\chi_\ell}{dx_j} + x_iB_i(x_i)\left(\sum_{s=1}^{i-1} h_s\left(x_i\right)\right)\frac{d\chi_i}{dx_j}\right)$$

Then, the derivative of $R_i(\mathbf{x}_i)$ becomes

$$\frac{dR_i(\mathbf{x}_i)}{dx_j} = \sum_{\ell=1}^{i-1}\left(A_{\ell-1}^{i-1}\left(x_\ell B_\ell\left(x_\ell\right) - rA_0^{\ell-1}\right)h_\ell\left(x_\ell\right)\frac{d\chi_\ell}{dx_j}\right) + B_i(x_i)\frac{d\chi_i}{dx_j}$$
$$- \sum_{\ell=1}^{i-1}\left(A_\ell^{i-1}\left(\sum_{s=\ell+1}^{i-1} h_s\left(x_s\right)\right)\frac{d\chi_s}{dx_j}\int_{x_\ell}^{x_{\ell+1}} vdB_{\ell+1}\left(v\right)\right)$$

The last term can be rewritten as:

$$\sum_{\ell=1}^{i-1}\left(A_{\ell}^{i-1}\left(\sum_{s=\ell+1}^{i-1} h_s\left(x_s\right)\right)\frac{d\chi_s}{dx_j}\int_{x_\ell}^{x_{\ell+1}} vdB_{\ell+1}\left(v\right)\right)$$
$$=\sum_{\ell=1}^{i-1}\left(\sum_{s=1}^{\ell-1} A_s^{i-1}\left(\int_{x_s}^{x_{s+1}} vdB_{s+1}\left(v\right)\right)h_\ell\left(x_\ell\right)\frac{d\chi_\ell}{dx_j}\right)$$

Re arranging and using $A_{\ell-1}^{i-1}h_\ell\left(x_\ell\right)=A_\ell^{i-1}f_\ell\left(x_\ell\right)$ we obtain

$$\frac{dR_i(\mathbf{x}_i)}{dx_j}=\sum_{\ell=1}^{i-1}\left(A_{\ell-1}^{i-1}R_\ell(\mathbf{x}_\ell)f_\ell(x_\ell)\frac{d\chi_\ell}{dx_j}\right)+B_i(x_i)\frac{d\chi_i}{dx_j}$$

and equation (A.4) follows.

# E A Weaker Sufficient Condition for Uniqueness

In this section we show that, in the two-group model, if the expected profit gain (see equation (7) in the main text) satisfies a condition that is analogous to (but stronger than) supermodularity, we can weaken the sufficient conditions for uniquneness in Proposition 4.

**Proposition 5.** *Let* $\Delta_{i,j}(\mathbf{x})=F_j(x_j)\hat{\Delta}_{i,j}(\mathbf{x})$. *Suppose that for every vector of group-symmetric cutoff strategies* $\mathbf{x}$, *the expected profit gain satisfies the following property*[29]

$$\hat{\Delta}_{1,1}(\mathbf{x})\hat{\Delta}_{2,2}(\mathbf{x})\geq\hat{\Delta}_{1,2}(\mathbf{x})\hat{\Delta}_{2,1}(\mathbf{x}) \tag{E.1}$$

*Then, the game has a unique equilibrium if for every firm* $i$ *and each opponent* $j$, *the following condition holds*

$$\frac{f_i(x_i)}{F_i(x_i)}\frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})}<1 \tag{E.2}$$

*hold for every vector* $\mathbf{x}$ *such that each dimension* $k$ *satisfies* $x_k\in[\underline{v}_{g(k)},\overline{v}_{g(k)}]$.

Before proving the result we note that, when $n_g=1$, $\Delta_{i,i}(\mathbf{x})$ is not defined; i.e., firm $i$'s profit gain when a firm of group $g(i)$ exits when there only is one firm in group $g(i)$. This, however, can be corrected if in property (E.1) we substitute $\hat{\Delta}_{i,i}(\mathbf{x})$ for $\Pi_i'(\mathbf{x})f_i(x_i)$. Under sufficient condition (E.2) this substitution is a bit more demanding than (E.1), as condition (E.2) implies $\hat{\Delta}_{i,i}(\mathbf{x})<\Pi_i'(\mathbf{x})/f_i(x_i)$. Below, we show that both the SPA and the linear model satisfy condition (E.1) for any $n_g\geq 1$.

**Proof.** By the proof of Proposition 4 we know that a herculean equilibrium exists. We need to prove that it is unique. By Lemma B.3 we know that firms will play group symmetric strategies. As in the proof of Proposition 4, let $\hat{\mathbf{x}}=$

[29] Condition (E.1) is equivalent to $\Delta_{1,1}(\mathbf{x})\Delta_{2,2}(\mathbf{x})\geq\Delta_{1,2}(\mathbf{x})\Delta_{2,1}(\mathbf{x})$.

$(x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2)$ be a vector of group-symmetric cutoff strategies. Pick any firm in group $i \in \{1,2\}$ and let $\Pi_i^{gs}(x_1, x_2) = \Pi_i(\hat{\mathbf{x}})$ —where $gs$ stands for group-symmetric— represent the expected profit of a firm belonging to group $i$ entering with a valuation $x_i$, when opponents play group-symmetric strategies $x_1$ and $x_2$. Observe that the function $\Pi_i^{gs}(x_1, x_2)$ has a two-dimensional domain, taking as input the group-symmetric strategy of each group.

Define $\chi_1(x)$ to be the function that solves $\Pi_1^{gs}(\chi_1(x), x) = 0$. Thus, $\chi_1(x)$ corresponds to group 1's symmetric best response to group 2 playing the group-symmetric cutoff $x$. By Lemma H.1, the value $\chi_1(x)$ exists and is unique; i.e., $\chi_1(x)$ is well defined.

Using implicit differentiation, the chain rule, that groups members are symmetric, and equation (B.2)

$$\chi_1'(x) = -\frac{\frac{\partial \Pi_1^{gs}(\chi_1(x),x)}{\partial x_2}}{\frac{\partial \Pi_1^{gs}(\chi_1(x),x)}{\partial x_1}} = -\frac{\sum_{j\in G_2} \frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_j}}{\sum_{j\in G_1} \frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_j}} \tag{E.3}$$

$$= \frac{-n_2 h_2(x_2)\Delta_{1,2}(\hat{\mathbf{x}})}{\Pi_1'(\hat{\mathbf{x}}) + (n_1-1)h_1(x_1)\Delta_{1,1}(\hat{\mathbf{x}})} > -\frac{n_2 h_2(x_2)\Delta_{1,2}(\hat{\mathbf{x}})}{n_1 h_1(x_1)\Delta_{1,1}(\hat{\mathbf{x}})} \tag{E.4}$$

where $x_1 = \chi(x_2)$ and $h_i(v) = f_i(v)/F_i(v)$ is the reversed hazard rate. The inequality in (E.4) follows from substituting sufficient condition (E.2) in the denominator.

To prove uniqueness $\hat{\Pi}_2'(x) > 0$ so that $\hat{\Pi}_2(x)$ single crosses zero from below. Recall $\hat{\mathbf{x}} = (\chi_1(x), \ldots, \chi_1(x), x, \ldots, x)$. Differentiating $\hat{\Pi}_2(x)$, using the chain rule, and that firms play group-symmetric strategies, we obtain[30]

$$\begin{aligned}
\hat{\Pi}_2'(x) &= \sum_{j\in G_2} \frac{\partial \Pi_2(\hat{\mathbf{x}})}{\partial x_j} + \chi_1'(x) \sum_{j\in G_1} \frac{\partial \Pi_2(\hat{\mathbf{x}})}{\partial x_j} \\
&= \Pi_2'(\hat{\mathbf{x}}) + (n_2-1)h_2(x)\Delta_{2,2}(\hat{\mathbf{x}}) + \chi_1'(x) n_1 h_1(\chi_1(x))\Delta_{2,1}(\hat{\mathbf{x}}) \\
&> n_2 h_2(x)\left[\Delta_{2,2}(\hat{\mathbf{x}}) - \frac{\Delta_{2,1}(\hat{\mathbf{x}})\Delta_{1,2}(\hat{\mathbf{x}})}{\Delta_{1,1}(\hat{\mathbf{x}})}\right] > 0.
\end{aligned}$$

The second equality follows from using Lemma B.2. The first inequality follows from using condition (E.2) in $\Pi_2'(\mathbf{x})$ and using the lower bound (E.4) for $\chi'(x)$. The last inequality follows from property (E.1). Proving that the derivative is always positive and equilibrium uniqueness. ■

**Example.** We show that property (E.1) holds in the linear model and in SPAs.

[30] If $n_1 = 1$ simply use $\chi_1'(x) = -n_2 h_2(x_2)\Delta_{1,2}(\hat{\mathbf{x}})/\Pi_1'(\hat{\mathbf{x}})$. Then,

$$\hat{\Pi}_2'(x) > n_2 h_2(x)\left[\Delta_{2,2}(\hat{\mathbf{x}}) - h_1(x_1)\frac{\Delta_{2,1}(\hat{\mathbf{x}})\Delta_{1,2}(\hat{\mathbf{x}})}{\Pi_1'(\hat{\mathbf{x}})}\right] > 0.$$

The last inequality follows from the property (E.1) modified. Similar steps can be applied if $n_2 = 1$.

