RAND Journal of Economics
Vol. 50, No. 3, Fall 2019
pp. 568–590

Sequential innovation, patent policy, and the dynamics of the replacement effect

Álvaro Parra*

I study how patent policy—characterized by patent length and forward protection—affects Research and Development (R&D) dynamics, leadership persistence, and market structure. Firms' R&D investments increase as the patent's expiration date approaches. Through forward protection, followers internalize the leader's replacement effect. In protective systems, this internalization is substantial, reversing Arrow's traditional result: followers invest less than leaders at every moment of the patent's life. I study the policy that maximizes innovative activity. Overly protective policies decrease innovation pace through two mechanisms: delaying firms' investments toward the end of the patent's life and decreasing the number of firms performing R&D.

1. Introduction

■ Consider the incentives that a technology leader faces when deciding whether to improve upon its currently patented technology. When new technologies cannibalize rents from existing products, the cannibalization reduces the leader's incentives to invest in replacing its patented technology (i.e., Arrow's *replacement effect*). The replacement effect is *nonstationary*, patents lose value when the patent's expiration date approaches. A leader's incentives to invest in R&D, therefore, increase as the patent term runs out. Time mitigates the replacement effect.

Firms that are behind in the technology race (or followers) are also affected by the leader's replacement effect. Stronger patent protection against future innovations increases the followers' probability of infringing on existing patents. When an infringement occurs, license fees equal to the leader's profit-loss (damages) must be paid in order to commercialize the new innovation. Through these fees, followers internalize the cost of replacing the leader, discouraging them

* University of British Columbia; alvaro.parra@sauder.ubc.ca.

I thank two anonymous referees and the Editor, Kathryn Spier, for their excellent comments and suggestions. I am indebted to Michael Whinston and Wojciech Olszewski for their advice and support. I also thank Nancy Gallini, Joshua Gans, Jorge Lemus, Igor Letina, Fernando Luco, Guillermo Marshall, Alessandro Pavan, Javiera Pumarino, Bruno Strulovici, Ralph Winter, and seminar participants at Bates and White, CEMFI, ESEM 2013 (Gothenburg), FTC Microeconomics Conference 2013 (Washington DC), IESE, IIOC 2015 (Boston), LAMES 2013 (Mexico DF), Mannheim University, Northwestern University, TOI 2013 (Viña del Mar), University of California at Davis, University of British Columbia, and Washington University. This research was supported by Northwestern's Center of Economic Theory and by the Social Sciences and Humanities Research Council of Canada. A previous version of this article was titled "Sequential Innovation and Patent Policy." All remaining errors are my own.

from investing in R&D. The extent of this internalization can be substantial. Under sufficiently high protection against future innovations, Arrow's result reverses: followers have less incentive to invest than leaders at every moment of the patent's life. As the patent's expiration date approaches, the profit-loss of a replaced leader fades away and the expected license fees paid by an infringing follower decrease. Consequently, the followers' incentives to improve upon existing technologies are also nonstationary. Both followers and leaders have greater incentive to invest in R&D toward the end of a patent's life.

Patents of different length and strength against future innovation thus induce different innovation patterns among technology leaders and followers; patent policy plays a crucial role in determining the *magnitude* and *timing* of R&D investments as well as the degree of *leadership persistence* that exists in the market. This article studies how patent policy—through its dynamic impact on the replacement effect—shapes firms' R&D incentives and market structure. Using these results, I study optimal patent design in the context of a quality-ladder model (Grossman and Helpman, 1991; Aghion and Howitt, 1992; Aghion et al., 2001).

Innovations come from a technology leader trying to prolong its lead or from followers aiming to become the new leader. A patent is represented by a two-dimensional policy determining how long a leader will be able to exclude others from using its current technology—*patent length*—and how enforceable its patent will be against future innovations—*forward protection*. Following Lemley and Shapiro (2005) and Farrell and Shapiro (2008), I treat forward protection as probabilistic, capturing both the uncertainty that exists when a replaced leader tries to enforce its patent against a new innovation and the possible leniency of courts toward new innovators. When a follower develops a new innovation, the replaced leader files an infringement lawsuit against the innovating follower. The patent authority—for example, a US federal court—may decide, with certain probability, to uphold the claim or to declare it invalid.[1] In the former case, a compulsory license fee, equal to the damages caused by the commercialization of the new innovation, must be paid by the infringing firm before the firm can commercialize the new invention and obtain economic profits.

This article contributes to the literature on leadership persistence. In the context of winner-takes-all models, Arrow (1962) shows that, when an innovating leader cannibalizes part of its existing rents, leaders have lower incentives to innovate than followers. In contrast, when an innovating follower shares the market with the existing leader, Gilbert and Newbery (1982) show that incumbents have an incentive to preempt followers, persisting as leaders. Using a stochastic innovation model, Reinganum (1983) shows that the preemptive effect in Gilbert and Newbery can dominate but only under nondrastic innovation. Building on these articles, I connect patent policy to leadership persistence. In particular, using a stochastic model that can accommodate drastic innovations, I show that the infringement of existing patents introduces market-sharing effects into the model. Unlike Gilbert and Newbery, where the reversal occurs due to preemptive motives, followers invest less than leaders due to the internalization of the (leader's) replacement effect through license fees. Strong forward protection increases expected license fees, internalizing the profit-loss of the leader, discouraging followers from investing in R&D.[2]

The value of possessing a patent, the number of competing firms, the extent of the replacement effect, and firms' investment decisions are *endogenously* determined by patent policy. In contrast with the previous literature—discussed further in the next section—the finiteness of patent protection induces nonstationary investments that are increasing throughout the patent's life. Although patents are necessary to incentivize innovation, longer protection intensifies the replacement effect, which induces technology leaders to *delay* their investments toward the end

[1] Allison and Lemley (1998) find that in 46% of the litigated cases, the suing patent is found invalid. PricewaterhouseCoopers (2018) documents that 37% of infringement claims were successful in US federal courts between 1998–2017, and that this success rate varies across sectors.

[2] Other factors that affect leadership persistence are: productivity differentials (Segerstrom and Zolnierek, 1999; Segerstrom, 2007), leader's ability to precommit to its R&D (Etro, 2004), and the technology gap among firms (Harris and Vickers, 1987; Denicolò and Zanchettin, 2012).

of the patent's life. In protective systems, because followers internalize the cost of replacing the leader, they also delay investments when longer protection is offered.[3]

To explore the policy consequences of the dynamic incentives induced by the replacement effect, I examine the combination of patent length and forward protection that maximizes the speed of innovative activity in a given market. I show that the optimal policy has a positive but *finite* length and, in contrast with previous findings (cf. O'Donoghue and Zweimüller, 2004; Denicolò and Zanchettin, 2012), that forward protection can be *desirable*. The optimal level of length and forward protection varies with the market's R&D productivity. In particular, among markets in which innovations take longer to produce or are costlier to develop, such as the pharmaceutical sector, patent length is a more effective tool for promoting innovation; long patents with little forward protection maximize innovative activity.[4] In contrast, markets in which innovations are less costly to produce or are more frequently generated, such as the software industry, forward protection is a more effective tool for promoting innovation; short patents that are protective against future breakthroughs maximize the innovation pace.[5]

Patent policy also plays an important role in determining market structure. Overly protective policies can disincentivize innovation by discouraging entry of followers. Greater forward protection has an immediate effect on entry by decreasing followers' innovation rents via higher expected license fees. Perhaps surprisingly, longer patent protection may also discourage entry, depending on the degree of forward protection. In a system with weak forward protection, longer patents increase innovation rents, encouraging entry. With strong forward protection, however, longer patents encourage entry up to a point. Under strong forward protection, longer patents delay followers' investments and, consequently, the arrival of their innovation rents; that is, longer patents can decrease followers' benefit from participating in the market, inducing their exit. This article, in short, formalizes two mechanism under which overly protective policies may decrease innovation rates: delaying firms' investments and reducing the number of firms investing in R&D.

By explicitly studying patent length and forward protection, I can explore the asymmetric incentives that patent policy gives to leaders and followers. I show that the effectiveness of a policy tool strongly depends on the level of protection granted by the other tool. This dependency creates a trade-off between patent length and forward protection, making the optimal policy to vary significantly across industries. Hopenhayn and Mitchell (2001) identify a sufficient single-crossing condition under which firms self-select into the optimal patent mechanism. Although their condition does not directly apply to this sequential environment—due, in part, to the asymmetric incentives that patent policy provides to leaders and followers—the optimal policy prescribed here trades patent length and forward protection in an analogous way to their optimal mechanism. This trade-off suggests that a market-dependent policy is not only desirable, but potentially self-implementable.

The remainder of this section contextualizes the article within the literature. Section 2 introduces the model, and Section 3 explores R&D dynamics, establishing the reversal-of-Arrow's result. Section 4 analytically studies an approximation of the model. There, I show how patent policy affects the replacement effect, its impact on R&D dynamics, and the policy that maximizes the innovation rate across industries. Section 5 performs a numeric analysis of the main model, showing the robustness of the previous results, and studying the impact that patent policy has in market structure. Section 6 explores various extensions of the model, and Section 7 concludes. All proofs are relegated to the Appendix.

[3] Although a different mechanism to that identified here, Koo and Wright (2010) showed that followers have incentives to delay R&D in order to pay lower license fees.

[4] In a study on the rewards necessary to induce the development of a new drug, Dubois et al. (2015) find that, at the mean market size, an additional \$1.8 billion in revenue is required.

[5] Consistent with this result, Bessen and Hunt (2007) empirically study the impact of the extension of patent rights within the software industry. They find that R&D expenditure (relative to sales) declined between 1987 and 1996.

□ **Related literature.** The model is a sequential extension of traditional (stochastic) patent races à la Loury (1979), Lee and Wilde (1980), and Reinganum (1982). Its departure from previous work is the consideration of the nonstationary incentives induced by patents with a finite length, its interaction with the replacement effect, and the internalization of the replacement effect by followers.[6] The goal is to better understand how patent policy affects R&D and market dynamics.

Early work on dynamic R&D incentives focused on models of a sequence of two innovations. These theories recognize that patent protection can hinder innovation by creating tension between the incentives given to develop first-generation technologies and those given to develop innovations that build upon (or complement) a first-generation technology (Scotchmer and Green, 1990; Scotchmer, 1991; Green and Scotchmer, 1995; Denicolò, 2000; Denicolò and Zanchettin, 2002; Bessen and Maskin, 2009). Although insightful, these models are unable to explain how this tension resolves in a sequential context, where every innovation builds upon previous technologies *and* enables future inventions. The finding that longer patents delay investments is a direct consequence of how this tension is resolved.

