Second-Price Auctions with Participation Costs*

José-Antonio Espín-Sánchez† Álvaro Parra‡ Yuzhou Wang§

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Abstract

We study equilibria and efficiency in second-price auctions with public participation costs. We generalize previous results by allowing arbitrary heterogeneity in the bidders' distribution of valuations and in their participation costs. We develop the notion of bidder *strength*, and show that a *herculean* equilibrium in which *stronger* bidders have a lower participation threshold than weaker bidders exists in general environments. We provide a sufficient condition for equilibrium uniqueness. Even though all equilibria are *ex-post* inefficient, an *ex-ante* efficient equilibrium always exists. Therefore, under the uniqueness condition, the herculean equilibrium is the unique equilibrium of the game and is *ex-ante* efficient.

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†Department of Economics, Yale University, jose-antonio.espin-sanchez@yale.edu.

‡Sauder School of Business, University of British Columbia, alvaro.parra@sauder.ubc.ca.

§Department of Economics, Yale University, yuzhou.wang@yale.edu.

1 Introduction

In this article, we study participation in a second-price auction with independent private values and public participation costs. Our framework expands existing models by accommodating rich forms of bidder heterogeneity, which enables a wide range of empirical applications facilitating policy analysis of auction markets. Our main contributions are a general characterization of the game's equilibria and identification of sufficient conditions that guarantee both equilibrium uniqueness and efficient outcomes.

We develop a notion of relative competitiveness called *strength* to characterize the auction's set of equilibria. Using the publicly known characteristics of bidders, strength ranks potential bidders by their ability to endure competition. The strength of a bidder is computed by the unique symmetric strategy profile in which the bidder is best-responding; that is, strength is the hypothetical participation-cutoff value that would make a bidder indifferent to participate, conditional on all other bidders using the same strategy as the bidder. Although this strategy is generally not an equilibrium, it neatly captures a bidder's ability to endure competition. Relative to another bidder's strength, a lower strength value—i.e., a stronger bidder—indicates that the bidder is willing to participate in the auction at a lower valuation even while facing competitors who are also participating at lower valuations; i.e., the bidder is more willing to participate despite facing more competition.

We define a *herculean* equilibrium as an equilibrium in which stronger bidders are more willing to participate—i.e., the bidders' participation strategies are ordered by their relative strength. We show that, in general environments, a herculean equilibrium always exists. Despite the *ex-post* inefficiencies that are created by costly participation, we also show that an *ex-ante* efficient equilibrium exists. In addition, we show that when the distributions of valuations are concave, the game has a unique equilibrium. We also provide a sufficient condition for uniqueness when the distributions of valuations are not concave. Therefore, when any of these sufficiency conditions hold, the unique equilibrium is both herculean and efficient.

A bidder's strength can be computed in *any* game, allowing to rank every potential auction participant. Strength allows to generalize existing work on *quasi-symmetric* games to environments without restrictions on the distributions of valuations or on participation costs. Quasi-symmetric games are those in which: (i) bidders have identical participation costs and their valuations are ordered by first-order stochastic dominance (FOSD; Tan and Yilankaya, 2006); or (ii) bidders with symmetric distributions of valuations, ordered by their participation costs (Cao and Tian, 2013). Strength is specially suited for applied research, as it allows us to rank bidders, and consequently, characterize equilibria in games with *any* degree of bidder heterogeneity.

To illustrate the previous point, consider the US Forest Service timber auctions studied in Roberts and Sweeting (2013). They assume a quasi-symmetric environment in which large bidders' (mills) valuations FOSD those of small bidders (loggers) but have identical participation costs. Suppose that, after the model has been estimated, we want to evaluate a policy that recommends subsidizing the participation cost of loggers.[1] This counterfactual scenario is no longer quasi-symmetric, as the millers' advantage in drawing higher valuations might be offset by the loggers' lower cost of participating. Whether millers are still *stronger* than loggers depends on the size of the subsidy. Since bidders are not quasi-symmetric, existing models cannot predict which firms are more likely to participate, whether the game has a unique equilibrium and, consequently, whether counterfactual analyses are robust to the existence of other equilibria. This article characterizes which firms are more likely to participate in the new scenario and provides a sufficient condition for equilibrium uniqueness that can be easily checked.

This article contributes to the literature of auctions with participation costs. In this literature, there are two broad classes of models that describe bidders' own information about their valuations. Levin and Smith (1994) study auction participation in environments where participation decisions are made with no private information (see also McAfee and McMillan, 1987; Tan, 1992; Jehiel and Lamy, 2015). In this framework, participation becomes a coordination game, and generally leads to multiple equilibria. When signals are informative but public—i.e., observed by all bidders—environments also resemble coordination games, as in Levin and Smith (1994). By contrast, our framework builds upon Samuelson (1985), who studied a symmetric environment in which bidders learn their private information *prior* to the participation decision. Within this framework, Campbell (1998) studies coordinated entry, whereas Tan and Yilankaya (2007) examine collusive outcomes and Menezes and Monteiro (2000) study optimal auction design. Recent articles have allowed more general information structures in which bidders

[1] As example of this type of policy, Marion (2007) and Krasnokutskaya and Seim (2011) evaluate entry fees and subsidies in the context of first price auctions.

receive (private) *signals* about their valuations before participating in the auction (c.f. Gentry and Li, 2014; Sweeting and Bhattacharya, 2015; Roberts and Sweeting, 2016). We discuss such models in Section 6.

Tan and Yilankaya (2006) and Cao and Tian (2013) identify conditions for a unique equilibrium in the context of quasi-symmetric games. The restricted degree of bidder-heterogeneity in their frameworks, however, constrains bidders' behavior in meaningful ways, making quasi-symmetric environments inadequate for applied work. We show that, in quasi-symmetric games where bidders play an herculean equilibrium, bidders have the same ranking in their probability of participating, equilibrium cutoffs, and expected revenues. Hence, when a unique equilibrium exists, quasi-symmetric environments cannot accommodate a bidder who is more likely to participate, but receives lower expected profits on average than a competitor. In line with Maskin and Riley (2000), we show that high-participation low-profit behavior can emerge in models with richer degrees of bidder heterogeneity. This article, therefore, provides a theoretical framework that better meets applied researchers' needs to accommodate behavior observed in data.

Our welfare analysis expands the early work of Stegeman (1996) (see also Lu, 2009). Although every equilibrium is *ex-post* inefficient, Stegeman (1996) shows that SPA with participation costs have one equilibrium that is *ex-ante* efficient. We provide a direct proof of Stegeman's result. Furthermore, we show that each equilibrium corresponds to a (possibly local) maximum or a saddle point of the social welfare function. Finally, by identifying the equilibrium that survives when the uniqueness condition holds, we partially characterize the efficient equilibria.

Finally, Espín-Sánchez and Parra (2019) generalizes the ideas of strength and herculean equilibrium developed here to characterize entry into oligopolistic markets. Despite the similarities in goals, there are key differences in terms of methodologies and results that make the contributions of this article distinctive. This article crucially relies on the linear-payoff structure of second-price auctions. In particular, welfare results, the sharper sufficient conditions for uniqueness, and more importantly the induction argument used in the characterization of the n-bidder scenario do not extend to their environment. In contrast, the techniques in Espín-Sánchez and Parra (2019) rely on a post-entry strict payoff-monotonicity assumption that is not satisfied in second-price auctions.

The rest of the article is organized as follows. Section 2 presents the model. Section 3 characterizes all equilibria, establishes existence and discusses efficiency. Section 4 defines strength, herculean equilibria and presents the main results of the article. Section 5 discusses the importance of allowing models with more heterogeneity than quasi-symmetry. Section 6 extends the results to environments with a reserve price and environments with partially informed bidders. Section 7 concludes. All proofs are relegated to the Appendix.

2 Setup

Consider a sealed-bid second-price auction with no reservation price in an independent private values environment.[2] The auction consists of one seller, n potential bidders, and one indivisible good. Before making any participation decision, each bidder i observes her valuation for the object v_i which is drawn from an atomless distribution function F_i with full support on $\mathbb{R}_+$. We assume that each F_i is continuously differentiable and has a finite expectation.[3] Upon privately observing their own valuation, each bidder, independently and simultaneously, decides whether to participate in the auction. If bidder i decides to participate, she incurs a cost $c_i > 0$. The tuple $(F_i, c_i)_{i=1}^n$, which includes the number of potential bidders n, is commonly known by all the bidders.

Definition (Symmetric and quasi-symmetric games)**.** A game is called *symmetric* if $F_i = F$ and $c_i = c$ for all i. A game is called quasi-symmetric if either: (i) $F_i = F$ for all i, or (ii) $c_i = c$ for all i and F_i are ordered by first-order stochastic dominance (FOSD).

After bidders make participation decisions, they observe other participating agents' identities. Afterwards, every participant submits their bid simultaneously. We simplify the bidding stage by assuming that each player bids their valuation; i.e., bidders play their weakly dominant strategy.[4] Therefore, we restrict attention to participation strategies. A participation strategy for bidder i is a mapping from bidder i's valuation to a probability of participating in the auction $\tau_i : \mathbb{R}_+ \to [0, 1]$. We assume that bidder i's strategy is an integrable function with respect to her own type v_i. We study the Bayesian Equilibrium of the participation game.

[2]For results in a common value setting see Murto and Välimäki (2015).

[3]Our results would still hold if the support of F_i were the interval $[0, b_i]$ with $b_i > 0$. This, however, would complicate our exposition as we would have to consider corner solutions.

[4]Tan and Yilankaya (2006) model non-participation as submission of a zero bid. Technically, their model is a one-stage game in which a bidder's dominant strategy is not to bid their valuation. By contrast, we explicitly model the sequential bid process. Both formulations are equivalent.

Given a strategy profile $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$, define

$$T_i(v) = F_i(v) + \int_v^\infty (1 - \tau_i(s))\, dF_i(s)$$

to be the *ex-ante* probability that bidder i does not obtain the object when the highest bid among her opponents is v. Observe that $T_i(v) > 0$ whenever $v > 0$. The expected utility of a bidder who participates in the auction with probability $\tau_i(v)$, faces opponents playing τ_{-i}, and values the good by v is:

$$u_i(\tau, v) = \tau_i(v) \left[vG_i(v) - \int_0^v s dG_i(s) - c_i \right], \tag{1}$$

where $G_i(v) = \prod_{k \neq i} T_k(v)$ is the probability that bidder i obtains the object when her valuation is v. In other words, conditional on participating, the expected utility of bidder i is the expected value of getting the good $vG_i(v)$, minus the participation costs c_i, minus the expected price paid, which distributes according to $dG_i(v)$ and is equal to the second highest bid in the auction.

3 Preliminary Results

In this section, we provide a preliminary characterization of the equilibria and efficiency properties of the game. We establish the existence of an equilibrium and we show that, without loss of generality, we can restrict our attention to cutoff strategies. In addition, we prove that, although every equilibrium of the game is *ex-post* inefficient, an *ex-ante* efficient equilibrium always exists.

3.1 Equilibrium Existence

Definition (Cutoff strategy)**.** A strategy $\tau_i(v)$ is called *cutoff* if there exists $x > 0$ such that

$$\tau_i(v) = \begin{cases} 1 & \text{if } v \geq x \\ 0 & \text{if } v < x \end{cases}.$$

A cutoff strategy specifies whether a bidder participates in the auction with certainty depending on her valuation being above some given threshold. Lemma 1 below shows that, without loss of generality, we can restrict our attention to cutoff strategies.

Lemma 1 (Cutoff are best responses)**.** *For each profile of opponent's strategies* τ_{-i}*, bidder* i *has a unique best response. Bidder* i*'s best response is a cutoff strategy given by the unique value of* v *that solves* $u_i(\tau_i = 1, \tau_{-i}, v) = 0$*.*

Lemma 1 follows from showing that, conditional on participation, and regardless of their opponents' strategies, a bidder's (expected) utility is monotonically increasing with respect to their own valuation, v_i. Then, because a bidders' utility is linear in the participation probability, and since they want to participate whenever there is positive expected utility to do so, bidders best respond by playing a cutoff strategy. The cutoff is defined by the valuation that gives zero expected utility for participating in the auction. When a bidder's valuation is equal to its cutoff, the bidder is indifferent to whether or not to participate in the auction. We break this indifference by assuming that bidders participate. The main consequence of Lemma 1 is that each equilibrium, if any exists, must be in cutoff strategies.

From now on, we abuse notation by denoting a cutoff strategy in terms of the cutoff itself. In addition, and without loss of generality, we order the bidders' identities according to their equilibrium cutoffs, with x_1 being the bidder with the lowest cutoff and x_n the bidder with the highest. For a given vector of cutoff strategies $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ define $\mathbf{x}^i = (x_1, x_2, \ldots, x_i)$ to be the vector of cutoffs up to bidder i. Let $A_i^n = \prod_{j>i}^n F_j(x_j)$ be the probability that bidders playing cutoffs *above* x_i do not participate in the auction; let $B_i(v) = \prod_{j<i} F_j(v)$ be the probability that bidders playing cutoffs *below* bidder i obtain valuations lower than v, and; let

$$r_i(\mathbf{x}^i) = x_i B_i(x_i) - \sum_{j=1}^{i-1} \left(A_j^{i-1} \int_{x_j}^{x_{j+1}} s dB_{j+1}(s) \right), \tag{2}$$

be bidder i's expected *revenue* when bidder i plays the highest participation cutoff in a game with $n = i$ potential bidders and bidder i's valuation is equal to its cutoff.[5] The next lemma characterizes every equilibria in the participation game.

Lemma 2 (Cutoff Equilibrium)**.** *The vector* $\mathbf{x}$ *of cutoff strategies constitutes an equilibrium if and only if the following condition holds for each bidder* i*:*

$$A_i^n r_i(\mathbf{x}^i) = c_i \tag{3}$$

To understand equation (3) recall that, in equilibrium, if a bidder's valuation is equal to its cutoff, they must be indifferent to participating in the auction. For any

[5] The following notation is being used throughout the article: $\sum \emptyset = 0$, $\prod \emptyset = 1$, and $x_0 = 0$.

bidder i, if $v_i = x_i$, participation by any bidder with a higher cutoff would imply losing the object. This event occurs with probability $1-A_i^n$ and leaves bidder i with zero revenue. As a consequence, bidder i only makes revenue with probability A_i^n. In this scenario, bidder i is the participating bidder with the highest participation cutoff and receives revenue $r_i(\mathbf{x}^i)$. The expected revenue of bidder i is the expected revenue conditional on winning times the probability of winning. In equilibrium, when a bidder's valuation is equal to its participation cutoff, the expected revenue from participating is equal to the participation cost. This indifference condition must hold for each bidder.

Lemma 2 characterizes all equilibria of the game but does not provide any information about whether equilibria exist or about which bidder plays which cutoff. Section 4 links bidders' public characteristics to equilibrium cutoffs. The next proposition, which follows from Brouwer's fixed-point theorem, establishes equilibrium existence.