(i) **Linear model:** Consider the linear model

$$\pi_i(v_e) = \eta_i - \delta_i(n_e - 1) + v_i,$$

where $n_e$ is the number of entrants in market structure $e$. In this context, for a given vector of cutoff strategies $\mathbf{x}$, equation (6) is given by

$$\Pi_i(v_i; \mathbf{x}_{-i}) = \eta_i + v_i - \delta_i \mathbb{I}_{n_e > 1} \sum_{e \in E_i} \left\{ \left( \prod_{j \in O_i(e)} F_j(x_j) \right) \left( \prod_{\ell \in I_i(e)} (1 - F_\ell(x_\ell)) \right) \right\}$$

and $\Pi'(\mathbf{x}) = 1$. Similarly, noticing that $\pi(v_i, v_{e\setminus i}) - \pi(v_i, v_j, v_{e\setminus i}) = \delta_i$ we obtain $\hat{\Delta}_{i,j}(\mathbf{x}) = \delta_i$. Then, the sufficient conditions for uniqueness, which are independent of $n_g$, become

$$\frac{f_i(x_i)}{F_i(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} = \begin{cases} \delta_i f_i(x_i) < 1 & \text{if } j \in g(i), \\ \delta_i F_j(x_j) f_i(x_i)/F_i(x_i) < 1 & \text{if } j \notin g(i) \end{cases}.$$

Finally, property (E.1) when $n_g > 1$ holds as

$$\hat{\Delta}_{1,1}(\mathbf{x})\hat{\Delta}_{2,2}(\mathbf{x}) - \hat{\Delta}_{1,2}(\mathbf{x})\hat{\Delta}_{2,1}(\mathbf{x}) = \delta_1\delta_2 - \delta_1\delta_2 = 0.$$

When $n_1 = 1$ (similarly for $n_2$)

$$\frac{\Pi_1'(\mathbf{x})}{f_1(x_1)}\hat{\Delta}_{2,2}(\mathbf{x}) - \hat{\Delta}_{1,2}(\mathbf{x})\hat{\Delta}_{2,1}(\mathbf{x}) = \frac{1}{f_1(x_1)}\delta_2 - \delta_1\delta_2 = \delta_2\left[\frac{1}{f_1(x_1)} - \delta_1\right] > 0$$

where the inequality follows from sufficient condition (E.2).

(ii) **Second Price Auction** In a SPA, we already know that $vf_i(v)/F_i(v)$ is a sufficient condition for uniqueness. We show that the framework satisfies condition (E.1). When $x_1 \leq x_2$ we have (the proof when $x_2 \leq x_1$ is analogous)

$$\begin{aligned} \Delta_{1,1}(\mathbf{x}) &= \Pi_1(\mathbf{x}) & \Delta_{1,2}(\mathbf{x}) &= \Pi_1(\mathbf{x}) \\ \Delta_{2,1}(\mathbf{x}) &= (x_1 - r)F_1(x_1)^{n_1}F_2(x_2)^{n_2-1} & \Delta_{2,2}(\mathbf{x}) &= \Pi_2(\mathbf{x}) \\ \Pi_1'(\mathbf{x}) &= F_1(x_1)^{n_1-1}F_2(x_2)^{n_2} & \Pi_2'(\mathbf{x}) &= F_1(x_1)^{n_1}F_2(x_2)^{n_2-1} \end{aligned}$$

where $\Pi_1(\mathbf{x}) = (x_1 - r)F_1(x_1)^{n_1-1}F_2(x_2)^{n_2}$ and $\Pi_2(\mathbf{x}) = F_2(x_2)^{n_2-1}R_2(x_1, x_2)$. Then,

$$\begin{aligned} &\Delta_{1,1}(\mathbf{x})\Delta_{2,2}(\mathbf{x}) - \Delta_{1,2}(\mathbf{x})\Delta_{2,1}(\mathbf{x}) \\ &= \Pi_1(\mathbf{x})F_2(x_2)^{n_2-1}\left[R_2(x_1, x_2) - (x_1 - r)F_1(x_1)^{n_1}\right] \\ &= \Pi_1(\mathbf{x})F_2(x_2)^{n_2-1}\left[x_2F_1(x_2)^{n_1} - rF_1(x_1) - \int_{x_1}^{x_2} v dF_1(v)^{n_1} - (x_1 - r)F_1(x_1)^{n_1}\right]. \end{aligned}$$

$$= \Pi_1(\mathbf{x}) F_2(x_2)^{n_2-1} \left[ \int_{x_1}^{x_2} F_1(v)^{n_1} dv \right] > 0$$

where the last equality follows from integrating by parts. When $n_1 = 1$, sufficient condition for uniqueness implies $\Pi_1'(\mathbf{x})/h_1(x) > \Pi_1(\mathbf{x}) = \Delta_{1,1}(\mathbf{x})$, which implies condition (E.1). A similar argument applies for $n_2 = 1$.

# F Uniqueness with Partially Informed Bidders

In this section we show that, when $n_1 = n_2 = 1$, $x_i f_i(x_i)/F_i(x_i) < 1$ implies

$$\frac{f_i(x_i)}{F_i(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1$$

Thus, sufficient condition (3) implies sufficient conditions (8) and (9). Start by observing that, by construction, $\Delta_{i,j}(\mathbf{x}) \leq \Pi_i(\mathbf{x})$. Then, it is sufficient to show

$$\frac{f_i(x_i)}{F_i(x_i)} \Pi_i(\mathbf{x}) < \Pi_i'(\mathbf{x}).$$

Recall

$$\pi_i(x_e) = \int_{r/x_i}^{\infty} \left( \int_0^{x_i \varepsilon} (x_i \varepsilon - \max\{r, s\}) d\Psi_i\left(s, x_e\right) \right) dG(\varepsilon) - K_i,$$

where $\Psi_i(s, x_e) = \prod_{j \in e \setminus i} G(s/x_j)$. Then,

$$\pi_i'(x_e) = \int_{r/x_i}^{\infty} \varepsilon \Psi_i\left(\varepsilon x_i, v_e\right) dG(\varepsilon)$$

we show that $\frac{f_i(x_i)}{F_i(x_i)} \pi_i(x_e) < \pi_i'(x_e)$, which implies the result

$$\begin{aligned}
\frac{f_i(x_i)}{F_i(x_i)} \pi_i(x_e) &= \int_{r/x_i}^{\infty} \left( \int_0^{x_i \varepsilon} \left( \frac{f_i(x_i)}{F_i(x_i)} x_i \varepsilon - \max\{r, s\} \right) d\Psi_i\left(s, x_e\right) \right) dG(\varepsilon) - K_i \\
&< \int_{r/x_i}^{\infty} \left( \int_0^{x_i \varepsilon} \left( \frac{f_i(x_i)}{F_i(x_i)} x_i \varepsilon \right) d\Psi_i\left(s, x_e\right) \right) dG(\varepsilon) \\
&= \int_{r/x_i}^{\infty} \frac{f_i(x_i)}{F_i(x_i)} x_i \varepsilon \Psi_i\left(x_i \varepsilon, x_e\right) dG(\varepsilon) \\
&< \int_{r/x_i}^{\infty} \varepsilon \Psi_i\left(x_i \varepsilon, x_e\right) dG(\varepsilon) = \pi_i'(x_e)
\end{aligned}$$

where in the first inequality we took all the subtracting terms to zero, the second equality integrated the inner integral, and the second inequality used $x_i f_i(x_i)/F_i(x_i) < 1$. This proves the result.

# G Uniqueness in Ordered Games

In an entry game, there are two elements that determine payoffs: the distribution of types $F_i(v_i)$ and the profit function $\pi_i(v_e)$. A game is called *ordered* when firms are symmetric in one of these two dimensions and are ordered in the other. In this section, we extend our results to ordered environments.

**Definition** (Ordered games)**.** A game is *ordered by profit* when firms have symmetric distributions of types, $F_i(v_i) = F(v_i)$ for every $i$, and anonymous profit functions that, for any realization $v_e$, satisfy $\pi_i(v_i, \mathbf{v}_{n_e-1}) \geq \pi_{i+1}(v_i, \mathbf{v}_{n_e-1})$, where $n_e$ is the number of entrants in $e$ and $\mathbf{v}_{n_e-1}$ is an $(n_e-1)$-dimensional vector containing the types of $i$'s competitors.

An entry game is called *ordered by distributions* when firms have symmetric and anonymous profit functions, $\pi_i(v_e) = \pi(v_i, \mathbf{v}_{n_e-1})$ for every $i$, and their distributions of types, $F_i(v_i)$, are ordered in terms of first-order stochastic dominance.[31]

Without loss of generality, we order firms' identities so they satisfy $F_i(v) \leq F_{i+1}(v)$ for all $v$ when ordered by distributions, or $\pi_i(v, \mathbf{v}_{n_e-1}) \geq \pi_{i+1}(v, \mathbf{v}_{n_e-1})$ when ordered by profit.

**Lemma G.1.** *Suppose an entry game in which firms are ordered (either by profit or distributions). Then, the firms are also ordered by strength, with $s_i < s_{i+1}$; i.e., firm 1 is the strongest and firm $n$ the weakest.*

**Proof of Lemma G.1.** We start by showing the order in the context of ordered by profit. Let $s_i$ be the strength of firm $i$, using $\sigma_i(s) \equiv \Pi_i(s; s, \ldots, s)$ we obtain

$$0 = \sigma_i(s_i) = \sum_{e \in \mathcal{E}_i} \left\{ \left( \prod_{j \in e^c} F(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{b} \pi_i \left( s_i, v_{e \setminus i} \right) \phi(v_{e \setminus i}) d^{n_e - 1} v_{e \setminus i} \right\}$$
$$> \sum_{e \in \mathcal{E}_i} \left\{ \left( \prod_{j \in e^c} F(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{b} \pi_{i+1} \left( s_i, v_{e \setminus i} \right) \phi(v_{e \setminus i}) d^{n_e - 1} v_{e \setminus i} \right\} = \sigma_{i+1}(s_i),$$

where in the inequality we used $\pi_i(v, \mathbf{v}_{n_e-1}) > \pi_{i+1}(v, \mathbf{v}_{n_e-1})$. In the last equality, after changing the firm's identity, we used $\mathcal{E}_i = \mathcal{E}_{i+1}$. Then, by Lemma 5, $\sigma_{i+1}(s)$ is increasing in $s$ and $s_{i+1} > s_i$.