The study of R&D incentives in the context of an infinite sequence of innovations has focused on stationary environments. Stationarity has been attained by assuming an exogenous arrival of innovations (Hopenhayn, Llobet, and Mitchell, 2006); by restricting the policy space to patents of infinite length (O'Donoghue, 1998; Acemoglu and Cao, 2015; Marshall and Parra, 2019) or to patents that terminate stochastically in a Poisson fashion (Acemoglu and Akcigit, 2012; Kiedaisch, 2015); by restricting investment in R&D to only potential followers (Hunt, 2004; Segal and Whinston, 2007); or by restricting R&D to only market leaders (Horowitz and Lai, 1996). Although these studies have emphasized the role of the replacement effect on the firms' R&D incentives, the standard stationarity assumption shuts down the dynamic incentives that exist throughout the patent life. This dynamic incentives are the focus of this work.

The problem of optimal patent design has traditionally been studied under the assumption that more protective policies lead to a higher pace of innovation (see Krasteva, 2014 for an exception). The main focus has been to find the policy that balances enhanced R&D incentives, induced by protective policies, with the social cost (deadweight loss) associated with lack of competition due to patent protection.[7] The assumption that (more) protective patents increase R&D, however, has weak empirical support; consistent with the findings in this article, Qian (2007) and Lerner (2009) suggest that protective patents only encourage R&D up to a point, becoming detrimental to innovation when too protective.

2. A model of sequential innovation

□ **Setup.** Consider a continuous-time economy characterized by an infinitely long ladder of innovations. Firms compete by investing in R&D to (stochastically) achieve an innovation and temporarily reach the technological lead in the market. There are $n \geq 0$ endogenously determined symmetric *followers*, denoted by f, who have access only to obsolete technologies. The firm with the leading technology is called the *leader* and is denoted by l. The leader invests in R&D in order to extend its lead in the market, whereas followers invest in R&D to leapfrog the current leader and become the new technology leader. Payoffs are discounted at a rate $r > 0$.

□ **Patent policy.** Every innovation is protected by a patent. Each patent is characterized by a *statutory length* $T \in \mathbb{R}^+$, denoting the amount of time that a leader will be able to exclude others from commercializing its current technology; and a *forward protection* $b \in [0, 1]$, denoting the

[6] To my knowledge, Doraszelski (2003) is the only other work to analyze nonstationary incentives in the context of R&D races. In his article, nonstationarity is due to knowledge accumulation throughout the patent race, whereas here, it is due to the finiteness of patent protection.

[7] With this aforementioned trade-off in mind, Gilbert and Shapiro (1990), Klemperer (1990), and Denicolò (1999) study optimal length and breadth as a function of the market's demand shape. Scotchmer (1999), Cornelli and Schankerman (1999), and Hopenhayn and Mitchell (2001) study policy self-selection.

probability that a new innovation will be considered to infringe on the leader's patent. Whereas a leader's infringement of its own patent has no active consequences, followers must pay a compulsory license fee to be able to profit from any innovation that infringes on an active patent.[8] The license fee is assumed equal to the *damages* that the leader suffers from the commercialization of the new technology, which will be determined in equilibrium. For all of the participants in this market, the tuple (T, b) is considered common knowledge and exogenously given.

To illustrate the workings of the patent system, consider a patent that grants no forward protection ($b = 0$). Under such a system, the leader is able to preclude imitation of its current technology for T years. The leader, however, has no protection against innovations that advance through the technology ladder—no license fees can be collected from any innovation that improves upon the leader's technology.[9] In contrast, when forward protection is maximal ($b = 1$), every innovation that improves upon the leader's technology must pay a license fee.

While a patent is active, the leader receives a monopoly flow of profits $\pi > 0$. When the patent expires, competition in the product market drives the leader's profit flow to zero. As soon as an innovation occurs, the innovating firm patents its new technology, gaining the right to exclude others from using it, in exchange for making this new technology known to the public. As a consequence of this release of information, any innovation produced by a follower will build upon the latest technology, leapfrogging the current leader.

For ease in exposition, I assume that the patents of *obsolete* technologies that have not yet expired are too costly to enforce and are, therefore, imitated. This assumption implies that the technology leader is always one-step ahead of its competitors. It also implies that two consecutive innovations by a leader do not increase its stream of profits π, as the old technology gets imitated—that is, the only benefits that a leader derives from an innovation are extending the clock of its patent protection and, in equilibrium, discouraging followers from investing. Because in practice, a new innovation by a leader only partially cannibalizes existing profits, Section 6 studies the scenario in which consecutive innovations also increase the profit flow that the leader receives. In this section, it is shown that the main forces and intuitions derived in the one-step-lead model are still present in this scenario.

□ **Values and entry.** Denote by t the time that has passed since the last innovation; that is, $t = 0$ represents the arrival of a new innovation and the beginning of a new patent race. Let v_t represent the leader's value of possessing a patent that has been active for t years and w_t represent a follower's value of facing a leader whose patent has been active for t years. Similarly, let q be the value of competing in a market with no patent protection. The values of v_t, w_t, and q are endogenously determined and depend on the underlying patent system, the profit flow that the leader receives while holding the patent, and the R&D decisions of every firm in the market. At $t = T$, however, because the leader and followers become symmetric when patent protection expires, $v_T = w_T = q$.

At the beginning of a new patent race (i.e., at $t = 0$), nonsuccessful firms (followers) decide whether to enter the race by paying an entry cost K. The entry cost can be interpreted as the followers' cost of understanding the new technology and adjusting their labs to be able to develop the next technology in the ladder. Followers will enter the race as long as $w_0 > K$. Because the followers' value of participating in this market will be decreasing with the number of competitors, in equilibrium we will have $w_0 = K$.

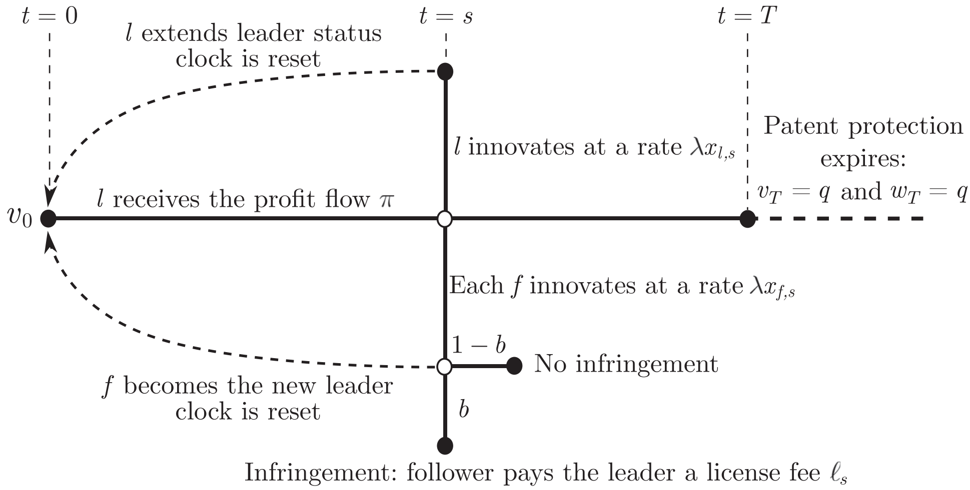
Alternatively, we could have assumed that either: the number of firms is exogenously given (no entry), or entry occurs at any moment in time (free entry). As shown in the online web

[8] In the case of an infringement, the leader may also choose to forbid the utilization of the new innovation. In order to show that overly protective policies slow the pace of innovation, I use the best-case scenario for patents by assuming compulsory licensing (Tandon, 1982).

[9] Concerns about forward protection as a policy tool may exists. Jaffe and Lerner (2011) argue that the creation of the US Court of Appeals for the Federal Circuit increased the number of infringement claims found to be valid. The Federal Trade Commission (2003) proposed weakening forward protection by lowering the requirements for patent invalidation.

FIGURE 1

TIMING OF THE GAME



Appendix, both assumptions deliver the same R&D dynamics as the entry-at-0 model. With respect to the no-entry model, the entry-at-0 model has an advantage in permitting the exploration of the impact that patent policy has on market structure. On the other hand, with respect to the free-entry model, the present model has the advantage of simplifying the exposition. In the entry-at-0 model, the number of competitors is fixed throughout the race; whereas, in the free-entry model, the number of followers is time-varying, adding an additional source of nonstationarity to the model.[10]

□ **R&D strategies.** In order to develop an innovation, firms invest in R&D. These investments lead to a stochastic arrival of innovations, which is an increasing function of the firms' investments. At every t, firms simultaneously choose their R&D investment flows $x_{k,t} \geq 0$ with $k \in \{l, f\}$. The investment flow $x_{k,t}$ represents how the R&D of firm k evolves through time. The instantaneous cost flow of this investment is given by the cost function $c(x) = x^2/2$.

Firm k's investment induces an arrival of innovations described by a nonhomogeneous Poisson process. The arrival rate of firm k at instant t is $\lambda x_{k,t}$. The parameter $\lambda > 0$ can represent either the firms' R&D productivity or a measure of the costs of producing an innovation.[11] The Poisson processes are independent among firms and generate a stochastic process that is memoryless but potentially nonstationary. The waiting time between two innovations is described by an exponential distribution where the probability of observing an innovation by instant t is equal to $1 - \exp(-z_{0,t})$, where $z_{\tau,t} = \lambda \int_\tau^t (x_{l,s} + n x_{f,s}) ds$ measures the accumulated innovation rates from instant τ to instant t. Because t is the only state variable of the model, I study the Markov-Perfect Equilibria of the game by restricting attention to strategies that are a mapping from the time since the last innovation occurred, t, to an R&D intensity.

□ **Timing.** The timing of the game, depicted in Figure 1, is as follows. When an innovation arrives, the time index t is reset to zero. From that time and onward, and while the leader's status

[10] Because the number of competitors is fixed throughout a patent race, the entry-at-0 model has the additional advantage of allowing natural comparison with the traditional patent-race literature, in which the number of competitors is also fixed.