Proposition 1 (Existence)**.** *For any game* $(F_i, c_i)_{i=1}^n$ *there exists an equilibrium.*

The intuition behind this result is simple. Because a bidder's valuation is unbounded above, and because the opponents' valuations have finite expected value, it is possible to find a cutoff for each bidder such that expected payoffs from participating in the auction are positive regardless of the opponents' behavior. With this upper bound, it is possible to compactify the set of feasible strategies. Due to the continuity of the best response functions, the existence result follows from Brouwer's Fixed-point theorem.

3.2 Welfare Analysis

We now discuss efficiency. As Stegeman (1996) pointed out, when participation is costly, *ex-ante* and *ex-post* efficiency are not equivalent. Moreover, when participation is costly, the revelation principle no longer applies because, in the equivalent direct mechanism, each bidder incurs a cost c_i to send a message—i.e., submit a bid—(Myerson, 1981). Thus, as there is no "cost free" way to elicit bidders' preferences. When more messages are solicited, any optimal mechanism trades off the direct cost of *ex-ante* soliciting more messages with the potential benefits from a better *ex-post* allocation. Although such a mechanism may be *ex-ante* optimal, any mechanism that does not solicit messages from all bidders in general produces *ex-post* misallocation with positive probability.

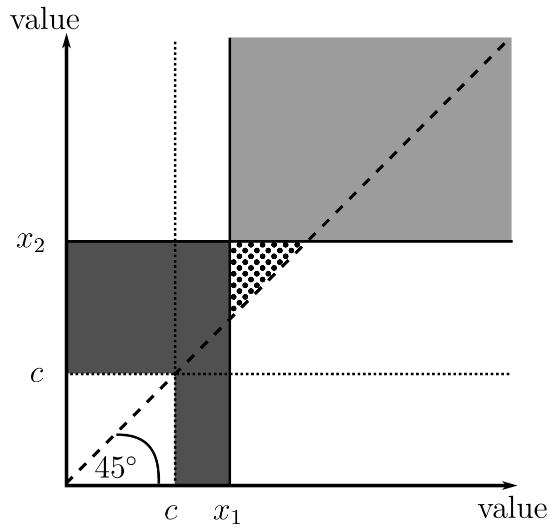


Figure 1: *Ex-post* inefficiency.

To illustrate this point, consider Figure 1, which depicts an equilibrium with two potential bidders, each with equal participation costs ($c_i = c$), but different cutoff equilibrium strategies ($x_1 < x_2$). Note that for an allocation to be *ex-post* efficient, only the bidder with the highest valuation, which must be above the participation cost, should participate in the auction.

In general, three types of inefficiencies arise. (i) *Insufficient Participation* (dark-shaded area): represents realizations of (v_1, v_2) in which there is at least one bidder whose valuation is greater than participation costs, but bidders stay out of the auction. (ii) *Excessive Participation* (lightly-shaded area): represents situations in which both bidders enter the auction, paying excessive participation costs. (iii) *Misallocation* (dotted area): realizations in which exactly one bidder participates, but is the bidder with the lowest valuation for the good. It is worth noticing that, conditional on participation, the bidder with the highest valuation wins the auction independent of the number of participants. Therefore, inefficiencies only arise due to miscoordinated participation.

From an *ex-ante* perspective, however, there is an efficient equilibrium. Consider the problem that a planner faces when choosing a strategy for each bidder conditional on the bidder's private information; i.e., the planner chooses a set of functions $\tau_i^* : \mathbb{R}_+ \to [0,1]$ determining the probability that bidder i participates given her valuation. Using similar arguments to those in Lemma 1 it can be shown that the planner only considers cutoff functions.[6] Therefore, the planner chooses

[6]Notice that in this case, the planner only solicits messages (bids) from bidders whose val-

the vector of cutoffs $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ that maximizes

$$W(\mathbf{x}) = \sum_{i=1}^{n} \left[\int_{x_i}^{\infty} (v_i \Omega_i(v_i, x_{-i}) - c_i) \, dF_i(v_i) \right] \tag{4}$$

where $\Omega_i(v_i, x_{-i}) = \prod_{k \neq i} F_k(\max\{v_i, x_k\})$ is the probability that bidder i obtains the object when her valuation is v_i. Notice that transfers between the winning bidder and the seller are irrelevant in terms of welfare. To explain (4) further, we focus on the planner's payoffs from bidder i. With probability $dF_i(v_i)$, bidder i draws the valuation v_i and participates in the auction whenever $v_i \geq x_i$, in which case she pays the participation cost c_i and wins the object with probability $\Omega_i(v_i, x_{-i})$. Total welfare is simply the aggregation of all possible values for a given bidder (integral), aggregated across all bidders (summation).

Proposition 2 (Welfare)**.** *There exists an equilibrium that is* ex-ante *efficient. Every critical point of the welfare function corresponds to an equilibrium of the game. That is, each equilibrium is either a (possibly local) maximum, or saddle point of* $W(\mathbf{x})$*.*

The intuition behind the proposition is as follows: consider the social contribution of a marginal decrease in bidder i's participation cutoff, x_i. By decreasing their cutoff, bidder i participates on a larger range of values, paying the participation cost c_i more often and, with probability Ω_i, becoming the highest valuation bidder. This latter effect also decreases the opponents' probability of obtaining the good, Ω_j. This decrease in probability occurs when an opponent who was winning the good is outbid by i. In these cases, bidder i's social contribution is the gap between bidder i's valuation and the second highest valuation. In a second-price auction, when the price paid is the valuation of the second-highest bidder, this gap is the same as bidder's i private gain. That is, the social trade-offs faced by the planner match the private trade-offs faced by a bidder. Because in an inflection point, the social (private) gain nets out from the entry cost c_i, every equilibrium matches an inflection point of the social welfare function. Notice, however, this equivalence may be broken if there is a reservation (minimum) bid.

This efficiency result is similar in spirit to Levin and Smith (1994), which shows that every participation equilibrium is *ex-ante* efficient when bidders are symmetric and have no private information at the moment of participation. Their findings do

uations are above the specified cutoff. That is, the planner does not solicit messages from all bidders with certainty.

not extend directly when private information exists. Privately informed bidders self-select according to their own characteristics, information, and expectations of other bidders' behavior. Different expectations may lead bidders to coordinate in inefficient equilibria, even if bidders are *ex-ante* symmetric.

To further illustrate the relation between uniqueness and efficiency, and to motivate the analysis that follows, we consider the case of $n = 2$ potential bidders. The Hessian of the planner's problem, evaluated at a critical point, is equal to:

$$H(\mathbf{x}) = -\begin{pmatrix} f_1(x_1)F_2(x_2) & x_1 f_1(x_1) f_2(x_2) \\ x_1 f_1(x_1) f_2(x_2) & f_2(x_2) F_1(x_2) \end{pmatrix}.$$

Observe that, under concavity of F_i, the second order condition for a maximum is satisfied at *every* critical point.[7] Therefore, only one critical point exists and the game has a unique, efficient equilibrium. This finding suggests that some form of concavity of the CDF may be sufficient to guarantee both uniqueness and efficient outcomes. As we show below, this intuition extends to a large set of models that are relevant for applied analysis.

4 Strength and Herculean Equilibrium

In this section, we present the main results of the article. In particular, we connect the bidders' public characteristics $(F_i, c_i)_{i=1}^n$ to the game's equilibrium strategies. The definition below, which only uses the information given in the game fundamentals, ranks bidders in terms of their ability to endure competition. We use this notion to further describe bidders' participation strategies.

Definition (Strength)**.** For a given game $(F_i, c_i)_{i=1}^n$, the strength of bidder i is the unique number $s_i \in \mathbb{R}_+$ that solves:

$$s_i \prod_{k \neq i} F_k(s_i) = c_i. \tag{5}$$

We say that bidder i is stronger than j if $s_i < s_j$.

[7]Concavity of the CDF implies $F_i(x) \geq x f_i(x)$ for every $x > 0$. Then, at every equilibrium $x_1 < x_2$, the first minor of $H(\mathbf{x})$ is always negative and

$$\det(H(\mathbf{x})) = f_1(x_1) f_2(x_2) \left(F_1(x_1)F_2(x_2) - (x_1)^2 f_1(x_1) f_2(x_2)\right) > 0.$$

Observe that the left hand side of (5) is strictly increasing in s_i, takes the value of 0 when $s_i = 0$, and is unbounded above. Therefore, strength is well defined. Each bidder i has a unique scalar s_i. Thus, we can always use strength to rank all bidders in the game. Notice that the index s_i is inversely related to strength of a bidder. A lower s_i means a stronger bidder.

Strength uses all public information from the game to elicit bidders' ability to endure competition. The strength of bidder i is defined as the cuttoff that bidder i plays in the unique symmetric strategy profile in which bidder i is best-responding (see equation (3) for the case of symmetric cutoffs). By computing this (symmetric) strategy in the context of asymmetric bidders, we can measure a bidder's willingness to participate in the auction relative to their ability to endure competition. To see this, start by noticing that facing a lower participation cutoff from a competitor means that a bidder faces *more* competition. Because of symmetric behavior, a lower measure of strength s_i means that a bidder is willing to participate at a lower valuation, even when her competitors participate more often—i.e., a bidder is more willing to participate despite facing more competition.

In order to further understand strength, the next lemma relates it with the notions of bidder competitiveness developed by the previous literature.

Lemma 3 (Strength in quasi-symmetric games)**.**

1. *If bidders have the same participation costs, and if their distributions of valuations are ordered by FOSD, then bidders who stochastically dominate other bidders are stronger.*
2. *If bidders have the same distributions of valuations but different participation costs, then bidders who have lower participation costs are stronger.*

The order provided by strength coincides with existing notions of relative competitiveness among bidders, such as FOSD or participation-cost order. Strength, however, extends the order to scenarios in which relative competitiveness is not self-evident.[8] Take, for example, a bidder whose distribution of valuations first-order stochastically dominates that of another bidder, but has a higher participation cost. This scenario is likely to arise in practice when the auctioneer subsidizes participation costs of small firms. In this case, although the former bidder may be stronger, as it is likely to draw a higher valuation, it may also be weaker than the

[8] Tan and Yilankaya (2006) calls the order induced by FOSD *intuitive*, whereas Cao and Tian (2013) calls the cost-order *monotone*. Because, in general, the order provided by strength might neither be intuitive nor monotone, we decided to avoid confusion and adopt the current nomenclature.

latter bidder, who also has lower participation costs. Strength not only ranks bidders in this (or any other) scenario but also, as is shown below, provides meaningful information about equilibrium behavior.

Definition (Herculean Equilibrium)**.** An equilibrium is called herculean if the equilibrium cutoffs are ordered by strength, with stronger bidders playing lower cutoffs. That is, $x_i < x_j$ if and only if $s_i < s_j$.

Because stronger bidders are more able to endure competition, they should be more inclined than weaker bidders to participate in the auction. In terms of equilibrium behavior, stronger bidders should play lower participation cutoffs. As we show in the following sections, this intuition is correct: in most applications a herculean equilibrium will exist.

Finally, notice that in symmetric games all bidders are equally strong; thus, in a herculean equilibrium bidders must play symmetric strategies. Furthermore, the strength of each bidder coincides with their symmetric equilibrium cutoff; i.e., $x_i = s_i$. Therefore, in symmetric games the notions of strength, symmetric equilibrium, and herculean equilibrium coincide. Because strength is well defined, this trivially implies that symmetric games have a unique symmetric equilibrium (of course, symmetric games may still have asymmetric equilibria.)

4.1 Herculean Equilibrium under two Bidders

In order to better illustrate our results, we start by presenting them in a two potential-bidder environment. From now on, unless otherwise noted, we order bidders' identities by their strength, with bidder 1 being the strongest bidder in the game. The following proposition is our main result in this context.

Proposition 3 (Existence and uniqueness)**.** *There always exists a herculean equilibrium. Every herculean equilibria is characterized by cutoffs $x_1 \leq x_2$ that jointly solve*

$$x_1 F_2(x_2) = c_1 \quad \textit{and} \quad x_2 F_1(x_2) - \int_{x_1}^{x_2} v dF_1(v) = c_2. \tag{6}$$

Moreover, a herculean equilibrium is the unique equilibrium of the game—and, therefore, efficient—if the following condition holds for each bidder

$$F_i(v) \geq v f_i(v) \quad \textit{for all } v \geq c_j. \tag{7}$$

Proposition 3 generalizes existing results in the literature in two ways. By introducing the notion of strength, we associate bidders' public characteristics with equilibrium-cutoffs order in *any* game $(F_i, c_i)_{i=1}^2$—that is, without limiting our attention to specific distributions of valuations and without restrictions on participation costs. The proposition also confirms the intuition that an equilibrium in which the strong bidder plays a lower participation cutoffs should exists. Perhaps more importantly, the proposition provides a sufficient condition on the shape of the distributions of valuations for the game to have a unique equilibrium. This result is particularly important for applied work, as it provides a testable condition that guarantees robust counterfactual analysis. Furthermore, as a consequence of Proposition 2, the sufficient condition for equilibrium uniqueness also guarantees efficient outcomes.[9]

In intuitive terms, condition (7) shapes the opponent's best-response so that payoffs are monotone in a bidder's strategy. It guarantees that bidder i's expected revenue is increasing in her cutoff x_i, even when bidder j best responds to the increase in x_i by decreasing x_j (increasing competition). This implies that only one cutoff makes bidder i indifferent to participate in the auction, leading to a unique equilibrium. From bidder i's perspective, bidder j best response is a function of i's distribution (see equations in (6)). Because condition (7) regulates best-response behavior, the condition only needs to hold for valuations that are above the opponents' entry costs, as no bidder would participate when her valuation is below her cost. The next lemma would help us to further characterize sufficient condition (7).

Lemma 4. *1) If (F_1, F_2) are concave, then (7) is satisfied and the equilibrium is unique. 2) If the distributions (F_1, F_2) become concave for high valuations, there exists a pair of entry costs (c_1, c_2) such that the game has a unique equilibrium.*[10]

Condition (7) is a weak form of concavity. In particular, auctions with concave distributions of valuations (e.g., Exponential, Generalized Pareto, or the standard Half-Normal distributions) always have a unique equilibrium. Many other distributions, such as Beta, Gamma, or Weibull are concave for certain parameter specifications. Most distributions used in applications are concave for sufficiently high valuations. The Lemma also show that for these eventually-concave distributions

[9]Observe, however, that the proposition does not tell us that a herculean equilibrium is always *ex-ante* efficient. For instance, in symmetric games, when there are multiple equilibria, the symmetric equilibrium need not be the efficient equilibrium.

[10]The proof of the lemma shows how to find the costs that guarantee equilibrium uniqueness.

there are sufficiently high participation costs guaranteeing equilibrium uniqueness. Example 3, below, illustrates this point.