For games ordered by distributions, rewriting $\sigma_i(s_i)$ we obtain

$$0 = \sigma_i(s_i) = \sum_{e \in \mathcal{E}_i \cap \mathcal{E}_{i+1}} \left\{ \left( \prod_{j \in e^c} F_j(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{b} \pi \left( s_i, v_{e \setminus i} \right) \phi(v_{e \setminus \{i, i+1\}}) f_{i+1}(v_{i+1}) d^{n_e - 1} v_{e \setminus i} \right\} +$$
$$\sum_{e \in \mathcal{E}_i \setminus \mathcal{E}_{i+1}} \left\{ \left( F_{i+1}(s_i) \prod_{j \in e^c \setminus i+1} F_j(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{b} \pi \left( s_i, v_{e \setminus i} \right) \phi(v_{e \setminus i}) d^{n_e - 1} v_{e \setminus i} \right\}$$

[31] Our results below also extend to environments in which firms are ranked consistently across both dimensions; i.e., $F_i(v) \leq F_{i+1}(v)$ for all $v$ and $\pi_i(v_i, \mathbf{v}_{n_e-1}) \geq \pi_{i+1}(v_i, \mathbf{v}_{n_e-1})$ for all $v_e$.

$$> \sum_{e \in \mathcal{E}_i \cap \mathcal{E}_{i+1}} \left\{ \left( \prod_{j \in e^c} F_j(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{b} \pi\left(s_i, v_{e \setminus i}\right) \phi(v_{e \setminus \{i,i+1\}}) f_i(v_{i+1}) d^{n_e - 1} v_{e \setminus i} \right\} +$$

$$\sum_{e \in \mathcal{E}_i \setminus \mathcal{E}_{i+1}} \left\{ \left( F_i(s_i) \prod_{j \in e^c \setminus i+1} F_j(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{b} \pi\left(s_i, v_{e \setminus i}\right) \phi(v_{e \setminus i}) d^{n_e - 1} v_{e \setminus i} \right\} = \sigma_{i+1}(s_i),$$

where the inequality uses two properties of FOSD. In the second term we used $F_i(v) < F_{i+1}(v)$. In the first term, we used $\int_{s_i}^{b} \varphi(v) f_i(v) dv \leq \int_{s_i}^{b} \varphi(v) f_{i+1}(v) dv$ for any non-increasing function $\varphi(x)$. Then, by Lemma 5, $\sigma_{i+1}(s)$ is increasing in $s$ and $s_{i+1} > s_i$. ■

The previous lemma shows that the firms' ranking provided by strength coincides with the order of the game. In ordered games, the firm ranking provided by strength is robust to adding competitors. That is, if we add a new firm to the game, the existing strength order between the firms remains unchanged (as illustrated in Figure 4a). Recall $\underline{v} = \min\{\underline{v}_i\}_{i=1}^{n}$ and $\overline{v} = \max\{\overline{v}_i\}_{i=1}^{n}$.

**Proposition 6.** *In ordered games, there always exists a* herculean *equilibrium. Moreover, the entry game has a unique equilibrium if the following condition holds*

$$(n-1) \frac{f_i(x_i)}{F_i(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi'_i(\mathbf{x})} < 1 \tag{G.1}$$

*for every pair of firms $\{i, j\}$ and every vector $\mathbf{x}$ such that each dimension satisfies $x_k \in [\underline{v}, \overline{v}]$, and the game is: i) ordered by profit or, ii) ordered by distributions and the profit gain does not depend on the type of competitors; i.e., $\delta_{i,j}(x_i, x_j, v_{e \setminus i}) = \delta_i(x_i, n_e)$.*

We postpone the proof to the next section. Observe that Proposition 6 is not a particular case nor a generalization of Proposition 4 in the main text. While the former can handle more than two groups of asymmetric firms, the latter allows for a larger degree of firm heterogeneity between the two groups. There are also differences in the sufficient condition for uniqueness.

Although Proposition 6 says that condition (G.1) needs to hold for every pair of potential entrants, for certain ordered structures it is sufficient to check the sufficient condition for a single pair of firms.

**Lemma G.2.** *1) If firms are ordered by distribution and belong to the exponentiated family, i.e., $F_i(v_i) = F(v_i)^{\theta_i}$, then condition (G.1) is satisfied for every firm if it holds for the strongest firm (that is, the firm with the highest $\theta_i$).*
*2) If firms are ordered by profits and satisfy $\pi_i(v_i, \mathbf{v}_{n_e - 1}) = \pi(v_i, \mathbf{v}_{n_e - 1}) + K_i$, then condition (G.1) is satisfied for every firm if it holds for any firm.*

**Proof of Lemma G.2.** In condition (G.1), when firms are ordered by distribution, the term $\Delta_{i,j}(\mathbf{x})/\Pi'_i(\mathbf{x})$ is common across firms and the term $f_i(x_i)/F_i(x_i) = \theta_i x_i f(x_i)/F(x_i)$ can be ordered using $\theta_i$. For the second claim, $f(x_i)/F(x_i)$ is common across firms, and the term $K_i$ cancels out from $\Delta_{i,j}(\mathbf{x})$ and $\Pi'_i(\mathbf{x})$. Thus, the same restriction applies to every firm. ■

## G.1 Proof of Proposition 6

We present the proof when firms are ordered by distributions. The proof when firms are ordered by profit is, basically, identical but we can drop the subindices from the distribution functions. Using Lemma G.1 we order firms using stochastic dominance, from stronger (firm 1) to weakest (firm $n$).

*Existence of an herculean equilibrium.* We prove existence by construction. For any vector of cutoff strategies $\mathbf{x}$ and $k \in \{2, \ldots, n\}$ let $\mathbf{x}^k = (x_k, x_{k+1}, \ldots, x_n)$. Construct the equilibrium vector sequentially, as follows:

- **Firm** 1: Define $\chi_1^1(\mathbf{x}^2)$ to be firm's 1 best response to $\mathbf{x}^2$; i.e., $\chi_1^1(\mathbf{x}^2)$ satisfies

$$\Pi_1(\chi_1^1(\mathbf{x}^2); \mathbf{x}^2) = 0.$$

where $\Pi_1(\mathbf{x})$ is defined in (6). By Lemma H.1 in the Auxiliary Result section, $\chi_1^1(\mathbf{x}^2)$ exists and (the best response) is unique and continuous.

- **Firm** 2: Let $\hat{\Pi}_2(\mathbf{x}^2) = \Pi_2(\chi_1^1(\mathbf{x}^2); \mathbf{x}^2)$; that is, $\hat{\Pi}_2(\mathbf{x}^2)$ represents firm's 2 profit after incorporating that firm 1 is best responding to $\mathbf{x}^2$. Define $\chi_2^2(\mathbf{x}^3)$ to be a solution to $\hat{\Pi}_2(\chi_2^2(\mathbf{x}^3), \mathbf{x}^3) = 0$. By Lemma H.1, $\chi_2^2(\mathbf{x}^3)$ exists and is continuous in each dimension of $\mathbf{x}^3$. This function represents firm's 2 best response when firms 1 and 2 are mutually best responding to each other and to $\mathbf{x}^3$. For ease in notation, denote firm's 1 best response after incorporating firm's 2 best response as $\chi_1^2(\mathbf{x}^3) = \chi_1^1(\chi_2^2(\mathbf{x}^3), \mathbf{x}^3)$.[32] This function is also continuous in each dimension of $\mathbf{x}^3$.

  **Claim 9.** For any $\mathbf{x}^3$, $\chi_2^2(\mathbf{x}^3) > \chi_1^2(\mathbf{x}^3)$.

  *Proof.* Fix $\mathbf{x}^3$ and find the value $\hat{x}$ that satisfies $\hat{x} = \chi_1^1(\hat{x}, \mathbf{x}^3)$. The value $\hat{x}$ exists by continuity of $\chi_1^1(\mathbf{x}^2)$ and by $\chi_1^1(\mathbf{x}^2)$ being bounded below and above by $\underline{v}_1$ and $\overline{v}_1$ respectively (by assumption A3). Then by Lemma H.2 in the auxiliary results section we have $\Pi_2(\hat{x}; \hat{x}, \mathbf{x}^3) < \Pi_1(\hat{x}; \hat{x}, \mathbf{x}^3) = 0$. Define a pair of sequences $\{y_m, z_m\}_{m \in \mathbb{N}}$ satisfying: (i) $y_1 = z_1 = \hat{x}$; (ii) $y_{m+1}$ is the unique (by Lemma H.1) value that solves $\Pi_2(z_m; y_{m+1}, \mathbf{x}^3) = 0$ (i.e., $y_{m+1}$ is firm's 2 best response to the cutoffs $(z_m, \mathbf{x}^3)$) and; (iii) $z_{m+1} = \chi_1^1(y_{m+1}, \mathbf{x}^3)$. By definition, $z_{m+1}$ solves $\Pi_1(z_{m+1}; y_{m+1}, \mathbf{x}^3) = 0$ and, by Lemma H.1, the value $z_{m+1}$ is also unique. We show that $\{y_m\}_{m \in \mathbb{N}}$ is increasing and $\{z_m\}_{m \in \mathbb{N}}$ decreasing. Because $\Pi_2(\hat{x}; \hat{x}, \mathbf{x}^3) < 0$ and $\Pi_2(\mathbf{x})$ being strictly increasing in the 2nd dimension, $y_2 > y_1 = \hat{x}$. Similarly, because (by Lemma B.2) $\Pi_1(\mathbf{x})$ is also increasing in the 2nd dimension, we have $\Pi_1(z_1; y_2, \mathbf{x}^3) > 0$, which implies $z_2 = \chi_1^1(y_2, \mathbf{x}^3) < z_1 = \chi_1^1(y_1, \mathbf{x}^3)$. This, in turn, implies (by Lemma H.2)

$$\Pi_2(z_2; y_2, \mathbf{x}^3) < \Pi_1(z_2; y_2, \mathbf{x}^3) = 0;$$

  which implies $y_3 > y_2$. By induction, the argument generalizes to an arbitrary step $m$ and the sequences $\{y_m, z_m\}_{m \in \mathbb{N}}$ are monotonically increasing and

[32] More generally, for $j < k$ the notation $\chi_j^k(\mathbf{x}^k)$ represents firm's $j$ best response to $\mathbf{x}^{j+1}$ (i.e., $\chi(\mathbf{x}^{j+1})$) after substituting subsequent best responses from firm $j+1$ up to firm $k$.