[11] To see this, assume that the leader's productivity is $\tilde{\lambda}$ and its costs are $\tilde{c}(x) = \rho c(x)$ for $\rho > 0$. We can redefine $\lambda = \tilde{\lambda}/\sqrt{\rho}$ and use the $(\lambda, c(x))$ formulation reinterpreting higher cost of innovation ρ as lower productivity λ.

lasts, the patent holder receives the monopoly profit flow π. Followers, on the other hand, obtain zero (product market) profit flow as they only have access to obsolete technologies. At every t, the leader and followers choose their investments simultaneously, determining the arrival rate of innovation for both types of firms.

When an innovation occurs, the succeeding firm becomes the new leader, and its technology renders the currently patented technology obsolete. In addition, if the innovating firm is a follower, with exogenous probability b, the follower's innovation is considered to infringe on the existing patent. In this case, the follower must pay the replaced leader a compulsory license fee (lump sum) of ℓ_t, equal to the damages caused to the leader due to the commercialization of the new innovation. If no innovation has occurred within the statutory length of the patent, the patent holder loses its leader status and becomes one of the many followers of the game. Consequently, no license fees can be charged for innovations that occur after T.

□ **Model interpretation.** The model admits a wide variety of applications commonly studied in the literature. A natural interpretation is to understand each breakthrough as a process (cost-saving) innovation in the context of a homogeneous good market under price competition. In this case, only the firm with the lowest marginal cost of production obtains positive profits.[12] Similarly, the model can also be interpreted as firms competing in price and through product-quality innovations. There, the consumers' willingness to pay equals the product's quality, and the leader's profit is a function of the quality gap between its product and that of the followers (see O'Donoghue, Scotchmer, and Thisse, 1998).

The model also accommodates drastic innovations or the traditional (Schumpeterian) creative destruction framework, in which each innovation completely replaces the old technology, rendering it obsolete—for example, the microprocessor industry. Finally, the profit flow π can be interpreted as a result of the direct commercialization of the innovation or the licensing of the technology to downstream markets.

3. R&D dynamics and arrow's reversal

□ **Competition after patent protection expires.** Patent protection expires after T years. For every $t \geq T$, the (no-longer) patented technology is imitated, the leader's profit flow is cannibalized to zero, and incentives are no longer time-varying. The market, therefore, becomes a stationary patent race with $n+1$ symmetric firms. Recall that, for any firm i, q represents the value of competing in a race without patent protection. Using stationarity, the Hamilton-Jacobi-Bellman (HJB) equation for q is given by:

$$rq = \max_{x_i}\{\lambda x_i(v_0 - q) + \lambda x_{-i}(w_0 - q - K) - c(x_i)\},$$

where $x_{-i} = \sum_{j \neq i} x_j$ is the sum of the R&D investments of i's competitors. At any instant of time t, firm i's flow value is given by the incremental benefit of an innovation $v_0 - q$, which occurs at a rate λx_i; minus the flow costs of its R&D investment $c(x_i)$; plus, at a rate λx_{-i}, i's opponents succeed and the firm obtains the incremental value of becoming a follower (net of entry costs) in the next patent race, $w_0 - q - K$.

Maximizing the HJB equation with respect to x_i and using the quadratic cost assumption, I obtain the optimal R&D investment rate $x_i^* = \lambda(v_0 - q)$. Imposing symmetry among firms and using the equilibrium-entry condition $w_0 = K$, I solve for q to obtain:

$$q = (r + \lambda^2(n+1)v_0 - \rho)/(\lambda^2(2n+1)), \tag{1}$$

[12] For example, consider firms price competing under the demand $q = 1/p$. Innovation decreases the marginal cost of production by $\beta \in (0, 1)$; that is, after n innovations, the marginal cost $c_n = \beta c_{n-1}$. The leader's profit is independent of n and equals $\pi = (p - c)q = (c_n - \beta c_n)/c_n = (1 - \beta)$.

where $\rho = ((r + \lambda^2 n v_0)^2 + 2\lambda^2 r v_0)^{1/2} > 0$. It can be verified that, if the value of a new innovation v_0 is positive, the value of competing in a patent race after patent protection expires is positive ($q > 0$), and that firms invest at a positive rate ($x_i^* > 0$). These solutions satisfy standard comparative statics: q and x_i^* increase in the value of a new patent v_0 and decrease in the number of competing firms n.

For any $t \leq T$, define $q_t = q \cdot \exp(-z_{t,T}) \cdot \exp(-r(T-t))$ to be the expected-discounted value of q at instant t; that is, the value of competing after patent protection expires from the perspective of instant t. The value q_t is computed by discounting q, at a rate r, for the $T - t$ years of protection left in the patent and by weighting it by the probability of not observing an innovation between instants t and T, $\exp(-z_{t,T})$. This expression will be relevant in defining payoffs when patent protection is in place.

□ **Competition under patent protection.** Given any sequence of investments by the followers $\{x_{f,t}\}_{t=0}^T$, from the perspective of time s, the leader's value of possessing a patent that has been active for s years, v_s, is equal to:

$$\max_{\{x_{l,t}\}_{t=s}^T} \int_s^T \big(\pi + \lambda x_{l,t} v_0 + n\lambda x_{f,t}(b\ell_t + w_0 - K) - c(x_{l,t})\big) e^{-z_{s,t}} e^{-r(t-s)} dt + q_s. \tag{2}$$

That is, with probability $\exp(-z_{s,t})$, no innovation has occurred between instant s and t, and the patent is still active at t. At that instant t, the leader receives the flow payoff π and pays the flow cost of its R&D investment $c(x_{l,t})$. The leader's investment results in an innovation at a rate of $\lambda x_{l,t}$, obtaining the benefit of a brand new patent v_0. On the other hand, each of the n followers investing at instant t may succeed at a rate of $\lambda x_{f,t}$. In this case, the innovating follower may infringe on the current patent with probability b, having to pay the leader a compulsory license fee of ℓ_t. Also, when replaced, the leader becomes a follower for the next race, obtaining the continuation value w_0 after paying the entry costs K. All of these payoffs are discounted at a rate r for $t - s$ years; that is, by $\exp(-r(t-s))$. Finally, the value of being the leader at instant s includes the expected-discounted value of competing after patent protection expires, q_s; taking into account that, at $t = T$, the leader obtains the continuation payoff q and ensuring value matching; that is, $v_T = q$.

Similarly, given a sequence of investments by the opponents $\{x_{-f,t}\}_{t=0}^T$, the value that a *follower* derives from competing at instant s, w_s, is given by:

$$\max_{\{x_{f,t}\}_{t=s}^T} \int_s^T (\lambda x_{f,t}(v_0 - b\ell_t) + \lambda x_{-f,t}(w_0 - K) - c(x_{f,t}))e^{-z_{s,t}} e^{-r(t-s)} dt + q_s, \tag{3}$$

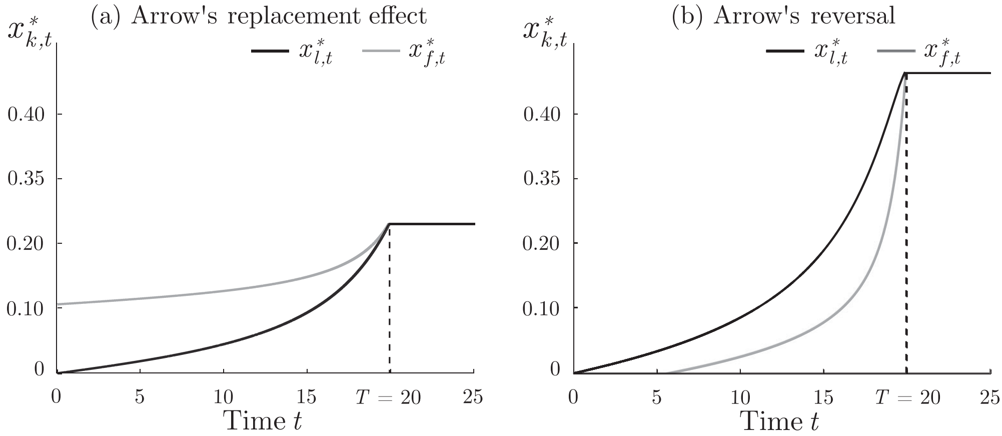
where $x_{-f,t} = x_{l,t} + (n-1)x_{f,t}$ is the R&D of all other firms in the market. As before, instant t is reached with probability $\exp(-z_{s,t})$. At every t, each follower pays the costs of its R&D $c(x_{f,t})$ and, at a rate $\lambda x_{f,t}$, receives the value of becoming the new leader v_0 minus, with an infringement probability b, the license fee ℓ_t. At a rate $\lambda x_{-f,t}$ opponents innovate, in which case the follower receives the value of becoming a follower facing a new leader w_0 minus the entry costs K. Finally, the value of being a follower at s also incorporates the expected-discounted continuation value of competing in a race after patent protection expires q_s.

□ **License fees.** License fees are assumed equal to the damages caused by the commercialization of a new innovation. From the leader's perspective, damages include the expected-discounted profits foregone and the loss in option value of improving upon its own technology. Both objects are included in the value of being the leader v_t. This value, however, also includes the (expected-discounted) rents obtained *after* patent protection expires, q_t. Because the infringer is not liable for the rents lost after patent protection expires, these rents have to be subtracted and, as a result, license fees are given by $\ell_t = v_t - q_t$.

By equation (2), $v_t > q_t$ for all $t < T$. License fees are positive while the patent is active. Also, as the value of a patent v_t decreases with time and q_t increases with t, license fees decrease over the patent life. Finally, because of value matching, $v_T = q$ and $\ell_T = 0$; as no damage

FIGURE 2

PATENT POLICY AND THE REPLACEMENT EFFECT



Note: Parameter values are $r = 5\%$, $\pi = 1/20$, $K = 1/30$, $\lambda = 1$, and $T = 20$. Forward protection in panel (a) $b = 1/3$ and in panel (b) $b = 3/4$.

occurs, an infringing follower does not pay license fees if the infringement occurs on the patent's expiration date.