Examples. To illustrate the usefulness of strength and herculean equilibria, and to illustrate the workings of the sufficient condition, we develop three examples. The first two examples make use of a Generalized Pareto distribution (GPD).[11] The choice of GPD yields a simple concave distribution with positive support that is flexible enough to change its mean and variance. Results and intuitions in the examples apply more generally. The third example corresponds to a log-normal distribution which is S-shaped.

1. **Second-order stochastic dominance.** Consider two asymmetric bidders whose distribution of valuations follows a GPD with shape parameter κ and scale parameter σ. Suppose both bidders have a symmetric participation cost c, but bidder 1 is characterized by $(\kappa_1, \sigma_1) = (0, 1)$ and bidder 2 by $(\kappa_2, \sigma_2) = (0.25, 0.75)$. Both distributions have the same mean but the second distribution has twice the variance. That is, the second distribution is a mean-preserving spread of the first. Because the CDFs cross, distributions are *not* ordered by FOSD. This game is not quasi-symmetric and it is not self-evident which bidder is stronger. Consequently, existing tools in the literature cannot characterize equilibrium behavior, nor determine whether the game has a unique equilibrium.

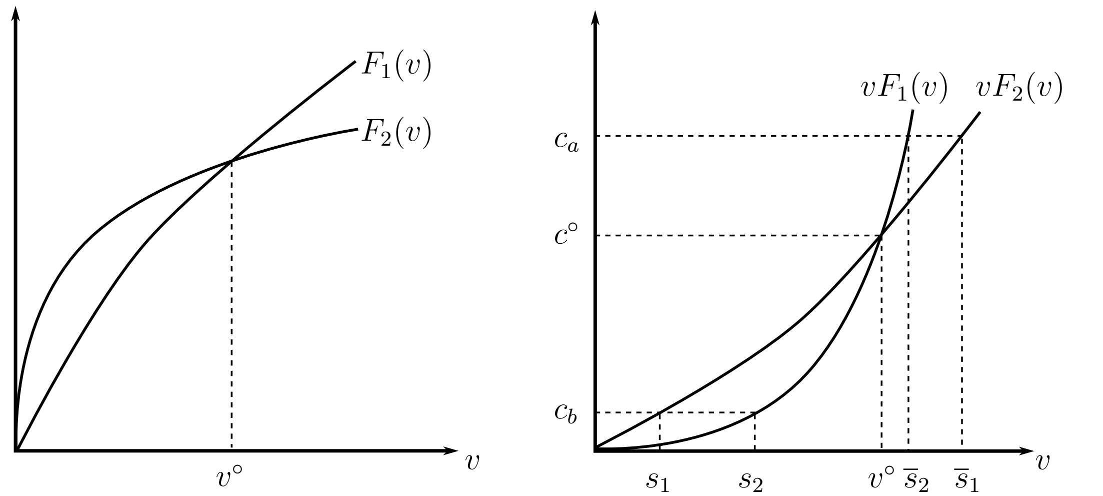
Intuitively, the stronger bidder would be the one whose distribution of valuations has more mass to the right of the equilibrium cutoffs strategies, as this implies the bidder is more likely to obtain higher valuations. If the equilibrium cutoff strategies are high, then bidder 2 would have more mass to the right of the cutoffs, and thus bidder 2 would be the stronger bidder. High equilibrium cutoff strategies are likely to occur when participation costs are high. Conversely, if the cutoff strategies are low, then bidder 1 would have more probability mass to the right of the cutoffs, and thus bidder 1 would be the stronger bidder. Low equilibrium cutoff strategies are likely to occur when participation costs are low.

This situation is illustrated in Figure 2. Panel (a) shows that both distributions are

[11] For $\kappa \in \mathbb{R}$ and $\sigma \in (0, \infty)$, the Generalized Pareto CDF is defined over $\mathbb{R}_+$ and given by

$$F(x|\kappa, \sigma) = \begin{cases} 1 - \left(1 + \frac{\kappa x}{\sigma}\right)^{-\frac{1}{\kappa}} & \kappa \neq 0 \\ 1 - e^{-\frac{x}{\sigma}} & \kappa = 0 \end{cases}.$$

The CDF is concave whenever $\kappa > -1$, its mean is well defined for $\kappa < 1$ and given by $\sigma/(1-\kappa)$, whereas its variance is defined for $\kappa < 1/2$ and given by $\sigma^2/(1-\kappa)^2(1-2\kappa)$.



(a) Distributions are concave

(b) Strength under different participation costs

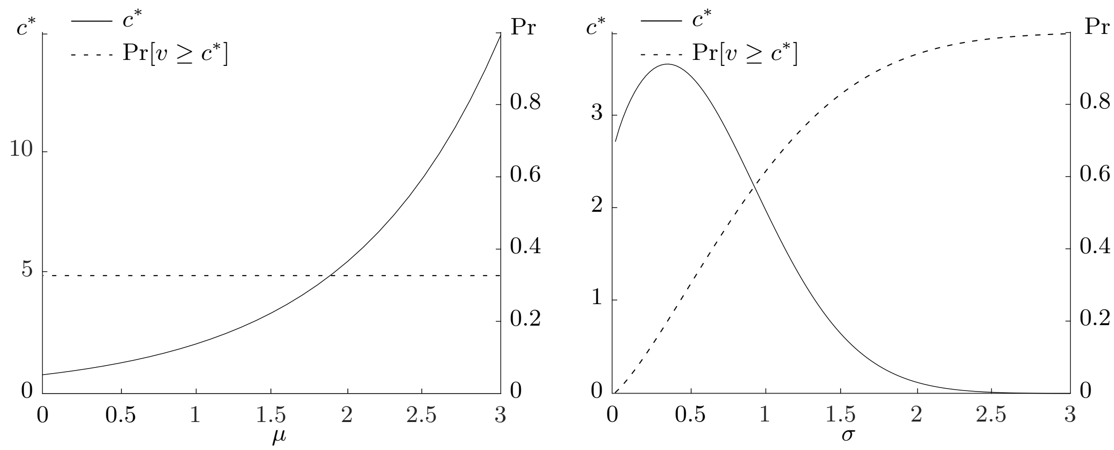
Figure 2: Strength under second-order stochastic dominance.

concave, thus Lemma 4 implies that the participation game has a unique equilibrium for any participation costs $c > 0$. Panel (a) also shows that both distributions cross at $v^\circ = 2.2007$. Panel (b) depicts the bidders' strength. It shows that bidders are equally strong when $c^\circ = 1.957$. For participation costs above c°, bidder 2 is stronger ($s_2 < s_1$) and, in the unique equilibrium, bidder 2 plays a lower cutoff strategy ($x_2 < x_1$). For instance, if $c_a = 2 > c^\circ$, then the vector of equilibrium cutoffs is $\mathbf{x} = (2.241, 2.238)$. Alternatively, when $c < c^\circ$, bidder 1 is stronger ($s_1 < s_2$) and plays a lower equilibrium cutoff strategy ($x_1 < x_2$). For example, if $c_b = 1 < c^\circ$, then the equilibrium is $\mathbf{x} = (1.281, 1.383)$.

The example above illustrates a simple but important point. In games that are not quasi-symmetric, the relative bidders' strength is not self-evident. Two games that only differ in the (symmetric) participation cost can generate different strength rankings and different predictions on which bidder would participate more often.

2. **Subsidized participation.** Suppose two asymmetric bidders whose valuations follow a GPD with shape parameter $\kappa = 0$. Bidder 1 is characterized by $(\sigma_1, c_1) = (1, 1)$ and bidder 2 by $(\sigma_2, c_2) = (2, 2)$. That is, bidder 2's valuation FOSD bidder 1's valuation, but bidder 1 has a lower participation cost. Situations like this may arise in a procurement auction when bidder 2 is a large firm and bidder 1 is a small (local) firm with subsidized participation. As before, the model is not quasi-symmetric and existing results do not apply.

We can use the notion of strength to characterize the equilibria in this game.



(a) c^* as a function of μ when $\sigma = 1$ (b) c^* as a function of σ when $\mu = 1$

Figure 3: Sufficiency with Log-Normal distribution

In this example, bidder 1 is stronger than bidder 2, as $s_1 = 1.73 < 2.24 = s_2$. Consequently, in a herculean equilibrium, bidder 1 plays a lower cutoff. In this case, $\mathbf{x} = (x_1, x_2) = (1.398, 2.511)$. Because both distributions are concave, this is the unique equilibrium of the game. Notice that $x_1 < s_1 < s_2 < x_2$. That is, the equilibrium cutoff strategies are "farther apart" than the strength values. This feature not only holds in this example, but in any herculean equilibrium with two types of bidders. This property can be useful when estimating auction models, as it can reduce the computing power necessary to find the equilibrium cutoffs. In particular, the strength of the weak bidder provides a lower bound for its cutoff and the strength of the strong bidder provides an upper bound.

3. **Uniqueness under a log-normal distribution.** To illustrate the intuition behind the sufficient condition for uniqueness (7), consider the case with two symmetric bidders under Log-normal valuations with parameters (μ, σ).[12] This distribution is not concave, therefore existing results in the literature do not apply. By Lemma 4, however, we can find a participation cost c^* that is sufficiently high, so that the sufficient condition (7) holds.

Figure 3 depicts the threshold c^* and the mass of valuations below c^*, as a function of μ and σ. Panel (a) shows that c^* increases in μ. This means that for distributions with higher medians, the minimum participation cost that guarantees uniqueness is higher. Notice, however, that the proportion of valuations below the entry costs is independent of μ. This observation implies that, under Log-normality,

[12] A Log-normal distribution with parameters (μ, σ) has mean $\exp(\mu + \frac{\sigma^2}{2})$ and median $\exp(\mu)$.

whether a game has a unique equilibrium only depends on the standard error of the distribution and the participation cost. To further understand the previous point, we show the relation between c^* and σ when $\mu = 1$. Panel (b) shows that the relation between c^* and σ is non-monotonic. In particular, c^* is maximal at 3.6493 when $\sigma = .3507$. This implies that any game with $c^* > 3.6493$ has a unique equilibrium, for any value of σ. When $\sigma > .3507$, c^* decreases with σ. Notice that the proportion of valuations below the entry costs is not independent of σ. The larger the variance of the distribution, the less demanding the condition for uniqueness becomes. In contrast, as $\sigma \to 0$, the mass of valuations above c^* converges to zero. That is, as the game converges to a complete information game—where equilibrium multiplicity is known to exists (c.f., Levin and Smith, 1994)—the sufficient condition for uniqueness is never met.

4.2 Herculean equilibrium for two groups of bidders

We now extend our previous results to environments with more than two bidders. Suppose first that there are two groups of bidders, say groups 1 and 2. Each group g consists of m_g bidders characterized by pairs (F_g, c_g), for $g = 1, 2$. Without loss of generality, assume that bidders in group 1 are stronger than those in group 2 ($s_1 \leq s_2$). Although bidders are symmetric within each group, the degree of asymmetry of the distribution of valuation or participation costs *across* groups is unrestricted. The two-group model is especially useful in applied work when bidders are divided by exogenous factors into two groups, such as incumbency (incumbent vs entrant) or size (small vs large). Examples of papers studying participation—not necessarily in the context of second price auctions—in environments with two groups of players include Athey *et al.* (2011), Krasnokutskaya and Seim (2011), Roberts and Sweeting (2013) among many others.

Proposition 4 (Two-groups equilibria)**.** *There always exists a herculean equilibrium. Every herculean equilibrium is characterized by the cutoffs $x_1 \leq x_2$ that jointly solve*

$$x_1 F_1(x_1)^{m_1-1} F_2(x_2)^{m_2} = c_1 \tag{8}$$

$$F_2(x_2)^{m_2-1}\left[x_2 F_1(x_2)^{m_1} - \int_{x_1}^{x_2} v\, d\left(F_1(v)^{m_1}\right)\right] = c_2. \tag{9}$$

Moreover, the herculean equilibrium is the unique equilibrium of the game and, *thus, efficient if for each bidder i*

$$F_i(v) \geq v f_i(v) \text{ for every } v \geq \min\{\mathbf{c}_k\}_{k \neq i}. \tag{10}$$

Proposition 4 generalizes Proposition 3 to the case in which there is more than one bidder in each group of bidders. The main difference between conditions (10) and (7) is that, with more than one opponent, the range of valuations under which the condition has to hold needs to include the participation cost of every opponent, including opponents within the same group.

Condition (10) for bidder i in the two-groups scenario differs from (7) (in the two-bidders scenario) when there is more than one bidder in i's group. In this case, the condition also has to hold for valuations above c_i, which might be lower than the cost of the other group. This is because the uniqueness proof has two steps. The first step shows that symmetric bidders—that is, bidders belonging to the same group—play symmetric strategies in equilibrium, so we can restrict our attention to group-symmetric strategies. This step only involves best-responses of bidders in i's group, making use of condition (10) starting at c_i. The second step shows that among the group-symmetric class of strategies, the only equilibrium is the herculean one. This step makes use of condition (10), for values higher than the other group entry cost. Finally, it is worth noting that Lemma 4 extends to this environment without modification.

4.3 Robust Strength Order among Bidders

The existence of herculean equilibrium in games with three or more groups of bidders is linked to the robustness of the ranking provided by strength. In particular, existence depends on whether the strength order between two bidders depends on the behavior of other bidders.

Consider an scenario with three bidders such that $s_1 < s_2 < s_3$. The construction of strength assumes symmetric behavior among bidders. Suppose instead that bidder 3 is constrained to participate at some given cutoff $\hat{x}_3$. Given this restriction, we can recalculate the strength of bidders 1 and 2 and find a reversal in their strength order. This reversal is associated with non-existence of herculean equilibrium. To illustrate this, suppose we constructed best-response cutoffs for bidders 1 and 2, as a function of bidder 3's cutoff. Because $s_1 < s_2$, the initially constructed cutoffs satisfy $x_1(s_3) < x_2(s_3)$. For different values of x_3, for instance at $x_3 = \hat{x}_3$, the strength order between bidders 1 and 2 reverses, reversing their best responses; i.e., $x_1(\hat{x}_3) > x_2(\hat{x}_3)$. In order to establish our existence and uniqueness results, we need to impose further structure to guarantee the robustness of strength. The next definition and lemma are instrumental to that purpose.

Definition (Cutoff upper bound)**.** Let $\overline{v}_i$ be the unique scalar that solves

$$\overline{v}_i G_i(\overline{v}_i) - \int_0^{\overline{v}_i} s dG_i(s) = c_i \tag{11}$$

where $G_i(v) = \prod_{k \neq i} F_k(\max\{v, c_k\})$ is the probability that bidder i obtains the object when her valuation is v and other bidders participate in the auction for valuations above their participation cost.