decreasing respectively. By assumption A3, $\{y_m\}_{m\in\mathbb{N}}$ is bounded above by $\overline{v}_2$ and $\{z_m\}_{m\in\mathbb{N}}$ is bounded below by $\underline{v}_1$. Thus, the sequences converge to $y_\infty$ and $z_\infty$, respectively. By convergence, we have: (i) $z_\infty = \chi_1^1(y_\infty, \mathbf{x}^3)$ and; (ii) $\Pi_2(z_\infty; y_\infty, \mathbf{x}^3) = \hat{\Pi}_2(y_\infty, \mathbf{x}^3) = 0$ (i.e., $y_\infty = \chi_2^2(\mathbf{x}^3)$). Thus, $\chi_1^1(y_\infty, \mathbf{x}^3) = \chi_1^2(\mathbf{x}^3)$ and, as $z_\infty < \hat{x} < y_\infty$, we have $\chi_2^2(\mathbf{x}^3) > \chi_1^2(\mathbf{x}^3)$. □

- **Firm** $k \leq n$: Suppose we have shown the existence of $\chi_\ell^\ell(\mathbf{x}^{\ell+1})$ for every $\ell \in \{1, \ldots, k-1\}$, have recursively defined $\chi_j^\ell(\mathbf{x}^{\ell+1}) = \chi_j^{\ell-1}(\chi_\ell^\ell(\mathbf{x}^{\ell+1}), \mathbf{x}^{\ell+1})$ for $j \in \{1, \ldots, \ell\}$, and that both constructions are continuous. Let $\hat{\Pi}_k(\mathbf{x}^k) = \Pi_k(\chi_1^{k-1}(\mathbf{x}^k), \ldots, \chi_{k-1}^{k-1}(\mathbf{x}^k), \mathbf{x}^k)$ represent firm's $k$ profit after incorporating that every firm $j \in \{1, \ldots, k-1\}$ is mutually best responding to each other and to $\mathbf{x}^k$. Define $\chi_k^k(\mathbf{x}^{k+1})$ (observe that $\chi_n^n$ is a number, not a function, as $\mathbf{x}^{k+1}$ is empty when $k = n$) to be a solution to $\hat{\Pi}_k(\chi_k^k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}) = 0$. By Lemma H.1, $\chi_k^k(\mathbf{x}^k)$ exists and is continuous in each dimension of $\mathbf{x}^k$. This function represents firm's $k$ best response to $\mathbf{x}^{k+1}$ when every firm $j \in \{1, \ldots, k-1\}$ is mutually best responding to each other and to $\mathbf{x}^k$.

**Claim 10.** For any $\mathbf{x}^{k+1}$, if firm $k-1$ is stronger than $k$ the solution $\chi_k^k(\mathbf{x}^{k+1})$ satisfies $\chi_k^k(\mathbf{x}^{k+1}) > \chi_{k-1}^k(\mathbf{x}^{k+1})$.

*Proof.* Fix any $\mathbf{x}^{k+1}$ and let $\chi_k^k(\mathbf{x}^{k+1})$ be one of the solutions found in the previous step. Then define the vector of cutoffs $\mathbf{x} = \left(\chi_1^k(\mathbf{x}^{k+1}), \ldots, \chi_k^k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}\right)$. Throughout the proof, the vector of strategies for every firm except firm $k$ and $k-1$, $\mathbf{x}_{E\setminus\{k,k-1\}}$, remains fixed (i.e., they are numbers not functions). Define $\hat{x}$ to be a value satisfying $\hat{x} = \chi_{k-1}^{k-1}(\hat{x}, \mathbf{x}^{k+1})$. The value $\hat{x}$ exists by continuity of $\chi_{k-1}^{k-1}(\mathbf{x}^k)$ and by $\chi_{k-1}^{k-1}(\mathbf{x}^k)$ being bounded below and above by $\underline{v}_{k-1}$ and $\overline{v}_{k-1}$ respectively (by assumption A3). By definition of best response $\hat{x}$ satisfies $\Pi_{k-1}(\hat{x}; \hat{x}, \mathbf{x}_{E\setminus\{k,k-1\}}) = 0$. Then, by Lemma H.2, we have

$$\Pi_k(\hat{x}; \hat{x}, \mathbf{x}_{E\setminus\{k,k-1\}}) < \Pi_{k-1}(\hat{x}; \hat{x}, \mathbf{x}_{E\setminus\{k,k-1\}}) = 0.$$

Define a pair of sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ satisfying: (i) $y_1 = z_1 = \hat{x}$; (ii) $y_{m+1}$ is the unique (by Lemma H.1) value that solves $\Pi_k(z_m; y_{m+1}, \mathbf{x}_{E\setminus\{k,k-1\}}) = 0$ (i.e., $y_{m+1}$ is firm's $k$ best response to the cutoffs $(z_m, \mathbf{x}_{E\setminus\{k,k-1\}})$) and; (iii) $z_{m+1} = \chi_{k-1}^{k-1}(y_{m+1}, \mathbf{x}^{k+1})$. By definition, $z_{m+1}$ solves $\Pi_{k-1}(z_{m+1}; y_{m+1}, \mathbf{x}_{E\setminus\{k,k-1\}}) = 0$ and, Lemma H.1, the value $z_{m+1}$ is also unique. We show that $\{y_m\}_{m\in\mathbb{N}}$ is increasing and $\{z_m\}_{m\in\mathbb{N}}$ decreasing. Because $\Pi_k(\hat{x}; \hat{x}, \mathbf{x}_{E\setminus\{k,k-1\}}) < 0$ and $\Pi_k(\mathbf{x})$ being strictly increasing in the $k$th dimension, $y_2 > y_1 = \hat{x}$. Similarly, because (by Lemma B.2) $\Pi_{k-1}(\mathbf{x})$ is also increasing in the $k$th dimension, we have $\Pi_{k-1}(\hat{x}; y_2, \mathbf{x}_{E\setminus\{k,k-1\}}) > 0$, which implies $z_2 = \chi_{k-1}^{k-1}(y_2, \mathbf{x}^{k+1}) < \chi_{k-1}^{k-1}(y_1, \mathbf{x}^{k+1}) = z_1$. This, in turn, implies (by Lemma H.2)

$$\Pi_k(z_2; y_2, \mathbf{x}_{E\setminus\{k,k-1\}}) < \Pi_{k-1}(z_2; y_2, \mathbf{x}_{E\setminus\{k,k-1\}}) = 0,$$

which, in turns, implies $y_3 > y_2$. By induction, the argument generalizes to an arbitrary step $m$ and the sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ are monotonically increasing and decreasing respectively. By assumption A3, $\{y_m\}_{m\in\mathbb{N}}$ is bounded above by

$\overline{v}_k$ and $\{z_m\}_{m\in\mathbb{N}}$ is bounded below by $\underline{v}_{k-1}$. Thus, the sequences converge to $y_\infty$ and $z_\infty$, respectively. By convergence, we have: (i) $z_\infty = \chi^{k-1}_{k-1}(y_\infty, \mathbf{x}^{k+1})$ and; (ii) $\Pi_k(z_\infty; y_\infty, \mathbf{x}_{E\setminus\{k,k-1\}}) = \hat{\Pi}_k(y_\infty; \mathbf{x}_{E\setminus\{k,k-1\}}) = 0$ (i.e., $y_\infty = \chi^k_k(\mathbf{x}^{k+1})$). Thus, $\chi^{k-1}_{k-1}(y_\infty, \mathbf{x}^{k+1}) = \chi^k_{k-1}(\mathbf{x}^{k+1})$ and, as $z_\infty < \hat{x} < y_\infty$, we have $\chi^k_k(\mathbf{x}^{k+1}) > \chi^k_{k-1}(\mathbf{x}^{k+1})$. ☐

Thus, we have constructed an equilibrium vector $\mathbf{x} = (\chi^n_1(x_n), \ldots, \chi^n_{n-1}(x_n), x_n)$ with the property that $x_i < x_{i+1}$; i.e., a Herculean equilibrium.

*Uniqueness within the herculean-equilibrium class*: We show that at each step $k$ of the previous construction there is a unique best response $x_k = \chi^k_k(\mathbf{x}^{k+1})$ to $\mathbf{x}^{k+1}$.

- **Firm** 1: The uniqueness of $\chi^1_1(\mathbf{x}^2)$ follows from Lemma H.1. Let $h_i(x) = f_i(x)/F_i(x)$ be the reversed hazard rate of firm $i$'s distribution of private information. The next result is needed for subsequent steps.

  **Claim 11.** Under condition (G.1), for every $j \in \{2, \ldots, n\}$, $\partial\chi^1_1(\mathbf{x}^2)/\partial x_j$ satisfies:

$$0 > \frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_j} = -h_j(x_j)\frac{\Delta_{1,j}(\mathbf{x})}{\Pi'_1(\mathbf{x})} > -\frac{h_j(x_j)}{h_1(x_1)}\frac{1}{n-1}. \tag{G.2}$$

$$\frac{1}{h_j(x_j)}\frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_j} < \frac{1}{h_q(x_q)}\frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_q}\frac{1}{n-1} \quad \text{for } q \in \{2, \ldots, j-1\} \tag{G.3}$$

  *Proof.* Let $\mathbf{x} = (\chi^1_1(\mathbf{x}^2), \mathbf{x}^2)$; using implicit differentiation and Lemma B.2 we obtain

$$\frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_j} = -\frac{\partial\Pi_1(\mathbf{x})/\partial x_j}{\partial\Pi_1(\mathbf{x})/\partial x_1} = -h_j(x_j)\frac{\Delta_{1,j}(\mathbf{x})}{\Pi'_1(\mathbf{x})}, \tag{G.4}$$

  which is negative as, $\Delta_{1,j}(\mathbf{x}) > 0$ and $\Pi_{1,j}(\mathbf{x}) > 0$ for every $\mathbf{x}$. The lower bound in equation (G.2) follows from applying condition (G.1) into equation (G.4). Property (G.3) follows from observing

$$\frac{1}{h_q(x_q)}\frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_q}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_j} = \frac{1}{\Pi'_1(\mathbf{x})}\left(\Delta_{1,j}(\mathbf{x}) - \frac{\Delta_{1,q}(\mathbf{x})}{n-1}\right) > 0,$$

  where the equality follows from substituting in equation (G.4), and the inequality follows from Lemma H.3 and the fact that $q \in \{2, \ldots, j-1\}$. ☐