□ **R&D dynamics.** I apply the Principle of Optimality to derive the following system of HJB equations describing the value of being a leader v_t and the value of being one of the n followers participating in the market, w_t (see the online web Appendix I for details on the derivation):

$$\begin{aligned} rv_t &= \max_{x_{l,t}\geq 0} \left\{\pi + \lambda x_{l,t}(v_0 - v_t) + n\lambda x_{f,t}(w_0 - v_t - K + b\ell_t) - c(x_{l,t}) + v_t'\right\} \\ rw_t &= \max_{x_{f,t}\geq 0} \left\{\lambda x_{f,t}(v_0 - w_t - b\ell_t) + \lambda x_{-f,t}(w_0 - w_t - K) - c(x_{f,t}) + w_t'\right\}. \end{aligned} \tag{4}$$

Conditions in (4) are both necessary and sufficient for a solution to be a maximum. Taking first-order conditions, the optimal R&D investment rates for the firms are:

$$x^*_{l,t} = \lambda(v_0 - v_t) \qquad \text{and} \qquad x^*_{f,t} = \max\{0, \lambda(v_0 - w_t - b\ell_t)\}. \tag{5}$$

Theorem 1 (R&D dynamics). At the beginning of a patent race ($t = 0$), leaders do not invest in R&D. As an active patent approaches its expiration date, leader and followers perform increasing investments over time. When patent protection expires, leader's and followers' investments converge.

Equations in (5) are very informative about the firms' R&D investment dynamics. As Arrow (1962) showed, firms' incentives to invest in R&D are driven by the *incremental value* they obtain from an innovation. For the instant t leader, the incremental value equals the expected profits from a new patent v_0 minus the cannibalized value from giving up the currently active patent v_t; or in Arrow's words, the costs of replacing itself. For the followers, the incremental value is equal to the profits from a new patent minus the cost of replacing itself w_t—which corresponds to the option value of continuing to compete in the race—and minus the expected license fee $b\ell_t$ that the follower has to pay in order to commercialize its innovation; that is, the benefit of a new innovation minus the cost of replacing the leader.

Because the value of having a patent, v_t, decreases as the patent expiration date approaches, the leader's investments increase over time (see Figure 2). Similarly, because damages are a function of the residual patent life, the expected license fees paid by a successful follower also diminish with time, inducing followers to increase their investments as the patent expiration date approaches. When the leader's patent expires ($t = T$), both leader and followers' values converge ($v_T = w_T = q$), as no license fee can be charged, the technology is imitated, and firms compete in a symmetric patent race. Consequently, the firms' investment rates converge to the rate when no patent protection exists; that is, $x^*_{l,T} = x^*_{f,T} = x^*_i$.

Theorem 2 (Arrow's reversal). Depending on forward protection parameter b, followers internalize the cost of replacing the leader through the license fee. When forward protection is sufficiently strong, followers do not invest at the beginning of the patent life, and then invest at a lower rate than the leader.

It is interesting to observe that Arrow's result—that followers have more incentives to innovate than leaders—may be reversed in a dynamic setting. To see this, re-write the followers' investment as:

$$x^*_{f,t} = \lambda \max\{0, \underbrace{v_0 - w_t}_{f\text{'s replacement}} - b\underbrace{(v_t - q_t)}_{l\text{'s profit loss}}\}. \tag{6}$$

Equation (6) connects two, often opposed, views on what drives leadership persistence in an industry. In a context where the innovator accrues all the rents in the market, Arrow (1962) stated that a leader had less incentive to invest in R&D than a follower, as a leader's innovation cannibalizes its existing rents (this cannibalization can be seen in the leader's R&D equation $x^*_{l,t} = \lambda(v_0 - v_t)$). Because followers have the option value of continuing to compete in the race—facing lower expected license fees—followers face their own replacement effect that discourages them from investing. When an innovating follower shares the market with the existing leader, Gilbert and Newbery (1982) show that the leader has more incentive to invest in R&D than a follower. This is because the leader's *profit loss* of sharing the market is larger than the followers' gain.[13] Although the model shares the leader-takes-all feature of Arrow, a market-sharing effect can be seen in (6). As license fees are equal to the damages that the leader suffers from an innovation, followers internalize the leader's profit-loss through the possibility of infringing. Unlike Gilbert and Newbery, where the profit-loss effect induces leaders to preempt followers, here, it discourages followers from investing in R&D. Under sufficiently strong forward protection, the followers' internalization of the leader's profit loss is substantial, potentially reversing Arrow's result.

The reversal-of-Arrow can easily be seen at the maximal forward protection. In this scenario, the followers' investments become $x^*_{f,t} = \max\{0, x^*_{l,t} - \lambda(w_t - q_t)\}$. Equation (3) implies $w_t > q_t$ for all $t < T$. Thus, at every $t < T$, followers invest at a lower rate than the leader. Also, because $x_{l,0} = 0$, followers make no R&D investments toward the beginning of the patent's life. By continuity, this is true not only at $b = 1$, but for a range of forward protection levels (see, e.g., Figure 2(b) when $b = 3/4$).

The combination of Theorems 1 and 2 provides clear and testable empirical predictions about industry dynamics, the persistence of leadership, and its relationship with patent policy. First, the probability that a leader innovates upon its own technology increases as the patent expiration date approaches. Second, the relative probability that an innovation is generated by a follower depends on the level of forward protection. In markets with strong forward protection, the innovation probability of a follower will be similar to or lower than that of the leader; whereas, under weak forward protection—or in markets in which situations of infringement are harder to determine—followers' innovations will be more prevalent. Last, leader and follower innovation rates tend to converge as the patent expiration date approaches. In order to better understand the

[13] Because monopoly profits are usually larger than twice the duopolistic profits ($\pi^m > 2\pi^d$), the leader's profit loss from becoming a duopoly $\pi^m - \pi^d$ is larger than the follower's value of innovating π^d.

role that patent policy plays in determining the replacement effect and leadership persistence, the next section studies how patent policy affects R&D dynamics.

To conclude this section, I show that the nonstationarity induced by finite patent protection is critical for the reversal-of-Arrow to occur. When patent protection is infinitely long, the value of having a patent does not diminish as time goes by. Consequently, firms face the same R&D incentives at any two instants in time. Because a new innovation merely replaces the leader's active patent with one of the same value, the leader's replacement effect is maximal, disincentivizing the leader to invest in R&D. Followers' investments, on the other hand, become stationary, as the license fees they have to pay in the case of an infringement do not decrease over time. Because leaders do not invest in R&D, followers' investments are (weakly) greater and the reversal does not occur.

Lemma 3 (Stationarity). When patent length is infinitely long ($T = \infty$), values become stationary, and the leader performs no R&D.

4. Short-run followers

■ Due to the complex dynamics generated by R&D investment strategies, the game does not have closed-form solutions for the value functions, making analytical results unattainable. In order to further understand the dynamics of the replacement effect, and to serve as a benchmark for the numerical analysis performed in Section 5, I approximate the model by assuming that the leader faces a sequence of short-run followers (cf. Fudenberg, Kreps, and Maskin, 1990). By doing so, I am able to retain the main economic forces behind the baseline model and obtain analytic solutions, thus, providing a better understanding of the dynamics of the replacement effect. As we shall see, the intuitions derived in the approximated model will carry through to the main model.

At each instant in time t, the leader faces a new follower. Each follower has a R&D productivity of μ ($\approx n\lambda$) and plays once in the game, thus eliminating dynamic considerations from their investment decisions. The follower playing at instant t decides how much to invest in R&D by maximizing its flow payoff. For the scenario with an active patent, the follower playing at instant t solves:

$$\max_{x_{f,t}}\{\mu x_{f,t}(v_0 - b\ell_t) - c(x_{f,t})\}.$$

That is, the follower maximizes the reward from a new innovation v_0, which occurs at a rate of $\mu x_{f,t}$, net of (expected) license fees $b\ell_t$, minus the flow cost of its R&D investment $c(x_{f,t})$. By maximizing the flow payoff, the instant t follower's investment rate is given by $x^*_{f,t} = \mu(v_0 - b\ell_t)$.

The main limitation of the approximated model is that followers do not internalize their own replacement effect. Because followers play only once in the game, they lack continuation value w_t; that is, their investment decisions do not incorporate the option value of competing in the future. Relative to the main model, this assumption leads followers to *overinvest* in R&D. For ease in exposition, it is also assumed that when the leader is replaced by a new innovation or when the leader's patent expires, it becomes one of the many short-run followers in the game. The latter assumption implies that the leader's terminal value of its patent is $v_T = 0$; that is, $q = 0$.[14] As before, when no patent is active, the firms' investments become stationary. For consistency and comparability with respect to the main model, after patent protection expires, total R&D is assumed equal to $x^* = (\lambda + \mu)v_0$.

The leader's payoffs and HJB equation remain identical to those in the main model. To obtain an analytic solution, start by assuming that the value of the next innovation is known and equal to

[14] By assuming that the leader remains a long-run player after its patent expires, it is possible to accommodate $v_T = q > 0$ and still obtain an analytic solution. This formulation does not alter the results but it does, however, substantially increase the complexity of the analysis.

$\hat{v}_0$. To be clear, v_0 represents the value of being a leader that just innovated, and $\hat{v}_0$ represents the value of being the next leader; that is, before an innovation occurs. In equilibrium, we will have $v_0 = \hat{v}_0$. Substituting the leader's and the followers' optimal R&D investments into the leader's HJB equation, the following ordinary differential equation is obtained:

$$-v_t' = av_t^2 - \theta v_t + \pi + \frac{1}{2}(\lambda \hat{v}_0)^2, \tag{7}$$

where $a = \lambda^2/2 + \mu^2 b(1-b)$ and $\theta = r + \lambda^2 \hat{v}_0 + \mu^2(1-b)\hat{v}_0$ are positive constants. This is a separable Riccati differential equation, which has a unique solution satisfying the boundary condition $v_T = 0$. The solution to equation (7), as a function of the conjectured value $\hat{v}_0$, is given by (see the online web Appendix II for details):

$$v_t(\hat{v}_0) = \frac{(2\pi + (\lambda \hat{v}_0)^2)(e^{\phi(T-t)} - 1)}{\theta(e^{\phi(T-t)} - 1) + \phi(e^{\phi(T-t)} + 1)}, \tag{8}$$

where $\phi = (\theta^2 - 2a(2\pi + (\lambda \hat{v}_0)^2))^{1/2}$. Equation (8) shows that the value of being the leader, v_t, depends on the conjectured value $\hat{v}_0$ and is a decreasing function of time t. In order to have a well-defined solution, it is necessary to show that a (unique) fixed-point $v_0(\hat{v}_0) = \hat{v}_0$ exists.