The value of $\overline{v}_i$ is well defined.[13] It provides an upper bound to bidder i's set of feasible best responses. The cutoff $\overline{v}_i$ corresponds to bidder i's best response assuming that the other bidders always enter whenever their valuations are above their entry costs; i.e., $\overline{v}_i$ is i's participation cutoffs under the highest level of feasible competition.

Lemma 5 (Robust strength order)**.** *Let $\underline{c} = \min\{c_i\}_{i=1}^n$ and $\overline{v} = \max\{\overline{v}_i\}_{i=1}^n$. Suppose that for any two bidders i and j, with $i < j$, the following condition holds:*

$$F_i(v)c_i \leq F_j(v)c_j \text{ for all } v \in [\underline{c}, \overline{v}]. \tag{12}$$

Then, bidders are ordered by strength with bidder 1 being the strongest bidder.

The set of models satisfying condition (12) includes quasi-symmetric games as particular cases. Condition (12) further extends the existing literature in participation in quasi-symmetric environments in two ways. First, it allows distribution functions that cross and allows for cost orders that do not coincide with distribution orders. Second, the condition does not restrict bidders to belong to one of two groups. In particular, if condition (12) holds with equality for some bidders, the condition allows for an arbitrary number of (strictly ordered) groups of bidders, with each group having an arbitrary number of members.

Example. To illustrate that models satisfying condition (12) might not be quasi-symmetric, consider a scenario in which bidders valuations belong to the Exponentiated distribution family; i.e., $F_i(x) = F(x)^{\theta_i}$ for any F satisfying our assumptions

[13] Is is well defined, as the right hand side of (11) is strictly increasing in $\overline{v}_i$, starts from zero when $\overline{v}_i = 0$, and is unbounded above due to the finite expectation assumption.

and $\theta_i > 0$. Observe that bidder i FOSD j if and only if $\theta_i > \theta_j$.[14] Suppose $\theta_i > \theta_j$, then using $\overline{v}$ we find that every $c_i \leq c_j F(\overline{v})^{\theta_j - \theta_i}$ satisfies condition (12). In particular, the game is not quasi-symmetric whenever $c_i \in (c_j, c_j F(\overline{v})^{\theta_j - \theta_i}]$, as firm i first order stochastically dominates j but has a higher participation cost.

Proposition 5 (n potential bidders)**.** *Under condition* (12)*, a herculean equilibrium always exits. Furthermore, the herculean equilibrium is the unique equilibrium of the game and, therefore, efficient if*

$$F_i(v) \geq v f_i(v) \text{ for all } v \in [\underline{c}, \overline{v}]. \tag{13}$$

Proposition 5 is neither a particular case, nor a generalization of our previous results. On the one hand, the proposition extends existence and uniqueness of herculean equilibrium to the case with n potential bidders. On the other hand, the proposition requires condition (12) to hold whereas previous propositions do not. The proposition generalizes the existence-of-equilibrium result in Miralles (2008), who studied 'intuitive' equilibria in a scenario with n-bidders ordered by FOSD and symmetric participation costs. More importantly, it extends those findings to a larger set of models and show, as in the previous scenarios, that our weak form of concavity is sufficient to guarantee equilibrium uniqueness and efficient outcomes. As before, Lemma 4 applies without modification.

5 On the Importance of Bidder Heterogeneity

In previous sections we emphasized that our results apply to environments that allow for more bidder heterogeneity than quasi-symmetric models. Here, we highlight the importance of allowing this type of heterogeneity. In particular, we show that quasi-symmetric models restrict the relation between observable outcomes and restrict bidder behavior under (exogenous) changes in competition.

Equilibrium, Revenues, and Entry probability In this section, we show that quasi-symmetric models limit the relation between the bidders' behavior—i.e., their participation cutoffs—and observed outcomes, such as the bidders' profitability and participation probability. For ease of exposition, we present the results with two bidders, but they could easily be extended to an arbitrary number of bidders.

[14] This example includes quasi-symmetric environments when two bidders $i < j$ satisfy $c_i < c_j$ but $\theta_i = \theta_j$ or when $c_i = c_j$ but $\theta_i > \theta_j$.

In precise terms, we study the relationship between: (i) the cutoff strategies, x_i; (ii) the *ex-ante* probability of participating in the auction, $1 - F_i(x_i)$; and (iii) the *ex-ante* expected payoff of each bidder; which, for a given vector of cutoffs strategies $\mathbf{x} = (x_1, x_2)$, is equal to:

$$U_i(\mathbf{x}) = \int_{x_i}^{\infty} \left(vF_j(\max\{v, x_j\}) - \int_{x_j}^{\max\{v,x_j\}} sdF_j(s) - c_i \right) dF_i(v). \tag{14}$$

That is, for each valuation v_i under which bidder i participates (i.e., for each $v_i > x_i$), the expected payoff of participating in the auction, weighted by the probability that v_i occurs.

The following proposition characterizes the relationship between these three objects as a function of the game's degree of bidder heterogeneity.

Proposition 6 (Cutoffs and revenue ranking)**.**

1. *In a symmetric game, a bidder playing a lower cutoff obtains higher (expected) payoffs and is more likely to participate.*
2. *In quasi-symmetric games where bidders play herculean equilibria, a bidder playing a lower cutoff obtains higher payoffs and is more likely to participate.*
3. *With general forms of bidder heterogeneity, cutoff, participation-probability, and payoff rankings may not coincide in a herculean equilibria.*

Consider a situation in which the data shows that distributions are concave, so that a unique equilibrium exists, and one bidder participates in an auction more often than another, but overall receives lower expected payoffs. Proposition 6 shows that symmetric and quasi-symmetric models cannot account for this type of behavior. The behavior, however, can be accommodated if more degree of bidder heterogeneity is allowed.

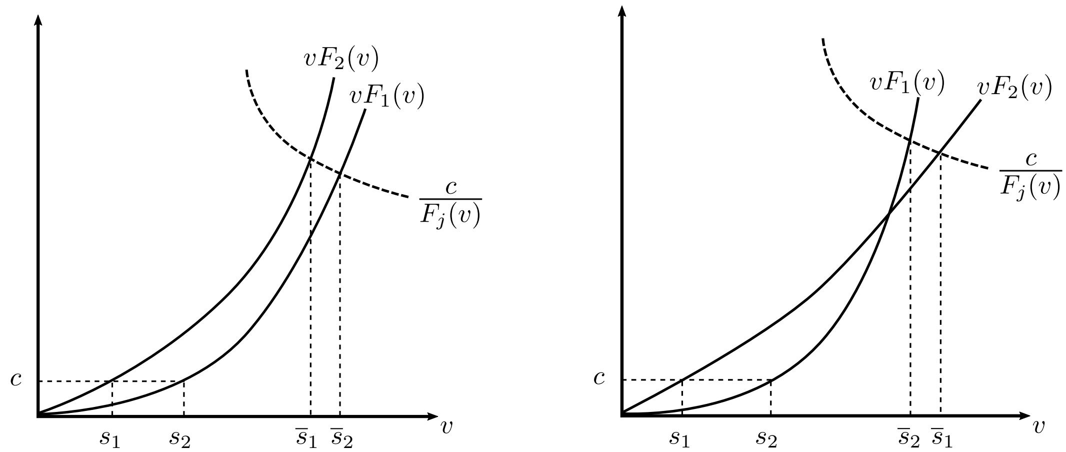
In quasi-symmetric games, when bidders play a herculean equilibrium, payoffs, cutoffs and participation probabilities are always ordered in the same way; i.e., bidders with lower cutoffs are more likely to participate and receive higher expected profits. To see the intuition behind this result, consider an environment in which bidders are quasi-symmetric in costs. In a herculean equilibrium, the low-cost bidder plays the lower cutoff, which implies—due to bidders having symmetric distributions of valuations—that she participates with higher probability. Suppose, for the sake of argument, that both bidders play the *same* cutoff strategy. Because the stronger bidder has a lower participation costs and bidders have symmetric distribution of valuations, the stronger bidder would receive a higher expected payoffs. This payoff order gets reinforced in equilibrium. Bidders participate whenever they have positive expected payoffs and the strong (low-cost) bidder participates at a larger range of valuations, obtaining even higher profits than the weaker bidder.[15] This construction strongly relies on quasi-symmetry. Once we allow for general forms of bidder asymmetries, the relation breaks even within the herculean equilibrium class, as shown in the examples below.

Example. Recall example 1 from section 4.1. There, bidders are not ordered by FOSD as bidder 2's CDF is a mean preserving spread of bidder 1's. When the participation cost is equal to c°, bidders are equally strong ($s_i = v^\circ$). Because the CDFs are concave, the unique equilibrium is given by the symmetric cutoffs equal to the bidders' strength ($x_i = v^\circ$). The expected payoff of bidder 2, however, is greater than the expected payoff of bidder 1. Using equation (14), we obtain $(U_1, U_2) = (0.103, 0.185)$. This means that although bidders' cutoffs are not ranked, their expected profits are. The intuition in this scenario follows from $F_2(v) < F_1(v)$ for every $v > v^\circ$. Relative to bidder 1, bidder 2's valuations (distributed according to $F_2(v)$) are skewed to the right tail of the distribution, whereas their expected payment price (distributed according to $F_1(v)$) is skewed towards the left (see Figure 2.(a)). In other words, for valuations greater than v°, bidder 2's conditional distribution of valuations FOSD the bidder 1's conditional distribution.

Beginning from the previous example, we construct an equilibrium in which bidder 1 receives a lower expected payoff than bidder 2, despite playing a lower participation cutoff and having a higher participation probability. By decreasing bidder 1's participation cost, bidder 1 becomes stronger than bidder 2 and will play a lower cutoff in the unique equilibrium of the game. By continuity, if the decrease in bidder 1's cost is small, we can construct an equilibrium with said characteristics. Take for example $(c_1, c_2) = (1.9, c^\circ)$, then bidder 1 is stronger and plays a lower cutoff—in this case $\mathbf{x} = (2.1327, 2.2196)$—but also receives lower expected payoffs $(U_1, U_2) = (1.11, 1.83)$. At a cutoff equal to v°, both bidders are equally likely to enter. Thus, $x_1 < v^\circ < x_2$ implies that bidder 1 is simultaneously more likely to participate and receive a lower expected payoff.

Finally, to show that cutoff order need not coincide with entry-probability order, modify the participation costs to $(c_1, c_2) = (1.1, 1)$. In this scenario, bidder 1

[15] Similar reasoning applies to quasi-symmetric games with FOSD distributions. If both bidders were to play the *same* participation cutoff, the stronger bidder would participate more often and receive a higher expected payoff due to FOSD. In a herculean equilibrium, the stronger bidder plays a lower cutoff, participating more often and obtaining even higher expected payoffs.



(a) First-order stochastic dominance.

(b) General bidder heterogeneity.

Figure 4: Competition and Strength.

plays a higher entry cutoff $x_1 = 1.434 > 1.313 = x_2$ while also participating more frequently $1 - F_1(x_1) = .238 > .234 = 1 - F_2(x_2)$.

Number of Competitors and Equilibrium Behavior We now discuss how an (exogenous) increase in the number of competitors generate different equilibrium predictions in quasi-symmetric and non-quasi-symmetric models. In particular, Lemma 6 below shows that, in non-quasi-symmetric models, increasing the number of competitors may change the relative strength position among existing bidders. Whereas, in quasi-symmetric models, this reversal cannot occur.

Lemma 6. *Adding a potential bidder to the game does not affect the existing strength-order among quasi-symmetric bidders but might change the order if bidders are not quasi-symmetric.*

Quasi-symmetric models restrict bidder behavior when faced with increased competition. Consider a baseline scenario with two asymmetric bidders with symmetric participation costs c. Suppose that bidder 1 is stronger than bidder 2 and, for simplicity, that $F_1(x)$ and $F_2(x)$ are concave so that a unique equilibrium exists. Because bidder 1 is stronger, it plays a lower participation cutoff than bidder 2. Using the definition of strength in equation (5), under symmetric participation costs, the strength of bidder 1 in a game with two bidders is given by $s_1 F_2(s_1) = c$. In Figure 4.(a), s_1 is solved by the intersection of the curve $vF_2(v)$ with the horizontal line c (and analogously for bidder 2). Suppose a new potential bidder j joins the game. When the third bidder is added to the game, the strength of bidder 1 is determined by $\bar{s}_1 F_2(\bar{s}_1) F_j(\bar{s}_1) = c$. To simplify comparison with the case of two bidders, we rearrange the previous equations that define the strength of bidder 1 as $\bar{s}_1 F_2(\bar{s}_1) = c/F_j(\bar{s}_1)$. In Figure 4.(a), $\bar{s}_1$ can be computed by the intersection of the curve $vF_2(v)$ and the curve $c/F_j(v)$ (and analogously for bidder 2).

Figure 4 shows how strength varies in two different scenarios. Panel (a) depicts a situation in which bidders are ordered by FOSD (a quasi-symmetric game). Panel (b) shows a scenario without this restriction. Notice how in Panel (a) the two functions $vF_i(v)$ never cross. However, without the FOSD restriction the two functions may cross in general. Intuitively, this means that adding a competitor in quasi-symmetric games does not alter the relative strength of existing bidders. In Panel (a), bidder 1 is always stronger than bidder 2. Whereas, in non-quasi-symmetric games, increased competition may affect the relative strength of bidders. As shown by Panel (b), bidder 1 is stronger in a two bidder scenario. But bidder 2 becomes stronger when a new potential bidder is added to the game, reversing equilibrium behavior among existing bidders.

6 Extensions and Discussion

In this section, we briefly discuss some assumptions in the model and extend our analysis in two important directions: scenarios with a reservation price and in which bidders are only partially informed about their valuation before making their participation decisions. The proofs in this section are relegated to the online Appendix.

Participation Costs In the model, participation costs could represent participation fees charged by the auctioneer, the cost of preparing and submitting a bid, the opportunity cost of attending the auction or, travel costs to the auction site. In all these cases, our results are a starting point for auction design with costly participation and heterogeneous agents (c.f. Menezes and Monteiro, 2000; Celik and Yilankaya, 2009; Moreno and Wooders, 2011, who study the case of symmetric agents).

Reserve Prices For ease in exposition we assumed throughout the paper that there is no reserve price. The notion of strength and our uniqueness result, however, can be easily extended to this setting. For simplicity we present results in a two-bidders context. Following similar steps, however, results can be generalized to environments with more than two bidders.