- **Firm** 2: Fix $\mathbf{x}^3$ and let $\mathbf{x} = (\chi^1_1(\mathbf{x}^2), \mathbf{x}^2)$, we need to show that the best response $\chi^2_2(\mathbf{x}^3)$ is unique. We do this by showing that $\hat{\Pi}_2(\mathbf{x}^2) = \Pi_2(\chi^1_1(\mathbf{x}^2); \mathbf{x}^2)$ is strictly increasing in $x_2$; so that, $\hat{\Pi}_2(x_2, \mathbf{x}^3)$ single crosses zero and there is a unique value $\chi^2_2(\mathbf{x}^3)$ satisfying $\hat{\Pi}_2(\chi^2_2(\mathbf{x}^3), \mathbf{x}^3) = 0$. Using the chain rule and equation (B.2)

$$\begin{aligned}\hat{\Pi}'_2(\mathbf{x}^2) = \Pi'_2(\mathbf{x}) + \frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_2}\frac{\partial\Pi_2}{\partial x_1} &= \Pi'_2(\mathbf{x}) + \frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_2}h_1(\chi^1_1(\mathbf{x}^2))\Delta_{2,1}(\mathbf{x}) \\ &> \Pi'_2(\mathbf{x}) - h_2(x_2)\frac{\Delta_{2,1}(\mathbf{x})}{n-1} > \Pi'_2(\mathbf{x})\left[1 - \frac{1}{(n-1)^2}\right] > 0,\end{aligned} \tag{G.5}$$

where in the first inequality follows from the lower bound in equation (G.2) and the second inequality follows from sufficient condition (G.1). This proves uniqueness of the best response. The next result is needed for the induction argument in the proof.

**Claim 12.** Let $\chi_1^2(\mathbf{x}^3) = \chi_1^1(\chi_2^2(\mathbf{x}^3), \mathbf{x}^3)$. Under condition (G.1), for every $j \in \{3, \dots, n\}$ and $\ell \in \{1, 2\}$, $\partial\chi_\ell^2(\mathbf{x}^3)/\partial x_j$ satisfies:

$$\frac{\partial\chi_2^2(\mathbf{x}^3)}{\partial x_j} = -\frac{h_j(x_j)\Delta_{2,j}(\mathbf{x}) + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_j} h_1(x_1)\Delta_{2,1}(\mathbf{x})}{\Pi_2'(\mathbf{x}) + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_2} h_1(x_1)\Delta_{2,1}(\mathbf{x})} \tag{G.6}$$

$$0 > \frac{\partial\chi_\ell^2(\mathbf{x}^3)}{\partial x_j} > -\frac{h_j(x_j)}{h_\ell(x_\ell)}\frac{1}{n-1} \quad \text{and,} \tag{G.7}$$

$$\frac{1}{h_j(x_j)}\frac{\partial\chi_2^2(\mathbf{x}^3)}{\partial x_j} < \frac{1}{h_q(x_q)}\frac{\partial\chi_2^2(\mathbf{x}^3)}{\partial x_q}\frac{1}{n-1} \quad \text{for } q \in \{3, \dots, j-1\} \tag{G.8}$$

*Proof.* To show equation (G.6) use implicit differentiation, the chain rule, and equation (B.2) to obtain

$$-\frac{\partial\chi_2^2(\mathbf{x}^3)}{\partial x_j} = \frac{\frac{\partial\hat{\Pi}_2}{\partial x_j}}{\frac{\partial\hat{\Pi}_2}{\partial x_2}} = \frac{\frac{\partial\Pi_2}{\partial x_j} + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_j}\frac{\partial\Pi_2}{\partial x_1}}{\Pi_2'(\mathbf{x}) + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_2}\frac{\partial\Pi_2}{\partial x_1}} = \frac{h_j(x_j)\Delta_{2,j}(\mathbf{x}) + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_j}h_1(x_1)\Delta_{2,1}(\mathbf{x})}{\Pi_2'(\mathbf{x}) + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_2}h_1(x_1)\Delta_{2,1}(\mathbf{x})}$$

Observe, by equation (G.5), that the denominator is positive. Using lower bound (G.2) and Lemma H.3 we can see that the numerator is also positive, implying that $\partial\chi_2^2(\mathbf{x}^3)/\partial x_j$ is negative; which proves the upper bound of (G.7) when $\ell = 2$. For the lower bound of equation (G.7) when $\ell = 2$, using equation (G.6), observe that equation (G.7) holds if and only if the following expression is positive (replace (G.6) into (G.7) and work out the inequality):

$$h_j(x_j)\left[\left(\frac{1}{h_2(x_2)}\frac{\Pi_2'(\mathbf{x})}{n-1} - \Delta_{2,j}(\mathbf{x})\right) + h_1(x_1)\left(\frac{1}{h_2(x_2)}\frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_2}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_j}\right)\Delta_{2,1}(\mathbf{x})\right].$$

The first round bracket is positive by sufficient condition (G.1). The second round bracket is positive by property (G.3). Thus, the expression is indeed positive and the lower bound in equation (G.7) holds.

We now prove the bounds of (G.7) when $\ell = 1$. Using $\chi_1^2(\mathbf{x}^3) = \chi_1^1(\chi_2^2(\mathbf{x}^3), \mathbf{x}^3)$, observe

$$\frac{\partial\chi_1^2(\mathbf{x}^3)}{\partial x_j} = \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_j} + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_2}\frac{\partial\chi_2^2(\mathbf{x}^3)}{\partial x_j}. \tag{G.9}$$

Using (G.4) to substitute for $\partial\chi_1^1(\mathbf{x}^2)/\partial x_\ell$ with $\ell \in \{2, j\}$ and using the lower bound in equation (G.7) when $\ell = 2$, we obtain the following upper bound:

$$\frac{\partial \chi_1^2(\mathbf{x}^3)}{\partial x_j} < \frac{h_j(x_j)}{\Pi_1'(\mathbf{x})} \left[ \frac{\Delta_{1,2}(\mathbf{x})}{n-1} - \Delta_{1,j}(\mathbf{x}) \right] < 0,$$

the inequality follows from Lemma H.3; proving the upper bound. The lower bound in equation (G.7) follows from using equation (G.9) and observing

$$\frac{\partial \chi_1^2(\mathbf{x}^3)}{\partial x_j} > \frac{\partial \chi_1^1(\mathbf{x}^2)}{\partial x_j} > -\frac{h_j(x_j)}{h_1(x_1)} \frac{1}{n-1},$$

where the inequalities follow from $\partial \chi_2^2(\mathbf{x}^3)/\partial x_j \cdot \partial \chi_1^1(\mathbf{x}^2)/\partial x_2 > 0$ and equation (G.2), respectively.

Finally, to prove property (G.8) use equation (G.6) to write

$$\frac{1}{h_q(x_q)} \frac{\partial \chi_2^2(\mathbf{x}^3)}{\partial x_q} \frac{1}{n-1} - \frac{1}{h_j(x_j)} \frac{\partial \chi_2^2(\mathbf{x}^3)}{\partial x_j} = \frac{1}{D_2} \left[ \Delta_{2,j}(\mathbf{x}) - \frac{\Delta_{2,q}(\mathbf{x})}{n-1} + h_1(x_1) \left( \frac{1}{h_j(x_j)} \frac{\partial \chi_1^1(\mathbf{x}^2)}{\partial x_j} - \frac{1}{h_q(x_q)} \frac{\partial \chi_1^1(\mathbf{x}^2)}{\partial x_q} \frac{1}{n-1} \right) \Delta_{2,1}(\mathbf{x}) \right],$$

where $D_2 = \Pi_2'(\mathbf{x}) + \frac{\partial \chi_1^1(\mathbf{x}^2)}{\partial x_2} h_1(x_1) \Delta_{2,1}(\mathbf{x}) > 0$. We show that a lower bound of this expression is positive. Taking $-\partial \chi_1^1(\mathbf{x}^2)/\partial x_q > 0$ to zero, we obtain

$$\frac{1}{D_2} \left[ \Delta_{2,j}(\mathbf{x}) - \frac{\Delta_{2,q}(\mathbf{x})}{n-1} + \frac{h_1(x_1)}{h_j(x_j)} \frac{\partial \chi_1^1(\mathbf{x}^2)}{\partial x_j} \Delta_{2,1}(\mathbf{x}) \right] > \frac{1}{D_2} \left[ \Delta_{2,j}(\mathbf{x}) - \frac{\Delta_{2,q}(\mathbf{x})}{n-1} - \frac{\Delta_{2,1}(\mathbf{x})}{n-1} \right] > \frac{1}{D_2} \left[ \Delta_{2,j}(\mathbf{x}) - \frac{2\Delta_{2,q}(\mathbf{x})}{n-1} \right] > 0.$$

The first inequality follows from using lower bound (G.2). The other two inequalities follow from Lemma H.3 and the fact that $q \in \{2, \dots, j-1\}$. □

- **Firm** $k \in \{3, \dots, n\}$: Suppose that, for every $p \in \{1, \dots, k-1\}$ and $j \in \{p+1, \dots, n\}$, we have proven that: $\chi_p^p(\mathbf{x}^{p+1})$ is unique;

$$0 > \frac{\partial \chi_p^p(\mathbf{x}^k)}{\partial x_j} = -\frac{h_j(x_j)\Delta_{p,j}(\mathbf{x}) + \sum_{\ell=1}^{p-1} \frac{\partial \chi_\ell^{p-1}(\mathbf{x}^p)}{\partial x_j} h_\ell(x_\ell)\Delta_{p,\ell}(\mathbf{x})}{\Pi_p'(\mathbf{x}) + \sum_{\ell=1}^{p-1} \frac{\partial \chi_\ell^{p-1}(\mathbf{x}^p)}{\partial x_p} h_\ell(x_\ell)\Delta_{p,\ell}(\mathbf{x})}; \tag{G.10}$$

$$0 > \frac{\partial \chi_q^p(\mathbf{x}^k)}{\partial x_j} > -\frac{h_j(x_j)}{h_q(x_q)} \frac{1}{n-1} \quad \text{for } q \in \{1, \dots, p\} \text{ and;} \tag{G.11}$$

$$\frac{1}{h_j(x_j)} \frac{\partial \chi_p^p(\mathbf{x}^{p+1})}{\partial x_j} < \frac{1}{h_q(x_q)} \frac{\partial \chi_p^p(\mathbf{x}^{p+1})}{\partial x_q} \frac{1}{n-1} \quad \text{for } q \in \{p+1, \dots, j-1\}. \tag{G.12}$$

Fix $\mathbf{x}^{k+1}$ and let $\mathbf{x} = (\chi_1^{k-1}(\mathbf{x}^k), \dots, \chi_{k-1}^{k-1}(\mathbf{x}^k), \mathbf{x}^k)$. We show that the best response $\chi_k^k(\mathbf{x}^{k+1})$ is unique by showing that $\hat{\Pi}_k(\mathbf{x}^k)$ is strictly increasing in $x_k$. Differentiating,

$$\begin{aligned}\hat{\Pi}_k'(\mathbf{x}^k) &= \Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1} \frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k} h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x}) \\ &> \Pi_k'(\mathbf{x}) - h_k(x_k)\sum_{\ell=1}^{k-1}\frac{\Delta_{k,\ell}(\mathbf{x})}{n-1} > \Pi_k'(\mathbf{x}) - h_k(x_k)\frac{(k-1)\Delta_{k,k-1}(\mathbf{x})}{n-1} > 0,\end{aligned}$$

where the inequalities follow from lower bound (G.11), Lemma H.3, and sufficient condition (G.1), respectively. This proves uniqueness of the best response. The next result completes the induction argument.