Proposition 4 (Existence and uniqueness). There exists a unique value $\hat{v}_0 > 0$ such that $v_0(\hat{v}_0) = \hat{v}_0$.

The comparative statics for the leader's value with respect to the model's parameters are quite intuitive. The equilibrium value of being a leader with a patent that has lived t years, v_t, increases with the discounted flow of monopoly profits, with the leader's productivity, and with more protective patents. The effect of an increase in the followers' productivity, μ, is less straightforward, as it depends on forward protection. When the leader is fully protected against future innovations, an increase in follower productivity has no effect in the leader's value; whereas, with little forward protection, an increase in the followers' productivity harms the leader (see the online web Appendix IV for details).

□ **The replacement effect and R&D dynamics.** I now explore how patent policy affects the dynamics of the replacement effect. In particular, I study the asymmetric impact that the different policy tools may have on the leader's and followers' R&D decisions throughout the patent's life.

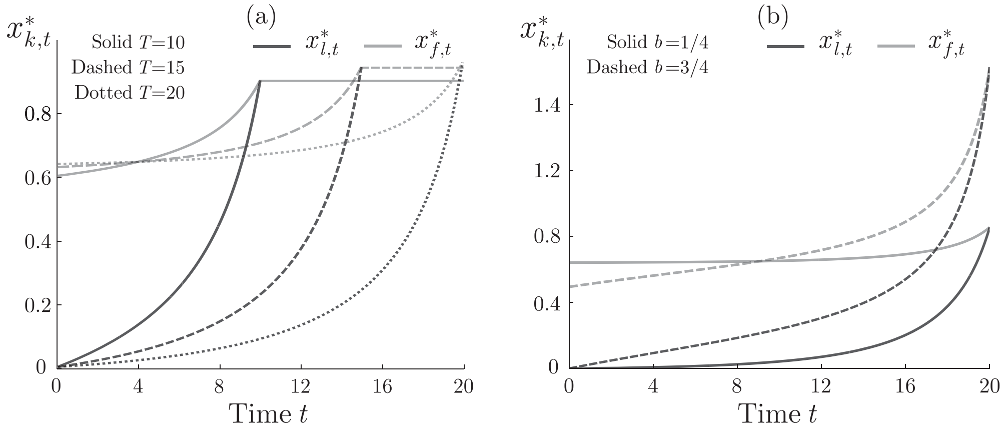
Theorem 5 (Patent length and leader R&D). An increase in patent length T delays the leader's investments; that is, it decreases the leader's R&D at the beginning of the patent's life, but increases it toward the end.

Theorem 5 explores how a change in patent length affects the leader's R&D throughout the patent life. At instant t, the leader's investment is a function of the incremental value it obtains from innovating, $v_0 - v_t$. Because an increase in patent length increases the value of being the leader v_t for all $t < T$, the equilibrium effect of an increase in patent length will depend on the change in magnitude of the increase in v_t throughout the patent life t. The driving force of Theorem 5 is that the increase in v_t becomes larger the closer the active patent is to its expiration date. As a consequence, the leader's *replacement effect* at instant t increases, reducing its incentives to invest in R&D. The delay effect follows from the observation that the leader's investment at the end of its patent life, $x_{l,T+dT} = \lambda v_0$, must be higher, as the value of a new patent is an increasing function of patent length T (see Figure 3(a)).

The intuition for why the leader's replacement effect increases with patent length follows from the observation that the *effective* duration of a patent generally differs from its statutory length. When longer patent protection is offered, the probability of actually reaching and benefiting from the patent extension is higher when the patent is close to its expiration date T. This implies

FIGURE 3

R&D INVESTMENTS UNDER DIFFERENT: (a) PATENT LENGTH AND (b) FORWARD PROTECTION



Note: Parameter values are $r = 5\%$, $\pi = 1/2$, $\lambda = \mu = 2/5$, and, when fixed, $T = 20$ and $b = 1/3$.

that the effective gain due to the increase in duration is larger the closer the patent is to its expiration date, reducing the incremental value of an innovation $v_0 - v_t$, thus decreasing investments at any instant $t < T$. Observe that the net effect of a change in patent length on the *sum* of the leader's R&D investments throughout the patent life is nonmonotonic in T, as both $T \in \{0, \infty\}$ induce leaders to perform no R&D. The impact that this nonmonotonicity has on patent design will be explored further below.

Theorem 6 (Patent length and follower R&D). The effect of an increase in patent length T on followers' investments depends on the level of forward protection. When patents offer no protection against future innovation ($b = 0$), followers' investments increase in T. When forward protection is maximal ($b = 1$), followers internalize the cost of replacing the leader, delaying their investments.

To analyze the effect of patent length on followers' investments, rewrite it as:

$$x^*_{f,t} = \mu[(1 - b)v_0 + b(v_0 - v_t)].$$

The total effect of an increase in patent length on the followers' investments is a convex combination of the impact it has on the value of a new patent v_0 and the impact it has on the leaders' incremental value of an innovation, $v_0 - v_t$. On the one hand, longer patent protection increases the value of an innovation, v_0, incentivizing followers to invest in R&D. On the other hand, stronger forward protection induces followers to internalize the leader's replacement effect, leading them to delay their investments. At the limit, when $b = 1$, followers fully internalize the replacement effect, delaying investments as much as the leader. Figure 3(a) suggests that the followers' internalization of the replacement effect is quite strong, dominating the increase in value of a new patent, even with low levels of forward protection.

Theorem 7 (Forward protection and R&D). An increase in forward protection b that increases the value of a new patent: (i) delays the followers' investments when $b \geq 1/2$, and (ii) increases the leader's R&D toward the end of the patent's life.

Forward protection has a direct negative effect on the followers' incremental value of an innovation due to higher expected license fees paid in the case of achieving a breakthrough. This leads to a decrease in the followers' investment rates at the beginning of the patent's life. As the patent expiration date approaches, expected licenses fees decrease to zero, and the discouragement effect of an increase in forward protection fades away. In particular, at $t = T$, the effect of an increase in forward protection in the value of an active patent v_T is zero, and the only effect left is the increase in value of a new innovation v_0. Hence, the market leader and followers increase their R&D investments toward the end of the patent's life. These effects are shown in Figure 3(b), which depicts firms' investment dynamics for different levels of forward protection b.

□ **Patent policy and the rate of innovation.** This section studies the policy that maximizes the rate of innovation and how this policy varies with the market's R&D productivity. The focus on innovation rates, as opposed to welfare, stems from the need to quantify the extent of the R&D delay induced by replacement effect under protective policies. Also, through innovation rates, we can better compare how patent length and forward protection perform and interact when providing R&D incentives to leaders and followers. Finally, the innovation rate is, by itself, an object of interest in the endogenous growth literature, applied work, and policy discussions. Nevertheless, below, I discuss how my results link with a policy that maximizes total welfare.

Let $\mu = n\lambda$, I study the policy that maximizes the rate of innovation as a function of the market's R&D productivity λ. For simplicity, from now on, I refer to (T^*, b^*) as the optimal policy, meaning the policy that maximizes the innovation rate. To define our measure of innovative activity, I leverage from the property that innovations follow a nonhomogeneous exponential distribution. In particular, I study the policy that minimizes the market's expected waiting time between innovations, which is given by[15]:

$$\mathbb{E}[t] = \int_0^\infty x_t t e^{-z_{0,t}} dt. \tag{9}$$

Theorem 8 (Long patents discourage R&D). The optimal policy consists of a finite patent length.

When innovation is sequential, longer patents promote R&D with diminishing returns and, at some point, become detrimental to innovation (see Figure 4(a) for an example). Under no protection ($T = 0$), innovation is not rewarded and no R&D is performed. On the other hand, although longer patents increase investment *after* patents expire, they also delay the leader's investments and, depending on forward protection, possibly those of the followers. Under infinitely long patents, the increase in R&D after the patent expires becomes irrelevant and the leader delays its investments perpetually, performing no R&D (see Lemma 3), thus decreasing the market's innovation rate.

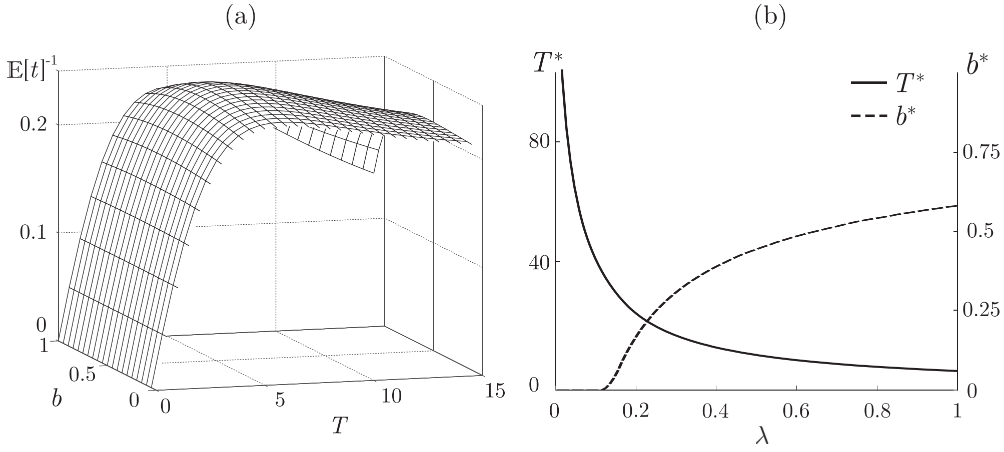
Theorem 8 builds on a literature that has shown different mechanisms through which long patent protection may be detrimental to innovation. In the context of a single innovation, Gallini (1992) shows that patents that last too long become ineffective in rewarding innovation, as they encourage entry by counterfeiters. Horowitz and Lai (1996) study an environment in which innovation dates are *deterministically* chosen by market leaders. They show that leaders will wait until the patent expires to introduce a new innovation and, therefore, infinitely long patents induce no innovation. Their result, however, is not robust to followers being able to perform R&D. Theorem 8 shows that, when innovation is stochastic, the discouragement effect of longer patent protection returns even when followers can perform R&D. Finally, Bessen and Maskin (2009) show that, when innovations are sequential and complementary—that is, when innovation increases the value of existing technologies—long patents hinder innovation incentives. Theorem 8 extends their findings to a scenario in which innovations cannibalize existing rents.