Assume that the auction has a reservation price of $r \geq 0$. We start by adapting the notion of strength to reflect the existence of the reserve price. In an scenario with two potential bidders, the strength of bidder i is the unique number that solves:

$$(s_i - r)F_j(s_i) = c_i. \tag{15}$$

As before, the strength of bidder i is defined as the unique symmetric strategy profile in which bidder i is best responding. The main difference with respect to the previous definition in equation (5) is that now the symmetric strategy profile takes into account the reserve price. Letting bidder 1 being the strong bidder of the game, our main results in the context of a reserve price is:

Proposition 7 (Existence and uniqueness with reserve price)**.** *There always exists a herculean equilibrium. Every herculean equilibria is characterized by cutoffs $x_1 \leq x_2$ that jointly solve*

$$(x_1 - r)F_2(x_2) = c_1 \quad \textit{and} \quad x_2 F_1(x_2) - rF_1(x_1) - \int_{x_1}^{x_2} v dF_1(v) = c_2. \tag{16}$$

A herculean equilibrium is the unique equilibrium of the game if for both bidders

$$F_i(v) \geq v f_i(v) \textit{ for all } v > r + c_j. \tag{17}$$

We can see from the proposition above that the existence of a reserve price weakens our sufficient condition for uniqueness. The lower bound for participating in the auction is now $r + c_i$. A higher the reservation price acts as an increase in the entry costs and, by Lemma 4, it becomes more likely that condition (17) is satisfied.

Partially Informed Bidders Thus far, we have studied environments in which bidders are *perfectly* informed about their valuations before making participation decisions. More generally, however, bidders could be *partially* informed about their valuations and only learn their true valuation after paying the participation cost. This model, also called the *selective entry* model, has been used in empirical applications by Gentry and Li (2014), Sweeting and Bhattacharya (2015), and Roberts and Sweeting (2016). We now show how our methodologies and results can be extended to this framework.

Consider the two bidder scenario of Section 4.1. Suppose that the valuation of bidder i is now given by $V_i = v_i \varepsilon_i$, where the *signal* v_i is observed before the participation decision and the *noise* ε_i is observed *after* paying the participation cost but *before* submitting a bid. We maintain our distributional assumptions over v_i and assume that ε_i is independent from v_i, distributed according to Φ_i, an atomless distribution with full support over $\mathbb{R}^+$, and $\mathbb{E}(\varepsilon_i) = 1$. At one extreme of *selective entry* models, is Levin and Smith (1994) where bidders are only informed after participating; i.e., v_i is degenerate at some point, conveying no private information. At the other extreme is Samuelson (1985) and our previous framework, in which bidders are fully and privately informed about their type before participating into the auction; i.e., ε_i is degenerate at 1.

For a given realization of the signal v_i, when bidder i is the sole participant in the auction, its *interim* expected payoff—that is, the expected payoff of bidder i right after the participation decisions have been (simultaneously) made, but before bidders receive their second signal ε_i and submit a bid—is equal to $v_i \int_0^\infty \varepsilon_i d\Phi_i(\varepsilon_i) = v_i$. Similarly, for a given realization of signals $\mathbf{v} = (v_1, v_2)$, the *interim* expected payoff of a bidder that faces competition from the other bidder is:

$$\pi_i(\mathbf{v}) = \int_0^\infty \left(\int_0^{v_i \varepsilon_i} (v_i \varepsilon_i - s) d\Phi_j \left(\frac{s}{v_j} \right) \right) d\Phi_i(\varepsilon_i) \tag{18}$$

where, after a change in variables, $\Phi_j(s/v_j)$ is the distribution of bidder j's bid, conditional on observing the signal v_j.

We can use the objects above to define the bidder's strength. Notice that strength is an object defined *ex-ante*. Therefore, strength is defined using the structure of the first signal, taking into account expectations over the second signal. For a given game $(F_i, \Phi_i, c_i)_{i=1}^2$, the strength of bidder i is the unique number $s_i \in \mathbb{R}_+$ that solves:

$$s_i F_j(s_i) + \int_{s_i}^\infty \pi_i(s_i, v_j) dF_j(v_j) = c_i. \tag{19}$$

The equation above finds the signal s_i that makes bidder i indifferent to participate in the auction when their opponent plays a cutoff strategy s_i.[16] The first term is bidder i's payoff when she is the sole entrant. In this case, bidder i's expected payoff matches her first signal, s_i, times the probability that bidder j does not participate when bidder j plays a cutoff equal to s_i, $F_j(s_i)$. The second term

[16] Recall that in the partially-informed-bidder model v_i is a signal and $V_i = v_i \varepsilon_i$ is the valuation.

includes the cases where bidder j participates; i.e., $v_j > s_i$. For each realization of j's signal, bidder i's expected payoff is given by equation (18). The expression integrates over every possible realization of v_j.

We say that bidder i is stronger than j if $s_i < s_j$. It can be readily verified that the left-hand side of (19) is strictly increasing and unbounded above; i.e., strength is well defined. As before, strength elicits a bidder's ability to endure competition by computing the symmetric strategy that makes each bidder indifferent between participating and not. A bidder with a lower value of strength s_i is willing to participate at a lower valuation. Without loss of generality, let bidder 1 be the strongest bidder of the game. The following proposition characterizes equilibrium in general selective entry models.

Proposition 8 (Partially informed bidders)**.** *There always exists an herculean equilibrium, which is characterized by cutoffs $x_1 \leq x_2$ that for bidder i solves*

$$x_i F_j(x_j) + \int_{x_j}^{\infty} \pi_i\left(x_i, y\right) dF_j\left(y\right) = c_i.$$

A herculean equilibrium is the unique equilibrium of the game if for both bidders

$$F_i(v) \geq v f_i(v) \text{ for all } v > \min\{c_1, c_2\}. \tag{20}$$

In this case, the weak concavity of the signal v_i distribution is also a sufficient condition for equilibrium uniqueness. Although the distributions of noise (Φ_i) do not show up in our sufficient condition, they may still affect the existence of multiplicity of equilibria in scenarios in which the distribution of signals (F_i) is *not* concave. Finally, observe that condition (20) is a bit stronger than (7). This is due to the non-linear relation that now exists between v_i and F_j when the other bidder participates.

7 Concluding Remarks

In this article we generalized existing results about second-price auctions with participation costs by allowing heterogeneity both in distributions of valuations and in participation costs. We developed the concept of strength, which uses bidders' public characteristics—here, distributions of valuations and participation costs—to rank bidders according to their ability to endure competition. We showed that an equilibrium with cutoffs ordered by strength—called herculean equilibrium— exists in situations of applied interest. Moreover, when the distributions of valuations are concave, we showed that the herculean equilibrium is the unique equilibrium of the game. Because there is always an *ex-ante* efficient equilibrium, when the conditions for uniqueness hold, the herculean equilibrium is both *ex-ante* efficient and the unique equilibrium of the game.

We believe that the methodology developed here can be extended to study second-price auctions with more general environments such as interdependent or affiliated values. Our methodology can also be extended to auction settings in which the auction designer creates endogenous heterogeneity, including when a bid handicap is imposed on a subset of bidders during the bidding stage (*e.g.*, bid preference programs for entrants). The tools developed in this article can be applied to estimate optimal participation fees and to compare revenue from participation fees and reserve prices. We regard such models as promising avenues for future research.

A Appendix: Omitted Proofs

This section presents the proofs omitted from the main text.

Proof of Lemma 1. Pick any τ_{-i}. Because i's utility is linear in τ_i, it is a best response to participate with probability 1 whenever there is a positive payoff of doing so. Hence, it is sufficient to show that, conditional on bidder i participating in the auction ($\tau_i(v) = 1$), i's utility crosses zero at a singleton point and from below. Differentiating $u_i(\tau_i = 1, \tau_{-i}, v) = \left[vG_i(v) - \int_0^v x dG_i(x) - c_i\right]$ with respect to v we obtain that $du_i/dv = G_i(v) > 0$ for all $v > 0$, which implies that i's utility is strictly increasing in v. By the finite expectation assumption on F_i, u_i is unbounded above in v. Therefore, because $u_i(\tau_i = 1, \tau_{-i}, 0) < 0$, there exist a unique best response which is given by the unique value of v that solves $u_i(\tau_i = 1, \tau_{-i}, v) = 0$. ■

Proof of Lemma 2. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a profile of cutoff strategies. Denote bidder i's expected utility of participating in the auction when her valuation is v, and the opponents play the cutoffs $\mathbf{x}_{-i}$ by $u_i(0, \mathbf{x}_{-i}, v)$. Lemma 1 shows that bidder i's best response to $\mathbf{x}_{-i}$ is given by the unique valuation x_i satisfying $u_i(0, \mathbf{x}_{-i}, x_i) = 0$. In particular, using equation (1) when every opponent uses a cutoff strategy, $u_i(0, \mathbf{x}_{-i}, x_i) = 0$ is equivalent to equation (3) which proves the lemma. ■

Proof of Proposition 1. We establish that the conditions of Brouwer Fixed Point Theorem are meet. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a collection of cutoffs. By Lemma 1, bidder i's best response to the profile of strategies $\mathbf{x}_{-i}$ is given by the unique valuation v that solves $u_i(0, \mathbf{x}_{-i}, v) = 0$. Since F_i is atomless and has full support, bidder i's best response is continuous in each of the opponent cutoffs. Moreover, since $u_i(0, \mathbf{x}_{-i}, v)$ is increasing in the opponents' cutoffs, the lowest utility for bidder i is achieved when each opponent participates with certainty (i.e., $\mathbf{x}_{-i} = \mathbf{0}_{-i}$). Let K_i be valuation of bidder i that satisfies $u_i(\mathbf{0}, K_i) = 0$. Hence, the vector of best responses is a continuous mapping from the compact and convex set$\times_{i=1}^{n}[0, K_i]$ to itself and all conditions of Brouwer Fixed Point Theorem are meet, proving existence of equilibrium. ■

Proof of Proposition 2. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ where, without loss of generality, we order the bidders identities from the lowest cutoff chosen by the planer, x_1, to the highest, x_n. Differentiating (4) with respect to x_i we obtain

$$W_{x_i}(\mathbf{x}) = f_i(x_i)(c_i - x_i\Omega_i(\mathbf{x})) + \sum_{k \neq i} \int_{x_k}^{\infty} s \left(\frac{d\Omega_k(s, \mathbf{x}_{-k})}{dx_i} \right) dF_k(s).$$

Observing that $d\Omega_k(v, \mathbf{x}_{-k})/dx_i = f_i(x_i) \prod_{\ell \neq k,i} F_\ell(\max\{v, x_\ell\})$ if $v \leq x_i$ and zero otherwise, we can write

$$W_{x_i}(\mathbf{x}) = -f_i(x_i)\left(x_i\Omega_i(\mathbf{x}) - \sum_{k=1}^{i-1}\left\{\int_{x_k}^{x_i} s\prod_{\ell\neq k,i} F_\ell(\max\{s,x_\ell\})dF_k(s)\right\} - c_i\right). \quad (21)$$

Corner solutions are not welfare maximizing as, when we take $x_i = 0$, $W_{x_i}(0,\mathbf{x}_{-i}) > 0$ for all $\mathbf{x}_{-i}$; and $\lim_{x_i\to\infty} W_{x_i}(x_i,\mathbf{x}_{-i}) < 0$ due to the unboundedness of $x_i\Omega_i(\mathbf{x})$. Therefore, an interior maximum exists, which is characterized by a value of x_i satisfying $W_{x_i}(\mathbf{x}) = 0$. The term inside the parenthesis in equation (21) is equal to zero whenever condition (3) holds.[17] Therefore, we conclude that there exists a cutoff equilibrium that is *ex-ante* efficient. Moreover, since every equilibrium satisfies $W_{x_i} = 0$, they are a critical point of W. Finally, because $W_{x_i,x_i}(\mathbf{x}) = -f_i(x_i)\Omega_i(\mathbf{x}) < 0$, the critical point cannot be a minimum.[18] Thus, every equilibria is either a local maximal or a saddle point. ■

Proof of Lemma 3. Both situations are particular cases of the proof of Lemma 5. ■

Proof of Proposition 3. See Proposition 4 in the case $m_1 = m_2 = 1$. ■

Proof of Lemma 4. The proof of both statements make use that a concave differentiable function is bounded above by its first-order Taylor approximation; i.e., for every x and y such that $x > y$

$$F(x) - F(y) \geq (x-y)f(x). \quad (22)$$

The first claim follows, from taking $y = 0$ and using $F(0) = 0$. For the second statement, let y in (22) be inflection point under which $F_i(v)$ becomes concave. Because of concavity, $f_i(x)$ is non increasing for every $x \geq y$. Because F_i is bounded above (by 1), $f_i(x)$ converges to zero as x goes to infinity. Thus, the cost $c_j > y$ such that condition (7) holds is implicitly given by $F_i(y) = f_i(c_j)y$. Then for every $x \geq c_j$ we have: $F_i(x) \geq xf_i(x) + F_i(y) - yf_i(x) \geq xf_i(x)$. ■

Proof of Proposition 4. Begin by observing that equations (8) and (9) define an equilibrium as they correspond to equation (3) for the case in which bidders play symmetrically within group.

[17] To see this observe that $x_i\Omega_i(\mathbf{x}) = x_iB_i(x_i)A_i^n$, that, for a given k,

$$\int_{x_k}^{x_i} s\prod_{\ell\neq k,i} F_\ell(\max\{s,x_\ell\})dF_k(s) = A_i^nA_k^{i-1}\int_{x_k}^{x_i} s\prod_{\ell<k} F_\ell(s)dF_k(s)$$

and $dB_{k+1}(s) = \sum_{j=1}^k dF_j(s)\prod_{\ell=1,\ell\neq j}^k F_\ell(s)$. Then, re-arrange the summation in (21) so that the limits of the integral are consecutive cutoffs (i.e., from x_k to x_{k+1}) instead of x_k to x_i.

[18] We would like to thank an anonymous referee for pointing this out.

Existence. By construction. If $s_1 = s_2 = s$ there is a herculean equilibrium with cutoffs $x_1 = x_2 = s$. Assume $s_1 < s_2$, let $g(x)$ the function implicitly defined by

$$g(x)F_1(g(x))^{m_1-1}F_2(x)^{m_2} = c_1.$$

The function $g(x) > c_1$ and represents the cutoff that bidders in group 1 have to play so that condition (8) is satisfied when everyone in group 2 plays the cutoff $x_2 = x$. Observe that $g(x)$ is strictly decreasing in x and satisfies $g(s_1) = s_1$. Define the function $h : [s_1, \infty) \to \mathbb{R}$ by

$$h(x) = F_2(x)^{m_2-1}\left[xF_1(x)^{m_1} - \int_{g(x)}^{x} yd(F_1(y)^{m_1})\right] - c_2.$$

The function $h(x)$ is continuous and corresponds to the payoffs that a member of group 2 obtains by playing the cutoff $x_2 = x$ when all other members of group 2 play x and all members of group 1 respond by playing $x_1 = g(x)$. A herculean equilibrium exists if there is x^* such that $h(x^*) = 0$ and $x^* > g(x^*)$. The next two claims prove the result.

Claim 1. $x^* \in (s_1, \infty)$ is necessary and sufficient for $x_1 < x_2$.