**Claim 13.** Under condition (G.1), for every $j \in \{k+1, \ldots, m\}$ and $p \in \{1, \ldots, k\}$, $\partial \chi_p^k(\mathbf{x}^{k+1})/\partial x_j$ satisfies

$$\frac{\partial \chi_k^k(\mathbf{x}^{k+1})}{\partial x_j} = -\frac{h_j(x_j)\Delta_{k,j}(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x})}{\Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x})} \tag{G.13}$$

$$0 > \frac{\partial \chi_p^k(\mathbf{x}^{k+1})}{\partial x_j} > -\frac{h_j(x_j)}{h_p(x_p)}\frac{1}{n-1} \quad \text{and,} \tag{G.14}$$

$$\frac{1}{h_j(x_j)}\frac{\partial \chi_k^k(\mathbf{x}^{k+1})}{\partial x_j} < \frac{1}{h_q(x_q)}\frac{\partial \chi_k^k(\mathbf{x}^{k+1})}{\partial x_q}\frac{1}{n-1} \quad \text{for } q \in \{k+1, \ldots, j-1\} \tag{G.15}$$

*Proof.* To show equation (G.13) use the implicit differentiation, the chain rule, and equation (B.2) to obtain

$$\begin{aligned}\frac{\partial \chi_k^k(\mathbf{x}^{k+1})}{\partial x_j} &= -\frac{\partial\hat{\Pi}_k(\mathbf{x})/\partial x_j}{\partial\hat{\Pi}_k(\mathbf{x})/\partial x_k} = -\frac{\frac{\partial \Pi_k(\mathbf{x})}{\partial x_j} + \sum_{\ell=1}^{k-1}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}\frac{\partial \Pi_k(\mathbf{x})}{\partial x_\ell}}{\Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{\partial \Pi_k(\mathbf{x})}{\partial x_\ell}}. \\ &= -\frac{h_j(x_j)\Delta_{k,j}(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x})}{\Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x})}.\end{aligned}$$

We already showed that the denominator is positive. We show that a lower bound of the numerator is positive, which immediately implies the upper bound in equation (G.14) for the case when $p = k$. Using equation (G.11) a lower bound for the numerator is

$$h_j(x_j)\left[\Delta_{k,j}(\mathbf{x}) - \sum_{\ell=1}^{k-1}\frac{\Delta_{k,\ell}(\mathbf{x})}{n-1}\right] > h_j(x_j)\left[\Delta_{k,j}(\mathbf{x}) - \frac{(k-1)\Delta_{k,k-1}(\mathbf{x})}{n-1}\right] > 0,$$

where both inequalities follows from Lemma H.3. Thus, the numerator is positive.

For the lower bound in equation (G.14) in the case $p = k$, replace (G.13) into (G.14) and observe that the inequality holds if and only if the following expression is positive

$$h_j(x_j)\left[\left(\frac{1}{h_k(x_k)}\frac{\Pi'_k(\mathbf{x})}{n-1} - \Delta_{k,j}(\mathbf{x})\right) + \sum_{\ell=1}^{k-1} h_\ell(x_\ell)\left(\frac{1}{h_k(x_k)}\frac{\partial\chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial\chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}\right)\Delta_{k,\ell}(\mathbf{x})\right]. \quad \text{(G.16)}$$

The first term in round brackets is positive due to sufficient condition (G.1). We now work with the summation and show that it is also positive. Before doing this, observe that, by definition, for every $\ell \in \{1, \ldots, k-1\}$

$$\chi_\ell^k(\mathbf{x}^{k+1}) = \chi_\ell^\ell(\chi_{\ell+1}^k(\mathbf{x}^{k+1}), \chi_{\ell+2}^k(\mathbf{x}^{k+1}), \ldots, \chi_k^k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}).$$

Then, for any $j \in \{k+1, \ldots, m\}$

$$\frac{\partial\chi_\ell^k(\mathbf{x}^{k+1})}{\partial x_j} = \frac{\partial\chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_j} + \sum_{q=\ell+1}^{k}\frac{\partial\chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_q}\frac{\partial\chi_q^k(\mathbf{x}^{k+1})}{\partial x_j}. \quad \text{(G.17)}$$

For a given $\ell$ in the summation in equation (G.16), we use equation (G.17) to write the round bracket as

$$\left(\frac{1}{h_k(x_k)}\frac{\partial\chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_k}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial\chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_j}\right) + \sum_{q=\ell+1}^{k-1}\frac{\partial\chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_q}\left(\frac{1}{h_k(x_k)}\frac{\partial\chi_q^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial\chi_q^{k-1}(\mathbf{x}^k)}{\partial x_j}\right). \quad \text{(G.18)}$$

Substitute equation (G.18) when $\ell = 1$ into the summation in equation (G.16) to obtain

$$\sum_{\ell=2}^{k-1}\left(h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x}) + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_2}a_1\right)\left(\frac{1}{h_k(x_k)}\frac{\partial\chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial\chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}\right) + a_1\left(\frac{1}{h_k(x_k)}\frac{\partial\chi_1^1(\mathbf{x}^k)}{\partial x_k}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial\chi_1^1(\mathbf{x}^k)}{\partial x_j}\right), \quad \text{(G.19)}$$

where $a_1 = \Delta_{k,1}(\mathbf{x})h_1(x_1) > 0$. Then, substituting (in increasing order) into equation (G.19) the expression (G.18) for $\ell = 2$, $\ell = 3$ until $\ell = k-1$, we obtain that the summation in equation (G.16) is equal to

$$\sum_{\ell=1}^{k-1} a_\ell\left(\frac{1}{h_k(x_k)}\frac{\partial\chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_k}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial\chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_j}\right) > 0, \quad \text{(G.20)}$$

where

$$a_\ell = h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x}) + \sum_{p=1}^{\ell-1} \frac{\partial \chi_p^p(\mathbf{x}^{p+1})}{\partial x_\ell} a_p \tag{G.21}$$

is defined recursively. The parenthesis in equation (G.20) is positive by equation (G.12). We show that each $a_\ell$ is positive, which proves the lower bound in equation (G.14) when $p = k$. By induction, suppose that for every $p \in \{1, \ldots, \ell-1\}$ we have shown that $0 < h_p(x_p)\Delta_{k,p}(\mathbf{x}) \leq a_p$ (we already showed this for $a_1$). We need to show that the same inequalities hold for equation (G.21). First, because $\partial \chi_p^p(\mathbf{x}^{p+1})/\partial x_\ell < 0$ and $a_p > 0$ (by the induction hypothesis) it is easy to see that $a_\ell < h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x})$. Using the lower bound in equation (G.11) and the upper bound for $a_p$ we obtain the following lower bound for equation (G.21)

$$a_\ell > h_\ell(x_\ell)\left[\Delta_{k,\ell}(\mathbf{x}) - \sum_{p=1}^{\ell-1} \frac{\Delta_{k,p}(\mathbf{x})}{n-1}\right] > h_\ell(x_\ell)\left[1 - \frac{(\ell-1)}{n-1}\right]\Delta_{k,\ell}(\mathbf{x}) > 0,$$

where the second inequality follows from Lemma H.3; which proves the result.

To prove the upper bound in equation (G.14) for $p \in \{1, \ldots, k-1\}$ we proceed by induction downwards. Suppose that for every firm $\ell \in \{p+1, \ldots, k\}$ we have proven

$$0 > \frac{\partial \chi_\ell^k(\mathbf{x}^{k+1})}{\partial x_j} > -\frac{h_j(x_j)}{h_\ell(x_\ell)} \frac{1}{n-1} \tag{G.22}$$

we prove that equation (G.14) holds for $p$. Observing that, in equation (G.17), $\partial \chi_p^p(\mathbf{x}^{p+1})/\partial x_\ell < 0$, we can construct an upper bound for $\partial \chi_p^k(\mathbf{x}^{k+1})/\partial x_j$ using the induction hypothesis (G.22)

$$\frac{\partial \chi_p^k(\mathbf{x}^{k+1})}{\partial x_j} < \frac{\partial \chi_p^p(\mathbf{x}^{p+1})}{\partial x_j} - \sum_{\ell=p+1}^{k} \frac{\partial \chi_p^p(\mathbf{x}^{p+1})}{\partial x_\ell} \frac{h_j(x_j)}{h_\ell(x_\ell)} \frac{1}{n-1}$$