Despite having a unique equilibrium with closed-form solutions for the value of being the leader v_t and firms' R&D investments x_t, the integral (9) cannot be analytically solved when $b > 0$

[15] For the purpose of illustration, if $x_t = \lambda$ for all t, the distribution of successes will follow an exponential distribution with an arrival rate equal to λ and $\mathbb{E}[t] = \lambda^{-1}$. Thus, the expected waiting time between innovations corresponds to the inverse of the market's R&D productivity.

FIGURE 4

(a) OPTIMAL POLICY HAS AN INTERIOR SOLUTION. (b) OPTIMAL POLICY AS A FUNCTION OF THE MARKET'S R&D PRODUCTIVITY λ.



Note: Parameter values are $r = 5\%$, $\pi = 1/20$, and $n = 1$. Figure (a) also uses $\lambda = 1$.

and $T \neq \infty$. This, added to changes in policy induce a change in the fixed-point $\hat{v}_0$, makes the analytical computation of the optimal policy unattainable. Therefore, I compute (9) numerically. Figure 4(a) shows that $\mathbb{E}[t]^{-1}$ is smooth on the model's parameters and that it possesses a unique maximum.

Result 9 (Optimal patent across markets). In the optimal policy, an increase in the market's R&D productivity λ decreases the optimal length T^* and increases the optimal forward protection b^*.[16]

Result 9 characterizes how the optimal policy changes across different markets according to the market's R&D productivity; see Figure 4(b). From the perspective of a policy maker, the result states that there is a trade-off between policy tools: one tool is effective at providing R&D incentives in markets where the other tool is not as effective. Long patents with weak forward protection are more effective in markets where innovations are costly to produce or are harder to achieve. Short patents with strong forward protection, in contrast, are more effective in markets where innovations either occur frequently or are not too costly to produce.

To understand the intuition behind this trade-off, compare the incentives present in markets with high productivity λ, such as the software industry, with those incentives present in markets with low productivity, such as the pharmaceutical sector. Under high R&D productivity, patent length is an ineffective tool to promote innovation, as the effective duration of a patent changes little when longer protection is offered. For instance, increasing patent length from 20 to 21 years in an industry in which innovations become obsolete every three years, does very little to increase the value of an innovation. Furthermore, because longer protection induces leaders to delay their investments, long patents decrease the market's innovation rate. In this context, strong forward protection can be used to reward innovation, and a short patent can be used to minimize the R&D delay.

In contrast, in markets with low R&D productivity, the statutory length of patents can affect the effective duration of patents for a wider range of patent lengths, making T a useful tool in promoting R&D. However, because longer patents induce leaders to delay their investments, followers' innovation is crucial to speed up innovative activity. Thus, weak forward protection

[16] The term Result is used to highlight that the proof of the statement is numerical.

must be offered in order to induce followers to perform R&D in the early stages of the patent life and increase the market's rate of innovation.

In an environment in which patent protection is infinitely long, O'Donoghue and Zweimüller (2004) and Denicolò and Zanchettin (2012) show that forward protection is undesirable, as it simply discourages follower investments. Result 9 shows that, once we allow for finite patents, the nonstationary incentives induced by patent length can make some forward protection desirable. Gilbert and Shapiro (1990), Klemperer (1990), and the work that builds on them, argue that a policy contingent on market characteristics—rather than a one-size-fits-all policy—incentivizes innovation at a lower social cost. The previous discussion, however, assumes that the *only* cost of providing protective patents is the deadweight loss associated with the market power granted by patent protection. Result 9, thus, adds a new layer to the discussion about patent design by identifying an extra cost of protective patents: a protective policy may lead to lower innovation rates. The result also sheds light on the potential implementability of a self-enforcing policy. The optimal policy trades patent length and forward protection, which has been shown to induce firms to self-select into the appropriate policy (cf. Hopenhayn and Mitchell, 2001).

To conclude, I briefly discuss how my results connect to a policy that seeks to maximize welfare. Because total welfare is affected by *both* the industry's innovation rate and the deadweight loss induced by patent protection, adding consumer welfare into the analysis will result in even shorter prescribed patents and, consequently, more forward protection. The magnitude of these effects will depend on the increase in consumer surplus that occurs with each innovation and the extent of the deadweight loss associated with patent protection, both of which strongly depend on the underlying model of competition, the assumed demand, and the nature of innovation.

5. A numerical example

■ In this section, I numerically study the model with long-run followers. The main objectives of this analysis are to show that the results derived in the short-run-follower model are robust and to explore the role that patent policy plays in determining market structure in innovative industries.

Replacing the optimal R&D investments rates in (5) into the HJB equations in (4), I derive the system of differential equations (B1) in Appendix B. There, I also describe the numeric method used to compute the equilibrium. Consistent with Proposition 4, a unique follower-symmetric equilibrium was found for each set of parameters. Figure 5 (a), (c), and (d) show that the main comparative statics presented in Theorem 5 to 7 remain: (i) protective policies increase the value of being the new leader; (ii) longer patent protection delays the leader's R&D investments and, when forward protection is strong, they also delay the followers' R&D; and (iii) forward protection increases the leader's R&D, but delays that of the followers.

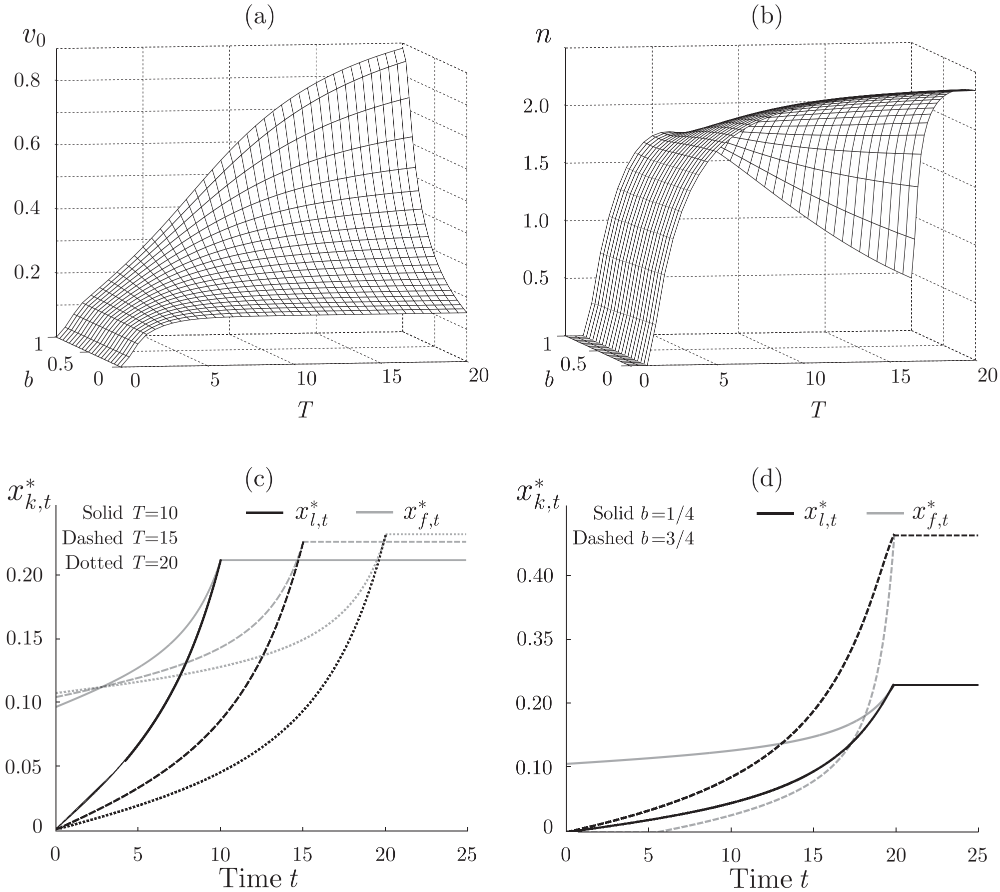
Figure 5(b) shows how the number of followers entering each race varies with patent policy. As expected, stronger forward protection discourages entry, as it simply decreases the rents that followers can accrue. Interestingly, the effect of longer patent protection on entry depends on the strength of forward protection.

Result 10 (Patent policy and entry). Patents that are too short induce no entry. An increase in patent length: (i) increases the number of competitors under weak forward protection, and (ii) under strong forward protection, increases the number of competitors up to a point and then reduces the number of competitors.

When patent protection is too short, no follower enters the market, as the value of participating in the patent race, w_0, is not high enough to compensate for the entry cost K. Under weak forward protection, longer patents induce more firms to enter the market. This also causes the value of a new patent, v_0, to not be very sensitive to changes in patent length (see Figure 5(a)). In particular, when no forward protection is offered, we can see that most of the effect of increasing patent

FIGURE 5

ENDOGENOUS MARKET STRUCTURE



Note: Parameter values are $r = 5\%$, $\pi = 1/20$, $K = 1/30$, $\lambda = 1$, and, when fixed, $T = 20$ and $b = 1/3$. Value functions were approximated to the fourth decimal point.

length is absorbed by the increase in the number of followers in the market, and the value of a new patent, v_0, increases only by a small amount (see Figure 5(b)).

As forward protection becomes stronger, we find an additional countervailing effect of offering long patent protection: it not only delays the firms' investments, but it also induces followers to exit the market (see Figure 5(b)). The exit of followers is due to three effects of patent length on the followers' value: (i) under strong forward protection, longer patents delay the followers' investments, delaying their expected rents, and decreasing the followers' value; (ii) the leader is able to charge license fees for a longer period of time; and (iii) longer patent protection also delays the arrival of the continuation value of competing in a race with no patent protection q. The combination of these three effects makes the market less attractive to followers, decreasing the number of competitors. Notice in Figure 5(a) that, when forward protection is strong, the value of a new innovation v_0 is very responsive to an increase in patent length, which is consistent with the leader simultaneously benefiting from longer patent protection and less competition.