Proof. Because $g(x)$ is weakly decreasing in x and $g(s_1) = s_1$, $x_1 = g(x^*) < x^* = x_2$ if and only if $x^* \in (s_1, \infty)$. ☐

Claim 2. $h(s_1) < 0$ and $h(x)$ is unbounded above.

Proof. Group 2 being weak (i.e., $s_1 < s_2$) implies

$$h(s_1) = s_1F_1(s_1)^{m_1}F_2(s_1)^{m_2-1} - c_2 < s_2F_1(s_2)^{m_1}F_2(s_2)^{m_2-1} - c_2 = 0.$$

On the other hand, $h(x)$ is unbounded above as $xF_1(x)^{m_1}F_2(x)^{m_2-1}$ is unbounded and the finite expectation assumption. ☐

By the intermediate value theorem, Claim 2 plus continuity imply that there exists $x^* \in (s_1, \infty)$ such that $h(x^*) = 0$. On the other hand, $h(x^*) = 0$ holds if and only if equations (8) and (9) are satisfied. Therefore, by Claim 1, we have a herculean equilibrium with $x_1 = g(x^*)$ and $x_2 = x^*$.

Uniqueness. From Lemma 9 in the Auxiliary Results section of the appendix we know that, under condition (10), symmetric bidders must play symmetric cutoffs. We need to show that there is no other herculean equilibrium, and that no non-herculean equilibria exists.

Claim 3. There exists a unique herculean equilibrium.

Proof. In a herculean equilibrium bidders are ordered by strength, thus we have to show there is no other equilibrium such that $x_1 < x_2$ and equations (8) and (9) hold; i.e., there exists a unique $x^* > s_1$ such that $h(x^*) = 0$. It is sufficient to show that $h'(x) > 0$ for all $x \geq s_1$, so that $h(x)$ single-crosses zero from below. Differentiating

$$h'(x) = F_2(x)^{m_2-1}\left\{(m_2-1)\frac{f_2(x)}{F_2(x)}\left[xF_1(x)^{m_1} - \int_{g(x)}^{x} yd\left(F_1(y)^{m_1}\right)\right]\right.$$
$$\left. + F_1(x)^{m_1} + m_1 g'(x)g(x)f_1(g(x))F_1(g(x))^{m_1-1}\right\}.$$

Because $F_2(x)^{m_2-1} > 0$, it is sufficient to show that the term in braces is non-negative for all $x \geq s_1$. Implicitly differentiating $g(x)$

$$g'(x) = -\frac{m_2 g(x)F_1(g(x))}{F_1(g(x)) + (m_1-1)g(x)f_1(g(x))}\frac{f_2(x)}{F_2(x)}$$

replacing into the expression in braces delivers

$$(m_2-1)\frac{f_2(x)}{F_2(x)}\left[xF_1(x)^{m_1} - \int_{g(x)}^{x} yd\left(F_1(y)^{m_1}\right)\right]$$
$$+\left[F_1(x)^{m_1} - \frac{m_1 m_2 g(x)^2 f_1(g(x))F_1(g(x))^{m_1}}{F_1(g(x)) + (m_1-1)g(x)f_1(g(x))}\frac{f_2(x)}{F_2(x)}\right]. \tag{23}$$

It is shown that a lower bound for the expression above is always positive. Maximize the subtracting term in the first square brackets by taking the upper bound $x\int_{g(x)}^{x} dF_1(y)^{m_1}$ in the integral. Using condition (10), maximize the subtracting term in the second square brackets by substituting $F_1(g(x))$ for $g(x)f_1(g(x))$ in the denominator (recall that $g(x) > c_1$). Then, equation (23) becomes

$$F_1(x)^{m_1} + \left[(m_2-1)x - m_2 g(x)\right]F_1(g(x))^{m_1}\frac{f_2(x)}{F_2(x)} \geq F_1(x)^{m_1}\left(1 - \frac{g(x)}{x}\right)$$

where $x \geq g(x)$ for $x \geq s_1$, and $f_2(x)/F_2(x) \leq x^{-1}$ were used to obtain the inequality. Hence the lower bound of (23) is non-negative if and only if $x \geq g(x)$, which is true as $x \geq s_1$. □

Claim 4. There is no equilibrium in which strong bidders play a higher cutoff than weak bidders.

Proof. To prove that the only equilibrium is the herculean, suppose we have a non-herculean equilibrium. By Lemma 9, in the Auxiliary Results section, symmetric bidders must play symmetric cutoffs under condition (10). Thus, the only possibility is to have

$x_1 > x_2$ but $s_1 < s_2$. Define $\bar{g}(x)$ to be the function that satisfies

$$\bar{g}(x)F_2(\bar{g}(x))^{m_2-1}F_1(x)^{m_1} = c_2.$$

Similarly, define

$$\bar{h}(x) = F_1(x)^{m_1-1}\left[xF_2(x)^{m_2} - \int_{\bar{g}(x)}^{x} yd(F_2(y)^{m_2})\right] - c_1.$$

The function $\bar{g}(x)$ is decreasing in x, satisfies $\bar{g}(s_2) = s_2$, and represents the cutoff that group 2 has to play so that condition (8) is satisfied when everyone in group 1 plays the cutoff $x_1 = x$. The continuous function $\bar{h}(x)$ corresponds to the payoffs that a member of group 1 obtains by playing the cutoff $x_1 = x$ when all other members of group 1 play x and all members of group 2 respond by playing $x_2 = \bar{g}(x)$. We show that there is no x such that $x_1 = x > \bar{g}(x) = x_2$ and $\bar{h}(x) = 0$, which implies that condition (9) does not hold and no non-herculean equilibrium exists.

Observe that $x > \bar{g}(x)$ if and only if $x \in (s_2, \infty)$ and that $s_1 < s_2$ implies that

$$\bar{h}(s_2) = s_2 F_1(s_2)^{m_1-1}F_2(s_2)^{m_2} - c_1 > s_1 F_1(s_1)^{m_1-1}F_2(s_1)^{m_2} - c_1 = 0.$$

By an analogous argument given in Claim 3, condition (10) implies $\bar{h}'(x) > 0$, and $\bar{h}(x) > 0$ for all $x \in (s_2, \infty)$. Therefore, there is no $x > s_2$ such that $\bar{h}(x) = 0$ and, by Lemma 2, no non-herculean equilibrium exists. □ ■

Proof of Lemma 5. By definition of i's strength $s_i \prod_{j\neq i} F_j(s_i) = c_i$. Equation (12) implies $c_{i+1}F_{i+1}(s_i)/F_i(s_i) > c_i$. Substituting for c_i on the RHS of i's strength and rearranging: $s_i \prod_{j\neq i+1} F_j(s_i) < c_{i+1}$. Since the LHS is increasing in s, $s_{i+1} > s_i$. ■

Proof of Proposition 5. *Existence.* For a given vector $\mathbf{v} = (v_1, \ldots, v_n)$, and following equation (2), define the family of functions $h_i^n(\mathbf{v}) = A_i^n r_i(\mathbf{v}^i) - c_i$.[19] This family of functions will be used in the proof of existence and uniqueness. Start by ordering bidders by strength, with bidder 1 being the strongest and n the weakest. By Lemma 2 a herculean equilibrium $\mathbf{x} = (x_1, \ldots, x_n)$ exists if and only if $h_i^n(\mathbf{v}) = 0$ for all i. We construct $\mathbf{x}$ recursively. Let $\tilde{\mathbf{v}}^i = (v_i, \ldots, v_n)$ represent the elements of $\mathbf{v}$ in the ith and higher positions.

Start constructing x_1. For any vector $\tilde{\mathbf{v}}^2$, define $x_1(\tilde{\mathbf{v}}^2)$ to be the value of v_1 that solves $h_1^n(v_1, \tilde{\mathbf{v}}^2) = 0$; i.e., $x_1(\tilde{\mathbf{v}}^2) = c_1/A_1^n$. We now construct x_2 recursively, by using the constructed $x_1(\tilde{\mathbf{v}}^2)$. By substituting $x_1(\tilde{\mathbf{v}}^2)$ in for the values of v_1 in $h_2^n(\mathbf{v})$, we can

[19] Recall that, for a given $\mathbf{v}$, $\mathbf{v}^i = (v_1, v_2, \ldots, v_i)$, $A_i^n = \prod_{j>i}^{n} F_j(v_j)$ and $r_i(\mathbf{v}^i)$ is given by (2).

write $h_2^n(\mathbf{v})$ as a function of $\tilde{\mathbf{v}}^2$ only. That is, $h_2^n(\tilde{\mathbf{v}}^2) = A_2^n r_2(\tilde{\mathbf{v}}^2) - c_2$ where

$$r_2(\tilde{\mathbf{v}}^2) \equiv r_2(x_1(\tilde{\mathbf{v}}^2), v_2) = v_2 F_1(v_2) - \int_{c_1/A_1^n}^{v_2} x dF_1(x)$$

is the revenue function $r_2(\mathbf{v}^2)$ after replacing the function $x_1(\tilde{\mathbf{v}}^2)$ for the value of v_1. The finite expectation assumption implies that $h_2^n(v_2, \tilde{\mathbf{v}}^3)$ is unbounded above in v_2. Define $\hat{v}_2$ to be the *largest* value of v_2 that satisfies $\hat{v}_2 = x_1(\hat{v}_2, \tilde{\mathbf{v}}^3)$. Observe that $\hat{v}_2$ always exists, as $v_2 \in \mathbb{R}_+$ and $x_1(v_2, \tilde{\mathbf{v}}^3)$ is a continuous function of v_2 with range in $(c_1, \overline{v}_1)$. Also, for every $v_2 > \hat{v}_2$, $v_2 > x_1(v_2, \tilde{\mathbf{v}}^3)$. Otherwise, v_2 and $x_1(v_2, \tilde{\mathbf{v}}^3)$ would cross again and $\hat{v}_2$ wasn't the largest point in which they cross.

Using $\hat{v}_2 = x_1(\hat{v}_2, \tilde{\mathbf{v}}^3) = c_1/(F_2(\hat{v}_2)A_2^n)$, we find $h_2^n(\hat{v}_2, \tilde{\mathbf{v}}^3) = c_1 F_1(\hat{v}_2)/F_2(\hat{v}_2) - c_2$. If the bidders are equally strong; i.e., condition (12) holds with equality, $h_2^n(\hat{v}_2, \tilde{\mathbf{v}}^3) = 0$. Then, we can define $x_2(\tilde{\mathbf{v}}^3) = \hat{v}_2$. If bidder 2 is strictly weaker, condition (12) implies, $h_2^n(\hat{v}_2, \tilde{\mathbf{v}}^3) < 0$. Thus, by the intermediate value theorem, there exists $x_2(\tilde{\mathbf{v}}^3) > \hat{v}_2$ such that $h_2^n(x_2(\tilde{\mathbf{v}}^3), \tilde{\mathbf{v}}^3) = 0$. Because $x_2(\tilde{\mathbf{v}}^3) > \hat{v}_2$, we have $x_2(\tilde{\mathbf{v}}^3) > x_1(x_2(\tilde{\mathbf{v}}^3), \tilde{\mathbf{v}}^3)$. Observe that by replacing $v_2 = x_2(\tilde{\mathbf{v}}^3)$ into $x_1(\tilde{\mathbf{v}}^2)$, we have written both x_1 and x_2 as functions of $\tilde{\mathbf{v}}^3$. Finally, by construction, the order between $x_1(\tilde{\mathbf{v}}^3)$ and $x_2(\tilde{\mathbf{v}}^3)$ is robust to any values of $\tilde{\mathbf{v}}^3$, implying that the order will not reverse when constructing cutoffs for weaker firms (though, the actual values of x_1 and x_2 do change when we change $\tilde{\mathbf{v}}^3$).

Suppose we have shown that, for any vector $\tilde{\mathbf{v}}^i$, $x_1(\tilde{\mathbf{v}}^i) \leq x_2(\tilde{\mathbf{v}}^i) \leq \cdots \leq x_{i-1}(\tilde{\mathbf{v}}^i)$ (strict whenever $s_{k-1} < s_k$). For each $k \leq i$, $x_k(\tilde{\mathbf{v}}^i)$ has been recursively constructed by finding a value v_k solving $h_k^n(v_k, \tilde{\mathbf{v}}^{k+1}) = 0$—which gives us $x_k(\tilde{\mathbf{v}}^{k+1})$—and, for every $j \in \{k+1, \ldots, i-1\}$, by replacing the solution of higher cutoffs $x_j(\tilde{\mathbf{v}}^{j+1})$ into $x_k(\tilde{\mathbf{v}}^{k+1})$. We show that there exists $x_i(\tilde{\mathbf{v}}^{i+1}) \geq x_{i-1}(x_i(\tilde{\mathbf{v}}^{i+1}), \tilde{\mathbf{v}}^{i+1})$ (strict if $s_{i-1} < s_i$) solving $h_i(x_i(\tilde{\mathbf{v}}^{i+1}), \tilde{\mathbf{v}}^{i+1}) = 0$. Notice that $h_{i-1}^n(x_{i-1}, \tilde{\mathbf{v}}^i) = 0$ implies $r_{i-1}(x_{i-1}, \tilde{\mathbf{v}}^i) = c_{i-1}/A_{i-1}^n$. Substituting the vector of solutions x^{i-1} we can write $h_i^n(\mathbf{v})$ as $h_i^n(\tilde{\mathbf{v}}^i) = A_i^n r_i(\tilde{\mathbf{v}}^i) - c_i$. Because of the finite expectation assumption, $h_i^n(\tilde{\mathbf{v}}^i)$ is unbounded above in v_i. Take $\hat{v}_i$ to be the largest value of v_i that satisfies $\hat{v}_i = x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1})$. This value exists by the same argument given to find $\hat{v}_2$ and it also satisfies $v_i > x_{i-1}(v_i, \tilde{\mathbf{v}}^{i+1})$ for $v_i > \hat{v}_i$. Using $\hat{v}_i = x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1})$ and Lemma 8.2 (see the Auxiliary Result section) we know[20]

$$r_i(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}) = F_{i-1}(x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}))r_{i-1}(x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}), \hat{v}_i, \tilde{\mathbf{v}}^{i+1}).$$

Then, using the property $r_{i-1}(x_{i-1}(\tilde{\mathbf{v}}^i), \tilde{\mathbf{v}}^i) = c_{i-1}/A_{i-1}^n$ and $\hat{v}_i = x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1})$, we can