Using equation (G.10), the upper bound for $\partial \chi_p^k(\mathbf{x}^{k+1})/\partial x_j$ is equal to

$$\frac{h_j(x_j)}{D_p} \sum_{\ell=p+1}^{k} \left( h_\ell(x_\ell)\Delta_{p,\ell}(\mathbf{x}) + \sum_{q=1}^{p-1} \frac{\partial \chi_q^{p-1}(\mathbf{x}^p)}{\partial x_\ell} h_q(x_q)\Delta_{p,q}(\mathbf{x}) \right) \frac{1}{h_\ell(x_\ell)} \frac{1}{n-1}$$
$$- \frac{h_j(x_j)}{D_p} \left( \Delta_{p,j}(\mathbf{x}) + \frac{1}{h_j(x_j)} \sum_{q=1}^{p-1} \frac{\partial \chi_q^{p-1}(\mathbf{x}^p)}{\partial x_j} \frac{\Delta_{p,q}(\mathbf{x})}{h_q(x_q)} \right)$$

where $D_p = \Pi_p'(\mathbf{x}) + \sum_{q=1}^{p-1} \frac{d\chi_q^{p-1}(\mathbf{x}^p)}{dx_p} h_q(x_q)\Delta_{p,q}(\mathbf{x}) > 0$. Taking $\partial \chi_q^{p-1}(\mathbf{x}^p)/\partial x_\ell < 0$ equal to zero and $\partial \chi_q^{p-1}(\mathbf{x}^p)/\partial x_j < 0$ to the lower bound in equation (G.11), we build the following upper bound for the previous expression (and omitting $D_p$, as it does not affect the sign)

$$h_j(x_j)\left[\sum_{\ell=p+1}^{k}\frac{\Delta_{p,\ell}(\mathbf{x})}{n-1}+\sum_{q=1}^{p-1}\frac{\Delta_{p,q}(\mathbf{x})}{n-1}-\Delta_{p,j}(\mathbf{x})\right]<h_j(x_j)\left[\frac{k-1}{n-1}-1\right]\Delta_{p,j}(\mathbf{x})$$
$$\leq 0.$$

The inequality follows from equation Lemma H.3; proving $\partial\chi_p^k(\mathbf{x}^{k+1})/\partial x_j<0$. The lower bound for $\partial\chi_p^k(\mathbf{x}^{k+1})/\partial x_j$ follows from equation (G.17) and observing

$$\frac{\partial\chi_p^k(\mathbf{x}^{k+1})}{\partial x_j}>\frac{\partial\chi_p^p(\mathbf{x}^{p+1})}{\partial x_j}>-\frac{h_j(x_j)}{h_p(x_p)}\frac{1}{n-1}$$

where the first inequality follows from $(\partial\chi_p^p(\mathbf{x}^{k+1})/\partial x_\ell)\cdot(\partial\chi_\ell^k(\mathbf{x}^{k+1})/\partial x_j)>0$ for every $\ell$, and the second from the lower bound in equation (G.11).

Finally, we prove equation (G.15) using equation (G.13) to write

$$\frac{1}{h_q(x_q)}\frac{\partial\chi_k^k(\mathbf{x}^{k+1})}{\partial x_q}\frac{1}{n-1}-\frac{1}{h_j(x_j)}\frac{\partial\chi_k^k(\mathbf{x}^{k+1})}{\partial x_j}=\frac{1}{D_k}\Bigg[\Delta_{k,j}(\mathbf{x})-\frac{\Delta_{k,q}(\mathbf{x})}{n-1}+$$
$$\sum_{\ell=1}^{k-1}h_\ell(x_\ell)\left(\frac{1}{h_j(x_j)}\frac{\partial\chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}-\frac{1}{h_q(x_q)}\frac{\partial\chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_q}\frac{1}{n-1}\right)\Delta_{k,\ell}(\mathbf{x})\Bigg],$$

where $D_k=\Pi_k'(\mathbf{x})+\sum_{\ell=1}^{k-1}\frac{\partial\chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x})>0$. We show that a lower bound of this expression is positive. Taking $-\partial\chi_\ell^{k-1}(\mathbf{x}^k)/\partial x_q>0$ to zero and $\partial\chi_\ell^{k-1}(\mathbf{x}^k)/\partial x_j<0$ to the lower bound in equation (G.11), we obtain

$$\frac{1}{D_k}\left[\Delta_{k,j}(\mathbf{x})-\frac{\Delta_{k,q}(\mathbf{x})}{n-1}-\sum_{\ell=1}^{k-1}\frac{\Delta_{k,\ell}(\mathbf{x})}{n-1}\right]>\frac{1}{D}\left[\Delta_{k,j}(\mathbf{x})-\frac{k\Delta_{k,q}(\mathbf{x})}{n-1}\right]>0.$$

The inequalities follow from Lemma H.3 and the fact that $q\in\{k,\dots,j-1\}$. □

Because at each step best responses are unique and at $k=n$ the firm has only one best response when every firm $k<n$ is best responding to $x_n$ and to each other, there is a unique Herculean equilibrium within the herculean class.

*No non-herculean equilibria exists*: By contradiction. Suppose $\mathbf{x}$ represents a non-herculean equilibrium. Order firms from smallest cutoff $x_1$ to largest, $x_n$. Let $p$ be the first instance (smallest cutoff) that a strength reversal occurs. That is, $x_p<x_{p+1}$ but $s_p>s_{p+1}$. Because every firm $k\in\{1,\dots,p\}$ is ordered by strength, they satisfy conditions (G.13), (G.11), and (G.12). We show that $x_{p+1}$ cannot lie above $x_p$ (i.e, a contradiction). Fix the strategies of all the firms but $p$ and $p+1$, i.e., $\mathbf{x}_{E\setminus\{p,p+1\}}$, and let $\hat{x}$ be the value that satisfies $\chi_p(\hat{x},\mathbf{x}_{E\setminus\{p,p+1\}})=\hat{x}$, where $\chi_p(\mathbf{x}_{-p})$ is firm's $p$ unique best response to $\mathbf{x}_{-p}$. This best response exists (and is unique) by Lemma H.1. The value $\hat{x}$ exists because $\chi_p(\mathbf{x}_{-p})$ is continuously decreasing in $x_{p+1}$. This implies that, for every $x_{p+1}>\hat{x}$, $\chi_p(\mathbf{x}_{-p})<x_{p+1}$. In addition,

following analogous steps to those in Claim 11, we can show that $\partial\chi_p(\mathbf{x}_{-p})/\partial x_{p+1} > -h_{p+1}(x_{p+1})/(h_p(x_p)(n-1))$. Then, by Lemma H.2, $\Pi_p(\hat{x},\hat{x},\mathbf{x}_{E\setminus\{p,p+1\}}) = 0 < \Pi_{p+1}(\hat{x},\hat{x},\mathbf{x}_{E\setminus\{p,p+1\}})$. Also, letting $\hat{\mathbf{x}} = (\chi_p(\mathbf{x}_{-p}),\mathbf{x}_{-p})$ observe that

$$\begin{aligned}\frac{d\Pi_{p+1}(\hat{\mathbf{x}})}{dx_{p+1}} &= \Pi'_{p+1}(\hat{\mathbf{x}}) + \frac{\partial\chi_p(\mathbf{x}_{-p})}{\partial x_{p+1}}\frac{\partial\Pi_{p+1}(\hat{\mathbf{x}})}{\partial x_p} \\ &> \Pi'_{p+1}(\hat{\mathbf{x}}) - h_{p+1}(x_{p+1})\frac{\Delta_{p+1,p}(\hat{\mathbf{x}})}{n-1} > 0\end{aligned}$$

Thus, $\Pi_{p+1}(\hat{\mathbf{x}})$ is strictly increasing in $x_{p+1}$ which implies that $\Pi_{p+1}(\hat{\mathbf{x}}) > 0$ for every $x_{p+1} \geq \hat{x}$, which implies that no equilibrium cutoff $x_{p+1} > \chi_p(\mathbf{x}_{-p}) = x_p$ exists. ■

# H Auxiliary Results

**Lemma H.1.** *Let $\Pi_i$ be defined by (6). Let $A$ and $B$ be disjoint sets of $k$ and $r$ firms, where $k+r<n$, such that $i \in A$. Define $f : [a,b]^{k+r} \to [a,b]^{n-k-r}$ to be a continuous function and let $\mathbf{x}_B$ be any vector of cutoff strategies for firms in set $B$. Then, there exist a value $\tilde{x}$ such that the symmetric $k$-dimensional vector $\tilde{\mathbf{x}}_A = (\tilde{x})_{i\in A}$ satisfies $\Pi_i(\tilde{\mathbf{x}}_A, f(\tilde{\mathbf{x}}_A,\mathbf{x}_B),\mathbf{x}_B) = 0$. The vector $\tilde{\mathbf{x}}_A$ is continuous in each dimension of $\mathbf{x}_B$. When the function $f$ is constant in $\tilde{x}$—i.e., when $\mathbf{x}_{E\setminus A} = (f(\tilde{\mathbf{x}}_A,\mathbf{x}_B),\mathbf{x}_B)$ does not change with $\tilde{\mathbf{x}}_A$—the value of $\tilde{x}$ is unique.*

**Proof.** Fix $\mathbf{x}_B$, because $f$ is continuous, the function $\Pi_i(\mathbf{x}_A, f(\mathbf{x}_A,\mathbf{x}_B),\mathbf{x}_B)$ is continuous in the input value $x$ of the symmetric vector $\mathbf{x}_A$. Let $\underline{\mathbf{v}}_A = (\underline{v}_i)_{i\in A}$ and $\overline{\mathbf{v}}_A = (\overline{v}_i)_{i\in A}$. Observe that assumptions A3 and A2 jointly imply

$$\Pi_i(\underline{\mathbf{v}}_A, f(\underline{\mathbf{v}}_A,\mathbf{x}_B),\mathbf{x}_B) \leq \pi_i(\underline{v}_i) < 0.$$

Similarly, assumption A3 and Lemma B.2 together imply,

$$\Pi_i(\overline{\mathbf{v}}_A, f(\overline{\mathbf{v}}_A,\mathbf{x}_B),\mathbf{x}_B) \geq \Pi_i(\overline{v}_i, a_{-i}) > 0.$$

Then, by the intermediate value theorem, there exist $\tilde{x} \in (\underline{v}_i, \bar{v}_i)$ such that

$$\Pi_i(\tilde{\mathbf{x}}_A, f(\tilde{\mathbf{x}}_A,\mathbf{x}_B),\mathbf{x}_B) = 0.$$