Table 1 shows the optimal patent under a different market's R&D productivity and quantifies, in percentage points, the cost of implementing the incorrect policy ($\mathbb{E}[t]/\mathbb{E}[t]^* - 1$). The optimal policy consists of a finite length and, depending on the market's productivity, positive forward protection. As in the approximated model, one tool is more effective in markets in which the other

TABLE 1 Optimal Patent under Different λ and a Quantification of the Delay in Innovation Pace $\mathbb{E}[t]/\mathbb{E}[t]^* - 1$ Induced By Implementing an Inefficient Policy

					$T = 10$		$T = 20$	
λ	T^*	b^*	$\mathbb{E}[t]^*$	n^*	$b = 1/3$	$b = 2/3$	$b = 1/3$	$b = 2/3$
0.5	33.6	0	6.26	3.10	19.2%	23.8%	6.6%	21.4%
0.75	14.4	0	4.42	2.57	2.4%	8.6%	5.8%	18.5%
1.0	9	0.02	3.48	2.18	1.7%	8%	7%	17.8%
1.25	5.7	0.22	2.87	1.81	3.8%	9.7%	9.36%	19.2%
1.50	4.1	0.24	2.45	1.55	6.6%	11.8%	12.2%	19.6%
1.75	3.2	0.25	2.14	1.35	9.6%	14.1%	15.1%	23.6%

Note: Parameters used: $r = 5\%$ and $\pi = 1/20$. (T^*, b^*) represents the optimal combination of length and forward protection. $\mathbb{E}[t]^*$ is the minimal waiting time between innovations, and $\mathbb{E}[t]/\mathbb{E}[t]^* - 1$ quantifies (in percent) the delay of implementing an inefficient policy.

tool is not. As the market's R&D productivity increases, less firms enter in equilibrium. Higher productivity increases the arrival of innovations, shortening the expected duration of the leader status, decreasing the value of an innovation v_0 and, consequently, entry. The cost of implementing the incorrect policy is quite substantial; the incorrect combination of patent length and forward protection can easily delay the expected arrival of innovations by 20%.

6. Extensions

■ In this section, I briefly discuss the robustness of previous results to extensions of the main model that, due to space limitations, are not fully developed here.

□ **Extending the leader's technological lead.** The reversal-of-Arrow and the investment delay caused by longer patent protection persists once we allow the leader to extend its technological lead in the market. To see this, modify the model by assuming that profits π_m are increasing in the number of consecutive innovations that a leader has achieved, m (i.e., the technology gap). For tractability, I also assume that the leader can extend the protection of its previous innovations with the arrival of a new innovation. Let $v_{m,t}$ be the value of achieving m consecutive innovations, with the latest innovation occurring t years ago. Similarly, let $w_{m,t}$ be the value of being a follower facing a leader that has made m consecutive innovations, with the latest innovation occurring t years ago. It can be shown that the value $v_{m,t}$ is increasing in the number of consecutive innovations m. In equilibrium, investments are given by:

$$x_{l,m,t} = \lambda(v_{m+1,0} - v_{m,t}),\quad x_{f,m,t} = \max\{0, \lambda(v_{1,0} - w_{m,t} - b\ell_{m,t})\},$$

where $\ell_{m,t} = v_{m,t} - q_t$ are the license fees paid at instant t when facing a leader with m innovations.

As in the main model, firms' investments increase as the patent expiration date approaches. Because $v_{m,t}$ is increasing in m, the cost (license fees) of replacing the leader increases with the technology gap between the leader and followers, discouraging followers from performing R&D, and increasing the magnitude of Arrow's reversal. In particular, the minimal level of forward protection for the reversal to occur decreases with the number of consecutive innovations m. In contrast, the leader experiences increased incentives to invest. For instance, $x_{l,m,0} = \lambda(v_{m+1,0} - v_{m,0})$ and investments are positive at $t = 0$, as the replacement effect does not completely cannibalize the value of the previous innovation. It can be shown that an increase in T initially increases the leader's R&D investments, then decreases the leader's investments toward the middle of the patent's life, and then increases investments when patent protection expires. In other words, the leader's incentive to delay exists but becomes weaker. For the followers, on the other hand, the incentive to delay (under strong forward protection) increases with the technology gap m.

Finally, because followers pay the entry cost K, the number of competing followers becomes cyclical. In particular, as the technology gap increases, the number of competing firms goes down. This is so because the cost of replacing the leader increases with the technology gap, decreasing the rents of participating in the next race. Once an innovation by a follower is achieved, reducing the technology gap to one, the number of competing followers increases and the cycle restarts.

□ **License fees: bargaining.** The proposed framework can accommodate the study of incentives provided by different forms of license fees. In particular, we can explore the effects of allowing a bargaining process between the leader and an infringing follower to determine license fees *beyond* the profit loss $v_t - q_t$; that is, $\ell_t = v_t - q_t + \beta(v_0 - v_t)$, where β can be interpreted as the Nash bargaining power of the leader or as the breadth of the patent. In this context, the followers' incentive to delay R&D investments when longer patents are offered under strong patent protection persists. In addition, as expected, when the leader can extract more rents, the reversal-of-Arrow occurs for a wider range of forward protection.

□ **License fees: undiscounted damages.** I have also examined the effects of computing the damages as the undiscounted sum of the stream of profit loss; that is, $\ell_t = (T - t)\pi$. Once again, this specification does not alter the incentives to delay induced by longer patent protection, nor the discouraging effect that stronger forward protection has on followers' investments. It is interesting to observe that, at $t = 0$, the expected license fee $T\pi$ may be larger than $v_0 - q_0$. In such a scenario, the reversal-of-Arrow result occurs for lower levels of forward protection.

7. Concluding remarks

■ This article studied how patent length and forward protection affect the innovation incentives of market leaders and followers. Finite patent protection induces firms' to increase investments with the nearing of the patent's expiration date. The possibility of infringing upon existing patents induces followers to internalize the leader's replacement effect. This internalization can be substantial, potentially reversing Arrow's result. Patent policy, therefore, plays an important role in the degree of leadership persistence that exists in an industry. Patent length and forward protection provide asymmetric R&D incentives between leaders and followers. Whereas longer patent protection delays the leader's investments toward the end of the patent term, followers' investments can be delayed or encouraged, depending on the strength of forward protection. In contrast, an increase in forward protection delays the follower's investments but encourages those of the leader.

Policies that aim to maximize innovative activity must balance the incentives that patent length and forward protection provide to the leader and followers. It was shown that short patents with strong forward protection are preferable in markets where innovations are relatively cheaper to produce. In contrast, long patents with weak forward protection are preferable in markets where innovations are costly. The cost of implementing an incorrect policy can be substantial and is larger in scenarios in which patent protection is both too long and protective against future innovations. Patent policy also affects the number of firms competing in the market. Although stronger forward protection always discourages entry, longer patent protection may encourage or discourage entry, depending on the level of forward protection. Under strong forward protection, longer patents not only delay the firms' investments but also decrease the number of firms investing.

Important questions about how patent policy affects innovation in a sequential context remain. The results presented here naturally open the question on whether there is a mechanism under which firms self-select into the appropriate policy. The trade-off that the optimal policy presents between length and forward protection suggests that such policy is feasible. The framework introduced here can also serve as a building block to study how patent policy can affect firms' decisions regarding adoption of new technologies, innovation quality choice, and disclosure of new innovations. In addition, this framework can be used to study the relationship that exists between patent policy and (endogenous) growth of different sectors in the economy. These question are regarded as future research.

References

ACEMOGLU, D. AND AKCIGIT, U. "Intellectual Property Rights Policy, Competition and Innovation." *Journal of the European Economic Association*, Vol. 10(1) (2012), pp. 1–42.

ACEMOGLU, D. AND CAO, D. "Innovation By Entrants and Incumbents." *Journal of Economic Theory*, Vol. 157 (2015), pp. 255–294.

AGHION, P., HARRIS, C., HOWITT, P., AND VICKERS, J. "Competition, Imitation and Growth with Step-By-Step Innovation." *The Review of Economic Studies*, Vol. 68(3) (2001), pp. 467–492.

AGHION, P. AND HOWITT, P. "A Model of Growth Through Creative Destruction." *Econometrica*, Vol. 60(2) (1992), pp. 323–351.

ALLISON, J.R. AND LEMLEY, M.A. "Empirical Analysis of the Validity of Litigated Patents." *American Intellectual Property Law Association Quarterly Journal*, Vol. 23(3) (1998), pp. 185–275.

ARROW, K. "Economic Welfare and the Allocation of Resources for Invention." In *The Rate and Direction of Inventive Activity: Economic and Social Factors*, NBER Chapters. National Bureau of Economic Research, Inc., 1962, pp. 609–626.

BESSEN, J. AND HUNT, R.M. "An Empirical Look at Software Patents." *Journal of Economics & Management Strategy*, Vol. 16(1) (2007), pp. 157–189.

BESSEN, J. AND MASKIN, E. "Sequential Innovation, Patents, and Imitation." *RAND Journal of Economics*, Vol. 40(4) (2009), pp. 611–635.

CORNELLI, F. AND SCHANKERMAN, M. "Patent Renewals and R&D Incentives." *RAND Journal of Economics*, Vol. 30(2) (1999), pp. 197–213.

DENICOLÒ, V. "The Optimal Life of a Patent When the Timing of Innovation Is Stochastic." *International Journal of Industrial Organization*, Vol. 17(6) (1999), pp. 827–846.

———. "Two-Stage Patent Races and Patent Policy." *RAND Journal of Economics*, Vol. 31(3) (2000), pp. 488–501.

DENICOLÒ, V. AND ZANCHETTIN, P. "How Should Forward Patent Protection Be Provided?" *International Journal of Industrial Organization*, Vol. 20(6) (2002), pp. 801–827.

———. "Leadership Cycles in a Quality-Ladder Model of Endogenous Growth." *The Economic Journal*, Vol. 122(561) (2012), pp. 618–650.

DORASZELSKI, U. "An R&D Race with Knowledge Accumulation." *RAND Journal of Economics*, Vol. 34(1) (2003), pp. 20–42.

DUBOIS, P., DE MOUZON, O., SCOTT-MORTON, F., AND SEABRIGHT, P. "Market Size and Pharmaceutical Innovation." *RAND Journal of Economics*, Vol. 46(4) (2015), pp. 844–871.

ETRO, F. "Innovation By Leaders." *The Economic Journal*, Vol. 114(495) (2004), pp. 281–303.

FARRELL, J. AND SHAPIRO, C. "How Strong Are Weak Patents?" *The American Economic Review*, Vol. 98(4) (2008), pp. 1347–1369.