[20]The equation above uses the recursion notation. The formulation from the lemma is

$$r_i(x^{i-2}, x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}), \hat{v}_i) = F_{i-1}(x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}))r_{i-1}(x^{i-2}, x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1})).$$

write $h_i^n(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}) = c_{i-1}F_{i-1}(\hat{v}_i)/F_i(\hat{v}_i) - c_i$. If bidders $i-1$ and i are equally strong, $h_i^n(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}) = 0$ by condition (12) and we can define $x_i(\tilde{\mathbf{v}}^{i+1}) = \hat{v}_i$. If bidder i is strictly weaker than $i-1$, condition (12) implies $h_i^n(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}) < 0$. Then, by the intermediate value theorem, there exists $x_i(\tilde{\mathbf{v}}^{i+1}) > \hat{v}_i$ such that $h_i^n(x_i(\tilde{\mathbf{v}}^{i+1}), \tilde{\mathbf{v}}^{i+1}) = 0$. Finally, because $x_i(\tilde{\mathbf{v}}^{i+1}) > \hat{v}_i$, we have $x_i(\tilde{\mathbf{v}}^{i+1}) > x_{i-1}(x_i(\tilde{\mathbf{v}}^{i+1}), \tilde{\mathbf{v}}^{i+1})$. Once again, the order between the cutoffs will be independent of the construction of cutoffs above. □

Uniqueness: Preliminaries. This proof uses induction. We begin by outlining the main argument. We order bidders from strongest to weakest. Define $H_k^n : \mathbb{R}^n \to \mathbb{R}^k$ to be the function equal to $h_i^n(\mathbf{v})$ (defined in the existence proof above) in the $i^{th} \leq k$ dimension. Fix a value k, by the existence proof we know there exists recursively defined functions $\mathbf{x}^k : \mathbb{R}^{n-k} \to \mathbb{R}^k$ satisfying $H_k^n(\mathbf{x}^k(\tilde{\mathbf{v}}^{k+1}), \tilde{\mathbf{v}}^{k+1}) = 0$. For any $i \leq k$, the total differential of $h_i^n(x^k(\tilde{\mathbf{v}}^{k+1}), \tilde{\mathbf{v}}^{k+1})$ with respect to v_j, $j > k$, is:

$$A_i^n \left[\sum_{s=1}^{i-1} A_s^{i-1} r_s(\mathbf{x}^s) f_s(x_s) \frac{dx_s}{dv_j} + B_i(x_i)\frac{dx_i}{dv_j} + r_i(\mathbf{x}^i) \left(\sum_{s>i}^{k} \frac{f_s(x_s)}{F_s(x_s)} \frac{dx_s}{dv_j} + \frac{f_j(v_j)}{F_j(v_j)} \right) \right]. \quad (24)$$

Using this equation and the implicit function theorem, we can write the vector of derivatives $d\mathbf{x}^k(\tilde{\mathbf{v}}^{k+1})/dv_{k+1}$ as the solution to the following system of linear equations:

$$A_i^n \left[M_k D_k + R_k \frac{f_{k+1}(v_{k+1})}{F_{k+1}(v_{k+1})} \right] = 0, \quad (25)$$

where (T denotes transpose):

$$D_k = \left(\frac{dx_1}{dx_{k+1}}, \frac{dx_2}{dx_{k+1}}, \ldots, \frac{dx_k}{dx_{k+1}} \right)^T, \quad R_k = (r_1(x_1), r_2(\mathbf{x}^2), \ldots, r_k(\mathbf{x}^k))^T$$

and

$$M_k = \begin{pmatrix} B_1(x_1) & r_1(x_1)\frac{f_2(x_2)}{F_2(x_2)} & r_1(x_1)\frac{f_3(x_3)}{F_3(x_3)} & \cdots & r_1(x_1)\frac{f_k(x_k)}{F_k(x_k)} \\ A_1^1 r_1(x_1) f_1(x_1) & B_2(x_2) & r_2(\mathbf{x}^2)\frac{f_3(x_3)}{F_3(x_3)} & \cdots & r_2(\mathbf{x}^2)\frac{f_k(x_k)}{F_k(x_k)} \\ A_1^2 r_1(x_1) f_1(x_1) & A_2^2 r_2(\mathbf{x}^2) f_2(x_2) & B_3(x_3) & \cdots & r_3(\mathbf{x}^3)\frac{f_k(x_k)}{F_k(x_k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1^{k-1} r_1(x_1) f_1(x_1) & A_2^{k-1} r_2(\mathbf{x}^2) f_2(x_2) & A_3^{k-1} r_3(\mathbf{x}^3) f_3(x_2) & \cdots & B_k(x_k) \end{pmatrix}.$$

If M_k is invertible, the the solution to (25) is given by:

$$D_k = -M_k^{-1} R_k f_{k+1}(v_{k+1})/F_{k+1}(v_{k+1}). \quad (26)$$

We will show that M_k is invertible. Then, using (26), we show that $dh_k^k(v_k)/dv_k > 0$. This implies that $h_k^k(v_k)$ single crosses zero, and x_k is uniquely defined. In words, in a game with $n = k$ bidders, in which every cutoff $\mathbf{x}^{k-1}(v_k)$ has been recursively constructed as a function of v_k, bidder k has a unique best response. Furthermore, we will show that $dh_k^k(v_k)/dv_k > 0$ also implies $dh_k^n(v_k, \tilde{\mathbf{v}}^{k+1})/dv_k > 0$ for any $n > k$ and any vector $\tilde{\mathbf{v}}^{k+1}$. This implies that in every game with $n > k$ bidders, for any $\tilde{\mathbf{v}}^{k+1}$, bidder k has a unique best response.[21] Then, by the induction argument, each step of the construction $x_k(\tilde{\mathbf{v}}^k)$ is uniquely defined and the herculean equilibrium is unique.

Claim 5. There exists a unique herculean equilibrium.

Proof. Fix a step k and assume there is $n \geq k+1$ bidders. For that given n, let $\mathbf{x}^k(\tilde{\mathbf{v}}^{k+1})$ to be the vector of functions constructed until step k in the recursion in the existence proof above. For any positive vector $\tilde{\mathbf{v}}^{k+2}$, we need to show that there is a unique value of x_{k+1} that solves $h_{k+1}^n(x_{k+1}, \tilde{\mathbf{v}}^{k+2}) = 0$. In particular, we show $dh_{k+1}^n(v_{k+1}, \tilde{\mathbf{v}}^{k+2})/dv_{k+1} > 0$, so that $h_{k+1}^n(v_{k+1}, \tilde{\mathbf{v}}^{k+2})$ single crosses zero from below.

Using (24),

$$\frac{dh_{k+1}^n(v_{k+1}, \tilde{\mathbf{v}}^{k+2})}{dv_{k+1}} = A_{k+1}^n(d_k D_k + B_{k+1}(v_{k+1}))$$

where $d_k = (A_1^k r_1(x_1) f_1(x_1),\ A_2^k r_2(\mathbf{x}^2) f_2(x_2),\ \ldots,\ A_k^k r_k(\mathbf{x}^k) f_k(x_k))$. Using (26), if M_k is invertible we can write $D_k = -M_k^{-1} R_k f_{k+1}(v_{k+1})/F_{k+1}(v_{k+1})$ and

$$\frac{dh_{k+1}^n(v_{k+1}, \tilde{\mathbf{v}}^{k+2})}{dv_{k+1}} = A_{k+1}^n\left(B_{k+1}(v_{k+1}) - q_k \frac{f_{k+1}(v_{k+1})}{F_{k+1}(v_{k+1})}\right)$$

where $q_k = d_k M_k^{-1} R_k$. Because $A_{k+1}^n > 0$ for all $n \geq k+1$, it is sufficient to show that the parenthesis (which corresponds to $dh_{k+1}^{k+1}(v_{k+1})/dv_{k+1}$) is positive for all relevant values of v_{k+1}. We show the previous statement and the invertibility of M_k by induction.

Observe $h_1^n(\mathbf{v}) = A_1^n v_1$, thus $dh_1^n(\mathbf{v})/dv_1 > 0$ and bidder 1 has a unique best response for any $n \geq 1$ (given by $x_1(\tilde{\mathbf{v}}^2) = c_1/A_1^n$). For bidder 2, observe $M_1 = B_1(x_1) = 1$ is invertible and $q_1 = (x_1)^2 f_1(x_1)$ is well defined. Then, $B_2(v_2) - q_1 f_2(v_2)/F_2(v_2) = F_1(v_2) - (x_1)^2 f_1(x_1) f_2(v_2)/F_2(v_2)$. Using condition (13) twice, $x_1 F_1(x_1)/v_2$ is an upper bound for the subtracting term. Since, by construction, we are interested in $v_2 \geq x_1$, $B_2(v_2) - q_1 f_2(v_2)/F_2(v_2) > 0$.

Suppose we have shown that M_{j-1} is invertible and $B_j(x_j) - q_{j-1} f_j(x_j)/F_j(x_j) > 0$ for all $j \leq k$. Let $l_k = (B_k(x_k) - q_{k-1} f_k(x_k)/F_k(x_k))^{-1}$ and observe that $l_k > 0$ by induction hypothesis; then, by the definition of M_k and using blockwise inversion,

$$M_k = \begin{pmatrix} M_{k-1} & R_{k-1}\frac{f_k(x_k)}{F_k(x_k)} \\ d_{k-1} & B_k(x_k) \end{pmatrix} \text{ and } M_k^{-1} = \begin{pmatrix} O & -\frac{f_k(x_k)}{F_k(x_k)} p_k(M_{k-1}^{-1} R_{k-1}) \\ -l_k(d_{k-1} M_{k-1}^{-1}) & l_k \end{pmatrix}$$

[21] Notice that this does not imply that for each $n > k$, bidder k's best response function is the same across different n.

where $O = M_{k-1}^{-1} + \frac{f_k(x_k)}{F_k(x_k)} l_k (M_{k-1}^{-1} R_{k-1} d_{k-1} M_{k-1}^{-1})$, and the inverse of M_k is well defined. We need to show $B_{k+1}(v_{k+1}) - q_k f_{k+1}(v_{k+1})/F_{k+1}(v_{k+1}) > 0$. Observing that $R_k = (R_{k-1}, r_k(\mathbf{x}^k))^T$, $d_k = (d_{k-1}F_k(x_k), r_k(\mathbf{x}^k) f_k(x_k))$, and using the definition of M_k^{-1} and l_k we can write:

$$q_k = F_k(x_k) q_{k-1} + f_k(x_k) \left(r_k(\mathbf{x}^k) - q_{k-1}\right)^2 \Big/ \left(B_k(x_k) - q_{k-1} f_k(x_k)/F_k(x_k)\right), \tag{27}$$

Thus, $B_{k+1}(v_{k+1}) - q_k \frac{f_{k+1}(v_{k+1})}{F_{k+1}(v_{k+1})} > 0$ is equivalent to show:

$$\left(B_k(v_{k+1}) \frac{F_k(v_{k+1}) F_{k+1}(v_{k+1})}{f_k(x_k) f_{k+1}(v_{k+1})} - q_{k-1} \frac{F_k(x_k)}{f_k(x_k)}\right) \left(B_k(x_k) - q_{k-1} \frac{f_k(x_k)}{F_k(x_k)}\right) > (r_k - q_{k-1})^2$$

where $B_{k+1}(v_{k+1}) = B_k(v_{k+1}) F_k(v_{k+1})$ was used. By the existence proof we are only interested in $v_{k+1} \geq x_i$; using this condition, that $B_k(v)$ is decreasing in v, and condition (13) we find that $(B_k(x_k) x_k - q_{k-1})^2$ is a lower bound for the LHS of the expression above. Lemma 8.1 shows $B_i(x_k) x_k \geq r_k(\mathbf{x}^k)$. Thus we just need to show that $B_k(x_k) x_k - q_{k-1} \geq 0$, which is done by proving $r_k(\mathbf{x}^k) - q_{k-1} \geq 0$. We do this by induction. Since q_0 is not defined, we begin with $i = 2$. Using integration by parts $r_2(\mathbf{x}^2) - q_1$ is equal to

$$x_1 F_1(x_1) + \int_{x_1}^{x_2} F_1(s) ds - (x_1)^2 f_1(x_1) > \int_{x_1}^{x_2} F_1(s) ds \geq 0$$

where condition (13) was used in the last step. Suppose we have shown $r_j(\mathbf{x}^j) \geq q_{j-1}$ for $j \leq i$. We show $r_{i+1}(\mathbf{x}^{i+1}) \geq q_i$. Using equation (27), this is equivalent to:

$$r_{i+1}(\mathbf{x}^{i+1})/F_i(x_i) - q_{i-1} - \left(r_i(\mathbf{x}^i) - q_{i-1}\right)^2 \Big/ \left(B_i(x_i) \frac{F_i(x_i)}{f_i(x_i)} - q_{i-1}\right) \geq 0.$$

Lemma 8.2 shows $r_{i+1}(\mathbf{x}^{i+1})/F_i(x_i) \geq r_i(\mathbf{x}^i)$. By induction hypothesis $r_i(\mathbf{x}^i) \geq q_{i-1}$ and we can rewrite the condition as

$$1 \geq \left(r_i(\mathbf{x}^i) - q_{i-1}\right) \Big/ \left(B_i(x_i) \frac{F_i(x_i)}{f_i(x_i)} - q_{i-1}\right).$$

The result follows from condition (13) and Lemma 8.1. Thus $r_{i+1}(\mathbf{x}^{i+1}) \geq q_i$, which proves $dh_{k+1}^{k+1}(v_{k+1})/dv_{k+1} > 0$ for all $v_{k+1} \geq x_k$. Notice that this result implies $dh_{k+1}^n(v_{k+1}, \tilde{\mathbf{v}}^{k+2})/dv_{k+1} > 0$ for all $\tilde{\mathbf{v}}^{k+2}$ and a unique herculean equilibrium exists. □

Claim 6. There is no non-herculean equilibria.

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be an ordered vector of equilibrium cutoffs. beginning from the lower cutoff, let i be the first bidder to play a smaller cutoffs than a stronger bidder $i+1$; i.e., $x_i < x_{i+1}$ but $s_i > s_{i+1}$. In other words, every bidder $k \leq i$ have their cutoffs in the same order as they strength. Because of this, we can use our recursive construction in the existence proof and our induction argument in the uniqueness proof up to bidder i, so that best responses are uniquely defined for any vector $\bar{x}^{i+1}$ that bidders may play.