Because the functions $\Pi_i$ and $f$ are continuous, the value $\tilde{\mathbf{x}}_A$ is continuous in each dimension of $\mathbf{x}_B$. For uniqueness when $f$ is constant, by the chain rule, $d\Pi_i/dx = \sum_{k\in A}\partial\Pi_i/\partial x_k > 0$ where the inequality follows from Lemma B.2. Therefore $\Pi_i(\mathbf{x}_A, f(\mathbf{x}_A,\mathbf{x}_B),\mathbf{x}_B)$, as a function of the value $x$ for the symmetric vector $\mathbf{x}_A$, is increasing and crosses zero once. ■

**Lemma H.2.** *Consider an ordered game, in which the firms' identities are ordered by strength, with firm 1 being the strongest. Then, for any firm $i<j$, valuation $y$,*

*and vector of strategies for the other firms $\mathbf{x}_{E\setminus\{i,j\}}$, we have*

$$\Pi_i(y; y, \mathbf{x}_{E\setminus\{i,j\}}) > \Pi_j(y; y, \mathbf{x}_{E\setminus\{i,j\}}).$$

**Proof.** If firms are ordered by profit, the inequality follows by definition. Recall $\phi(v_e) = \prod_{j\in e} f_j(v_j)$. For games ordered by distribution, observe

$$\begin{aligned}
\Pi_i(y; y, \mathbf{x}_{E\setminus\{i,j\}}) = &\sum_{e\in\mathcal{E}_i\setminus\mathcal{E}_j} \left\{ \left( F_j(y) \prod_{k\in e^c\setminus j} F_k(x_k) \right) \int_{(x_k)_{k\in e\setminus i}}^{b} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} + \\
&\sum_{e\in\mathcal{E}_i\cap\mathcal{E}_j} \left\{ \left( \prod_{k\in e^c} F_k(x_k) \right) \int_y^b \int_{(x_k)_{k\in e\setminus\{i,j\}}}^{b} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus\{i,j\}}) f_j(v) d^{n_e-1} v_{e\setminus i} \right\} \\
&> \sum_{e\in\mathcal{E}_i\setminus\mathcal{E}_j} \left\{ \left( F_i(y) \prod_{k\in e^c\setminus j} F_k(x_k) \right) \int_{(x_k)_{k\in e\setminus i}}^{b} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} + \\
&\sum_{e\in\mathcal{E}_i\cap\mathcal{E}_j} \left\{ \left( \prod_{k\in e^c} F_k(x_k) \right) \int_y^b \int_{(x_k)_{k\in e\setminus\{i,j\}}}^{b} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus\{i,j\}}) f_i(v) d^{n_e-1} v_{e\setminus i} \right\} \\
&= \Pi_j(y, y, x_{E\setminus\{i,j\}}),
\end{aligned}$$

where the inequality uses two properties of FOSD. The first term uses that $F_i(x) \leq F_j(x)$ for all $x$. The second term uses that $\int_y^b \varphi(x) f_i(x) dx \leq \int_y^b \varphi(x) f_j(x) dx$ for any non-increasing function $\varphi(x)$. ■

**Lemma H.3.** *Let firm $k$ be stronger than firm $j$. Suppose the firms play cutoffs $x_k < x_j$; then, for any firm $i$, $\Delta_{i,j}(\mathbf{x}) \geq \Delta_{i,k}(\mathbf{x})$ if: (i) firms are ordered by profits, or; (ii) firms are ordered by distribution and the profit gain only depends on the number of entrants.*

**Proof.** Start by observing that, in the expression for $\Delta_{i,j}(\mathbf{x})$ (see equation (7)), the sum over market structures $\mathcal{E}_i \setminus \mathcal{E}_j$ can be divided into two disjoints sets: $(\mathcal{E}_i \cap \mathcal{E}_k) \setminus \mathcal{E}_j$ and $\mathcal{E}_i \setminus (\mathcal{E}_j \cup \mathcal{E}_k)$. Using these sets subtract $\Delta_{i,j}(\mathbf{x}) - \Delta_{i,k}(\mathbf{x})$ to obtain

$$\begin{aligned}
&\sum_{e\in\mathcal{E}_i\setminus(\mathcal{E}_j\cup\mathcal{E}_k)} \left\{ \left( \prod_{\ell\in e^c} F_\ell(x_\ell) \right) \int_{(x_\ell)_{\ell\in e\setminus i}}^{b} \left(\delta_i(x_i, x_j, v_{e\setminus i}) - \delta_i(x_i, x_k, v_{e\setminus i})\right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} \\
&\quad + \sum_{e\in(\mathcal{E}_i\cap\mathcal{E}_k)\setminus\mathcal{E}_j} \left\{ \left( \prod_{\ell\in e^c\setminus j} F_\ell(x_\ell) \right) F_j(x_j) \int_{x_k}^b \int_{(x_\ell)_{\ell\in e\setminus\{i,k\}}}^{b} \delta_i(x_i, x_j, v_{e\setminus i}) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} \\
&\quad - \sum_{e\in(\mathcal{E}_i\cap\mathcal{E}_j)\setminus\mathcal{E}_k} \left\{ \left( \prod_{\ell\in e^c\setminus k} F_\ell(x_\ell) \right) F_k(x_k) \int_{x_j}^b \int_{(x_\ell)_{\ell\in e\setminus\{i,j\}}}^{b} \delta_i(x_i, x_k, v_{e\setminus i}) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} \qquad \text{(H.1)}
\end{aligned}$$

where we used $\mathcal{E}_i \setminus (\mathcal{E}_j \cup \mathcal{E}_k) = \mathcal{E}_i \setminus (\mathcal{E}_k \cup \mathcal{E}_j)$ and, by profits being anonymous, we dropped the second sub index from the profit gain $\delta_i(x_{\{i,j\}}, v_{e\setminus i})$. Equation (H.1) has three summations. For the first one, observe that the term inside the integral is non-negative as

$$\delta_i(x_i, x_j, v_{e\setminus i}) - \delta_i(x_i, x_k, v_{e\setminus i}) = \pi_i(x_i, x_k, v_{e\setminus i}) - \pi_i(x_i, x_j, v_{e\setminus i}) \geq 0$$

where the last inequality follows from assumption A2 and $x_k < x_j$. This implies that the first summation is non-negative.

For the last two summations in (H.1), we show that a lower bound of the first term is equal to the subtracting term. Thus, the difference is non-negative. Observe that, for each market structure $e \in (\mathcal{E}_i \cap \mathcal{E}_k) \setminus \mathcal{E}_j$ in the first term, we can remove firm $k$ and add firm $j$, i.e., $\hat{e} = (e \setminus j) \cup k$, and the new market structure satisfies $\hat{e} \in (\mathcal{E}_i \cap \mathcal{E}_j) \setminus \mathcal{E}_k$, which belongs to the second term. We show that a lower bound of payoffs in $e$ is equal to those in $\hat{e}$.

*(i) Ordered by profit*: When the game is ordered by profit, we can drop the sub-index from the distributions of types. Bounding the expression under market structure $e \in (\mathcal{E}_i \cap \mathcal{E}_k) \setminus \mathcal{E}_j$

$$\left(\prod_{\ell \in e^c \setminus j} F(x_\ell)\right) F(x_j) \int_{x_k}^{b} \int_{(x_\ell)_{\ell \in e\setminus\{i,k\}}}^{b} \delta_i(x_i, x_j, v_{e\setminus i}) \phi(v_{e\setminus\{i,k\}}) f(v_k) d^{n_e - 1} v_{e\setminus i}$$
$$> \left(\prod_{\ell \in e^c \setminus k} F(x_\ell)\right) F(x_k) \int_{x_j}^{b} \int_{(x_\ell)_{\ell \in e\setminus\{i,j\}}}^{b} \delta_i(x_i, x_k, v_{e\setminus i}) \phi(v_{e\setminus\{i,k\}}) f(v_j) d^{n_e - 1} v_{e\setminus i}$$

where in the inequality we used $x_j > x_k$ in three places: (i) in the probability of firm $j$ being out of the market; (ii) in the domain of integration over $k$'s types, which jointly with $\delta_i(x_i, x_j, v_{e\setminus i}) \geq 0$ implies that we are integrating over a smaller domain, decreasing the value of the integral, and; (iii) $\delta_i(x_i, s, v_{e\setminus i})$ being increasing in $s$ (by assumption A2). Finally, we inverted the roles of firm $k$ and $j$ in $e$ using that payoffs are anonymous to re-arrange indexes. Thus, we obtain that the lower bound equals the payoffs in the third summation of (H.1) under market structure $\hat{e} \in (\mathcal{E}_i \cap \mathcal{E}_j) \setminus \mathcal{E}_k$. Because the inequality holds for every market structure $e \in (\mathcal{E}_i \cap \mathcal{E}_k) \setminus \mathcal{E}_j$, the result follows.

*(ii) Ordered by distribution*: When the game is ordered by distributions and the profit gain only depends on the number of entrants, the expression under market structure $e \in (\mathcal{E}_i \cap \mathcal{E}_k) \setminus \mathcal{E}_j$ becomes

$$\left(\prod_{\ell \in e^c \setminus j} F_\ell(x_\ell)\right) F_j(x_j) \left((1 - F_k(x_k)) \prod_{\ell \in e\setminus\{i,k\}} (1 - F_\ell(x_\ell))\right) \delta_i(x_i, n_e)$$
$$> \left(\prod_{\ell \in e^c \setminus k} F_\ell(x_\ell)\right) F_k(x_k) \left((1 - F_j(x_j)) \prod_{\ell \in e\setminus\{i,j\}} (1 - F_\ell(x_\ell))\right) \delta_i(x_i, n_e).$$

The inequality uses stochastic dominance, the fact that $x_k < x_j$ (so that $F_j(x_j) \geq F_j(x_k) \geq F_k(x_k)$), and re-arranges indexes. The lower bound equals the payoffs in the third summation of (H.1) under market structure $\hat{e} \in (\mathcal{E}_i \cap \mathcal{E}_j) \setminus \mathcal{E}_k$. The inequality holds for every market structure $e \in (\mathcal{E}_i \cap \mathcal{E}_k) \setminus \mathcal{E}_j$, proving the result. ■