Let's analyze $h^n_{i+1}(\mathbf{v})$. Because $h^n_i(x_i, \tilde{\mathbf{v}}^{i+1}) = 0$ we know $r_i(x_i, \tilde{\mathbf{v}}^{i+1}) = c_i/A^n_i$. Substituting the vector of solutions $\mathbf{x}^i$ we can write $h^n_{i+1}(\mathbf{v})$ as $h^n_{i+1}(\tilde{\mathbf{v}}^{i+1}) = A^n_{i+1} r_{i+1}(\tilde{\mathbf{v}}^{i+1}) - c_{i+1}$. Take v_{i+1} to be value of that satisfies $v_{i+1} = x_i(v_{i+1}, \tilde{\mathbf{v}}^{i+2})$ and notice that Lemma 8.2 implies $r_{i+1}(x_i, \tilde{\mathbf{v}}^{i+2}) = F_i(x_i) r_i(x_i, x_i, \tilde{\mathbf{v}}^{i+2})$. Then, using $r_i(x_i, \tilde{\mathbf{v}}^{i+1}) = c_i/A^n_i$, we can write $h^n_{i+1}(x_i, \tilde{\mathbf{v}}^{i+2}) = c_i F_i(x_i)/F_{i+1}(x_i) - c_{i+1}$ which is *positive* under (12) and the condition that bidder $i+1$ is stronger than bidder i. We need to show that there is no $v^*_{i+1} > x_i$ such that $h^n_{i+1}(v^*_{i+1}, \tilde{\mathbf{v}}^{i+2}) = 0$. This follows from the proof of uniqueness as condition (13) implies $dh^n_{i+1}(v_{i+1}, \tilde{\mathbf{v}}^{i+2})/dv_{i+1} > 0$ for $v^*_{i+1} > x_i$, which implies the result. □ ■

Proof of Proposition 6. Define the *ex-ante* expected payoff of bidder i under the vector of cutoffs $\mathbf{x} = (x_1, x_2)$ as:

$$U_i(\mathbf{x}) = \int_{x_i}^{\infty} \left(vF_j(\max\{v, x_j\}) - \int_{x_j}^{\max\{v, x_j\}} s dF_j(s) - c_i \right) dF_i(v)$$

Let $x_1 < x_2$, computing $\Delta \equiv U_1(\mathbf{x}) - U_2(\mathbf{x})$ for the three different scenarios we obtain:

$$\Delta = \begin{cases} \int_{x_1}^{x_2} (v - c) dF(v) & \text{sym.} \\ \int_{x_1}^{x_2} (v - c_1) dF(v) + (c_2 - c_1)(1 - F(x_2)) & \text{cost} \\ \int_{x_1}^{x_2} (vF_2(x_2) - c) dF_1(v) + \int_{x_2}^{\infty} \Gamma_1(v, x_2) f_1(v) - \Gamma_2(v, x_1) f_2(v) dv & \text{FOSD} \end{cases}$$

where $\Gamma_i(v, x) = xF_{3-i}(x) + \int_x^v F_{3-i}(s) ds - c$ which is increasing in v and x and positive if the value of x corresponds to an equilibrium cutoff for bidder i.[22] The first case corresponds to symmetric bidders ($F_i(v) = F(v)$ and $c_i = c$ for all i). Since in equilibrium $x_1 F(x_2) = c_1$, $U_1(\mathbf{x}) > U_2(\mathbf{x})$ whenever $x_1 < x_2$. Thus, in a symmetric game, cutoff order implies expected payoff order. Similarly, for the second case, in a herculean equilibrium in which bidders are ordered by costs ($c_2 > c_1$) and using the same argument above, bidders expected payoff are ordered. Lastly, in the third case, when bidders play a herculean equilibrium and bidders are ordered by first order stochastic dominance ($F_1(v) \leq F_2(v)$ for all v) we have that $\Gamma_1(v, x) \geq \Gamma_2(v, x)$ for any v and x. Then,

$$\int_{x_2}^{\infty} \Gamma_1(v, x_2) f_1(v) dv > \int_{x_2}^{\infty} \Gamma_2(v, x_1) f_1(v) dv \geq \int_{x_2}^{\infty} \Gamma_2(v, x_1) f_2(v) dv$$

[22] Integration by parts was used to obtain $\Gamma_i(v, x)$

where the first inequality follows from the change in identity and herculean cutoffs ($x_1 < x_2$), and the last inequality follows from integrating monotonic functions under stochastic dominance. Which proves that $\Delta > 0$ in the three cases. For the order in the participation probability notice that when the distribution are symmetric, cutoff order and probability order are equivalent. In a herculean equilibrium $x_1 < x_2$ if and only if F_1 FOSD F_2. Thus, $F_1(x_1) < F_1(x_2) \leq F_2(x_2)$ and the order follows. ■

Proof of Lemma 6. Pick two quasi-symmetric bidders, say 1 and 2, such that $s_1 < s_2$. Suppose we add a new potential bidder j. Let $\alpha_i(v) = \prod_{k \neq i} F_k(v)$ which includes bidder j. $\alpha_i(v)$ is strictly increasing in v. For cost order: $s_1 < s_2$ implies $c_1 < c_2$ and $\alpha_1(v) = \alpha_2(v)$. Then, using (5), $\bar{s}_1 \alpha_1(\bar{s}_1) = c_1 < c_2 = \bar{s}_2 \alpha_2(\bar{s}_2)$, which implies $\bar{s}_1 < \bar{s}_2$ and the strength-order is preserved. For FOSD: $s_1 < s_2$ implies $c_1 = c_2 = c$ and $\alpha_1(v) > \alpha_2(v)$. Then $\bar{s}_1 \alpha_1(\bar{s}_1) = c = \bar{s}_2 \alpha_2(\bar{s}_2)$, which implies $\bar{s}_1 < \bar{s}_2$ and strength order is preserved. For an example of strength reversal for non-quasi-symmetric bidders see the main text. ■

B Auxiliary Results

Lemma 7. *If a differentiable function $f(x)$ satisfies $xf'(x) \leq f(x)$ for $x > c$, then, for every $x > c$, $f(x)/x$ is weakly decreasing in x.*

Proof. Define $\phi(x) = f(x)/x$. Then, $\phi'(x) = (f'(x)x - f(x))/x^2$, which is non-positive by the condition. □

Lemma 8. *Let $(x_1, x_2, \ldots, x_n)$ be an ordered vector from smallest, x_1, to largest, x_n. Then, the following properties hold.*

1. $x_i B_i(x_i) \geq r_i(\mathbf{x}^i)$ *and strict if exists $j < i$ such that $x_j < x_{j+1}$.*

2. $r_i(\mathbf{x}^i) > F_{i-1}(x_{i-1}) r_{i-1}(\mathbf{x}^{i-1})$ *and with equality if $x_i = x_{i-1}$.*

Proof. Recall the definition of $r_i(\mathbf{x}^i)$ in equation (2). For the first claim simply observe, $x_i B_i(x_i) - r_i(\mathbf{x}^i) = \sum_{k=1}^{i-1} \left(A_k^{i-1} \int_{x_k}^{x_{k+1}} s dB_{k+1}(s) \right)$ which is strictly positive if there exists a bidder $j < i$ such that $x_j < x_i$ or zero otherwise. For the second claim we show that $r_i(\mathbf{x}^i) = F_{i-1}(x_{i-1}) r_{i-1}(\mathbf{x}^{i-1}) + \int_{x_{i-1}}^{x_i} B_i(s) ds$, which proofs the claim. Rewriting (2):

$$r_{i-1}(\mathbf{x}^{i-1}) = x_i B_i(x_i) - F_{i-1}(x_{i-1}) \sum_{k=1}^{i-2} \left(A_k^{i-2} \int_{x_k}^{x_{k+1}} s dB_{k+1}(s) \right) - \int_{x_{i-1}}^{x_i} s dB_i(s).$$

Integrating by parts the last term we obtain:

$$r_{i-1}(\mathbf{x}^{i-1}) = x_{i-1} B_i(x_{i-1}) - F_{i-1}(x_{i-1}) \sum_{k=1}^{i-2} \left(A_k^{i-2} \int_{x_k}^{x_{k+1}} s dB_{k+1}(s) \right) + \int_{x_{i-1}}^{x_i} B_i(s) ds.$$

Since, by definition, $B_i(x_{i-1}) = B_{i-1}(x_{i-1})F_{i-1}(x_{i-1})$, the result follows. $\square$

Lemma 9. *Symmetric bidders with concave CDF's must play symmetric equilibrium cutoffs.*

Proof. By contradiction. W.l.o.g. order bidders identities in terms of their cutoff order, with bidder 1 being the bidder with the lower cutoff. Suppose there exists an equilibrium such that bidders $q < p$ are symmetric; i.e., $F_q = F_p = G$ and $c_q = c_p = c$, but play $x_q < x_p$. Integrating (3) by parts we obtain (see derivation in the Online Appendix):[23]

$$\sum_{j=1}^{i} \left\{ \prod_{k \geq j, k \neq i} F_k(x_k) \int_{x_{j-1}}^{x_j} \left(\prod_{\ell < j} F_\ell(y) \right) dy \right\} = c_i. \tag{28}$$

Subtracting (28) of q to that of p delivers

$$\begin{aligned} 0 = \sum_{j=q+1}^{p} & \left\{ \prod_{k \geq j, k \neq p} F_k(x_k) \int_{x_{j-1}}^{x_j} \left(\prod_{\ell < q} F_\ell(y) \right) G(y) \left(\prod_{\ell=q+1}^{j-1} F_\ell(y) \right) dy \right\} \\ & - (G(x_p) - G(x_q)) \sum_{j=1}^{q} \left\{ \prod_{k \geq j, k \neq q,p} F_k(x_k) \int_{x_{j-1}}^{x_j} \left(\prod_{\ell < j} F_\ell(y) \right) dy \right\} \end{aligned} \tag{29}$$

We show that a strict lower bound for the right-hand side of (29) is non-negative, a contradiction to (29). The first summation is strictly positive, we take a lower bound of this summation by taking a lower bound of its integrals in three steps: (i) for the terms in the first product ($\ell < q$), replace $F_\ell(y)$ by $F_\ell(x_q)$; (ii) substitute $G(y)$ by $G(x_q)$ and; (iii) for the terms in the second product (ranging from $q + 1$ to $j - 1$), replace $F_\ell(y)$ by $F_\ell(x_\ell)$. Hence, the following *strict* lower bound for the first summation is obtained[24]

$$(x_p - x_q)G(x_q) \prod_{\ell < q} F_\ell(x_q) \prod_{k > q, k \neq p} F_k(x_k)$$

Now we construct an upper bound to the subtracting term in (29) by substituting in the integral $F_\ell(x_j)$ for $F_\ell(y)$. Then, the second summation in equation (29) becomes

$$\prod_{k > q, k \neq p} F_k(x_k) \left(\sum_{j=1}^{q-1} \left\{ x_j \prod_{k=j+1}^{q-1} F_k(x_k) \left(\prod_{\ell \leq j} F_\ell(x_j) - \prod_{\ell \leq j} F_\ell(x_{j+1}) \right) \right\} + x_q \prod_{\ell < q} F_\ell(x_q) \right)$$

Since $x_j \leq x_{j+1}$, the summation in the previous expression is over non-positive terms. We can obtain an upper bound by replacing the summation with zero. Then, our strict

[23] Recall the notation conventions: $\sum_\emptyset = 0$, $\prod_\emptyset = 1$ and $x_0 = 0$.

[24] The strict inequality is guaranteed by taking $G(x_q)$ as lower bound of $G(y)$ over the range of integration x_q to x_p with $x_q < x_p$.

lower bound for the right-hand side of (29) is

$$(x_p G(x_q) - x_q G(x_p)) \prod_{\ell<q} F_\ell(x_q) \prod_{k>q, k\neq p} F_k(x_k).$$

Because the products are positive, the previous expression is non-negative if and only if $G(x_q)/x_q > G(x_p)/x_p$. The result follows from condition (10), Lemma 7 and $x_q < x_p$. $\square$

References

Athey, S., Levin, J. and Seira, E. (2011). Comparing open and sealed bid auctions: Evidence from timber auctions. *The Quarterly Journal of Economics*, **126** (1), 207–257.

Campbell, C. M. (1998). Coordination in auctions with entry. *Journal of Economic Theory*, **82** (2), 425 – 450.

Cao, X. and Tian, G. (2013). Second-price auctions with different participation costs. *Journal of Economics & Management Strategy*, **22** (1), 184–205.

Celik, G. and Yilankaya, O. (2009). Optimal auctions with simultaneous and costly participation. *The BE Journal of Theoretical Economics*, **9** (1), 1–33.

Espín-Sánchez, J.-A. and Parra, A. (2019). Entry games with private information, WP Cowles Fundation.

Gentry, M. and Li, T. (2014). Identification in auctions with selective entry. *Econometrica*, **82** (1), 315–344.

Jehiel, P. and Lamy, L. (2015). On discrimination in auctions with endogenous entry. *The American Economic Review*, **105** (8), 2595–2643.

Krasnokutskaya, E. and Seim, K. (2011). Bid preference programs and participation in highway procurement auctions. *The American Economic Review*, **101** (6), 2653–2686.

Levin, D. and Smith, J. L. (1994). Equilibrium in auctions with entry. *The American Economic Review*, **84**, 585–599.

Lu, J. (2009). Why a simple second-price auction induces efficient endogenous entry. *Theory and Decision*, **66** (2), 181–198.

Marion, J. (2007). Are bid preferences benign? the effect of small business subsidies in highway procurement auctions. *Journal of Public Economics*, **91** (7-8), 1591–1624.

Maskin, E. and Riley, J. (2000). Asymmetric auctions. *The Review of Economic Studies*, **67** (3), 413–438.

McAfee, R. P. and McMillan, J. (1987). Auctions with entry. *Economics Letters*, **23** (4), 343–347.

Menezes, F. M. and Monteiro, P. K. (2000). Auctions with endogenous participation. *Review of Economic Design*, **5** (1), 71–89.

Miralles, A. (2008). Intuitive and noncompetitive equilibria in weakly efficient auctions with entry costs. *Mathematical Social Sciences*, **56** (3), 448 – 455.

MORENO, D. and WOODERS, J. (2011). Auctions with heterogeneous entry costs. *The RAND Journal of Economics*, **42** (2), 313–336.

MURTO, P. and VÄLIMÄKI, J. (2015). Common Value Auctions with Costly Entry, Working Paper.

MYERSON, R. B. (1981). Optimal auction design. *Mathematics of operations research*, **6** (1), 58–73.

ROBERTS, J. W. and SWEETING, A. (2013). When should sellers use auctions? *American Economic Review*, **103** (5), 1830–61.

— and — (2016). Bailouts and the preservation of competition: The case of the federal timber contract payment modification act. *American Economic Journal: Microeconomics*, **8** (3), 257–88.

SAMUELSON, W. F. (1985). Competitive bidding with entry costs. *Economics Letters*, **17** (1-2), 53–57.

STEGEMAN, M. (1996). Participation costs and efficient auctions. *Journal of Economic Theory*, **71** (1), 228–259.

SWEETING, A. and BHATTACHARYA, V. (2015). Selective entry and auction design. *International Journal of Industrial Organization*, **43**, 189 – 207.

TAN, G. (1992). Entry and r & d in procurement contracting. *Journal of Economic Theory*, **58** (1), 41–60.

— and YILANKAYA, O. (2006). Equilibria in second price auctions with participation costs. *Journal of Economic Theory*, **130** (1), 205–219.

— and — (2007). Ratifiability of efficient collusive mechanisms in second-price auctions with participation costs. *Games and Economic Behavior*, **59** (2), 383–396.