FEDERAL TRADE COMMISSION. "To Promote Innovation: The Proper Balance Between Competition and Patent Law and Policy." 2003.

FUDENBERG, D., KREPS, D.M., AND MASKIN, E.S. "Repeated Games with Long-Run and Short-Run Players." *The Review of Economic Studies*, Vol. 57(4) (1990), pp. 555–573.

GALLINI, N.T. "Patent Policy and Costly Imitation." *RAND Journal of Economics*, Vol. 23(1) (1992), pp. 52–63.

GILBERT, R. AND NEWBERY, D.M.G. "Preemptive Patenting and the Persistence of Monopoly." *The American Economic Review*, Vol. 72(3) (1982), pp. 514–526.

GILBERT, R. AND SHAPIRO, C. "Optimal Patent Length and Breadth." *RAND Journal of Economics*, Vol. 21(1) (1990), pp. 106–112.

GREEN, J.R. AND SCOTCHMER, S. "On the Division of Profit in Sequential Innovation." *RAND Journal of Economics*, Vol. 26(1) (1995), pp. 20–33.

GROSSMAN, G.M. AND HELPMAN, E. "Quality Ladders in the Theory of Growth." *The Review of Economic Studies*, Vol. 58(1) (1991), pp. 43–61.

HARRIS, C. AND VICKERS, J. "Racing with Uncertainty." *The Review of Economic Studies*, Vol. 54(1) (1987), pp. 1–21.

HOPENHAYN, H., LLOBET, G., AND MITCHELL, M. "Rewarding Sequential Innovators: Prizes, Patents, and Buyouts." *Journal of Political Economy*, Vol. 114(6) (2006), pp. 1041–1068.

HOPENHAYN, H.A. AND MITCHELL, M.F. "Innovation Variety and Patent Breadth." *RAND Journal of Economics*, Vol. 32 (2001), pp. 152–166.

HOROWITZ, A.W. AND LAI, E.L.C. "Patent Length and the Rate of Innovation." *International Economic Review*, Vol. 37(4) (1996), pp. 785–801.

HUNT, R.M. "Patentability, Industry Structure, and Innovation." *The Journal of Industrial Economics*, Vol. 52(3) (2004), pp. 401–425.
JAFFE, A.B. AND LERNER, J. "Innovation and Its Discontents: How Our Broken Patent System Is Endangering Innovation and Progress, and What to Do About It." Princeton University Press, 2011.
KIEDAISCH, C. "Intellectual Property Rights in a Quality-Ladder Model with Persistent Leadership." *European Economic Review*, Vol. 80 (2015), pp. 194–213.
KLEMPERER, P. "How Broad Should the Scope of Patent Protection Be?" *RAND Journal of Economics*, Vol. 21(1) (1990), pp. 113–130.
KOO, B. AND WRIGHT, B.D. "Dynamic Effects of Patent Policy on Sequential Innovation." *Journal of Economics & Management Strategy*, Vol. 19(2) (2010), pp. 489–512.
KRASTEVA, S. "Imperfect Patent Protection and Innovation." *The Journal of Industrial Economics*, Vol. 62(4) (2014), pp. 682–708.
LEE, T. AND WILDE, L.L. "Market Structure and Innovation: A Reformulation." *The Quarterly Journal of Economics*, Vol. 94(2) (1980), pp. 429–436.
LEMLEY, M.A. AND SHAPIRO, C. "Probabilistic Patents." *Journal of Economic Perspectives*, Vol. 19(2) (2005), pp. 75–98.
LERNER, J. "The Empirical Impact of Intellectual Property Rights on Innovation: Puzzles and Clues." *The American Economic Review*, Vol. 99(2) (2009), pp. 343–348.
LOURY, G.C. "Market Structure and Innovation." *Quarterly Journal of Economics*, Vol. 93(3) (1979), pp. 395–410.
MARSHALL, G. AND PARRA, A. "Innovation and Competition: The Role of the Product Market." *International Journal of Industrial Organization*, Vol. 65 (2019), pp. 221–247.
O'DONOGHUE, T. "A Patentability Requirement for Sequential Innovation." *RAND Journal of Economics*, Vol. 29(4) (1998), pp. 654–679.
O'DONOGHUE, T., SCOTCHMER, S., AND THISSE, J.-F. "Patent Breadth, Patent Life, and the Pace of Technological Progress." *Journal of Economics & Management Strategy*, Vol. 7(1) (1998), pp. 1–32.
O'DONOGHUE, T. AND ZWEIMÜLLER, J. "Patents in a Model of Endogenous Growth." *Journal of Economic Growth*, Vol. 9(1) (2004), pp. 81–123.
PRICEWATERHOUSECOOPERS. Patent Litigation Study, 2018.
QIAN, Y. "Do National Patent Laws Stimulate Domestic Innovation in a Global Patenting Environment? A Cross-Country Analysis of Pharmaceutical Patent Protection, 1978–2002." *The Review of Economics and Statistics*, Vol. 89(3) (2007), pp. 436–453.
REINGANUM, J.F. "A Dynamic Game of R and D: Patent Protection and Competitive Behavior." *Econometrica*, Vol. 50(3) (1982), pp. 671–688.
———. "Uncertain Innovation and the Persistence of Monopoly." *The American Economic Review*, Vol. 73(4) (1983), pp. 741–748.
SCOTCHMER, S. "Standing on the Shoulders of Giants: Cumulative Research and the Patent Law." *The Journal of Economic Perspectives*, Vol. 5(1) (1991), pp. 29–41.
———. "On the Optimality of the Patent Renewal System." *RAND Journal of Economics*, Vol. 30(2) (1999), pp. 181–196.
SCOTCHMER, S. AND GREEN, J. "Novelty and Disclosure in Patent Law." *RAND Journal of Economics*, Vol. 21(1) (1990), pp. 131–146.
SEGAL, I. AND WHINSTON, M.D. "Antitrust in Innovative Industries." *The American Economic Review*, Vol. 97(5) (2007), pp. 1703–1730.
SEGERSTROM, P.S. "Intel Economics." *International Economic Review*, Vol. 48(1) (2007), pp. 247–280.
SEGERSTROM, P.S. AND ZOLNIEREK, J.M. "The R&D Incentives of Industry Leaders." *International Economic Review*, Vol. 40(3) (1999), pp. 745–766.
TANDON, P. "Optimal Patents with Compulsory Licensing." *Journal of Political Economy*, Vol. 90(3) (1982), pp. 470–486.

Appendix A

This Appendix contains the proofs that are omitted from the main text.

Proof of Theorem 1. At $t = 0$, $x_{l,0} = \lambda(v_0 - v_0) = 0$, and the first claim follows. Similarly, $x_{f,0} = \max\{0, \lambda((1-b)v_0 + q_0 - w_0)\}$. Equation (3) implies that $w_0 > q_0$ and, therefore, followers do not invest in R&D for sufficiently high b. Convergence of investments is given by $w_T = v_T = q$, therefore, $\ell_T = 0$ and $x_{f,T} = x_{l,T} = v_0 - q$. To show that investments are increasing toward the end of the patent life, replace (5) into (4), evaluate at $t = T$, and use equation (1) to obtain $v'_T = -\pi$ and $w'_T = 0$. Differentiating the firms' investment rates with respect to t, we obtain:

$$\frac{dx_{l,t}}{dt} = -v'_t \qquad \text{and} \qquad \frac{dx_{f,t}}{dt} = -(w'_t + b(v'_t - q'_t)).$$

Evaluating the derivatives at $t = T$ and using $q'_T = (r + n\lambda^2(v_0 - q))q > 0$, we obtain $x'_{l,T} = \pi > 0$ and $x'_{f,T} = \pi + q'_T > 0$, implying, by continuity, that both investments increase toward the end of the patent life. ■

Proof of Theorem 2. See discussion in the main text. ■

Proof of Lemma 3. Taking the limit of (2) and (3) when T goes to infinity and then using the Principle of Optimality, we obtain:

$$rv_\infty = \max_{x^l_{l,\infty}\geq 0} \{\pi + \lambda x_{l,\infty}(v_\infty - v_\infty) - n\lambda x_{f,\infty}(1-b)v_\infty - c(x_{l,\infty})\}.$$

$$rw_\infty = \max_{x_{f,\infty}\geq 0} \{\lambda x_{f,\infty}((1-b)v_\infty - w_\infty) - \lambda x_{-f,\infty}w_\infty - c(x_{f,\infty})\},$$

where v_∞ and w_∞ denote the value of being a leader and an follower under an infinitely long patent, and $x_{k,\infty}$ represents the investment of firm $k \in \{l, f\}$. Because leaders obtain no incremental value from an innovation, leaders perform no R&D; that is, the reversal-of-Arrow cannot occur. ■

Proof of Propostion 4. I start by proving the existence of a fixed-point. The online web Appendix II shows that there is a unique solution to the differential equation (7) satisfying the boundary condition. Thus, I just need to show that there is a fixed-point $v_0(\hat{v}_0) = \hat{v}_0$ for a positive value of $\hat{v}_0$.[17] Define $f(z) = v_0(z) - z$. Showing the existence of the fixed-point is equivalent to show that exists $\hat{v}_0 > 0$ such that $f(\hat{v}_0) = 0$.

I show existence by means of the Intermediate Value Theorem. Observe that ϕ and θ go to ∞ at a rate of z, when z goes to infinity. Then, it is easy to check that

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{\left(\frac{2\pi}{z} - z\mu^2(1-b) - r\right)\left(1 - \frac{1}{e^{\phi T}}\right) - \phi\left(1 + \frac{1}{e^{\phi T}}\right)}{\frac{\theta}{z}\left(1 - \frac{1}{e^{\phi T}}\right) + \frac{\phi}{z}\left(1 + \frac{1}{e^{\phi T}}\right)} = -\infty.$$

It remains to show that there is z such that $f(z) > 0$. The result follows from choosing $z = 0$. There, $f(0) = v_0(0) - 0$. Given the behavior of firms in an equilibrium, and because there is no benefit from developing a new innovation, we are in phase 0 (see the online web Appendix II) throughout the patent's life, so $v_0(0) = (\pi/r)(1 - exp(-rT)) > 0$.

To prove uniqueness, I make use of the fact that $f(z)$ is continuous and show that at any fixed-point $\hat{v}_0$, $f'(\hat{v}_0) < 0$; that is, $f(z)$ single-crosses zero from above. Define the function

$$\psi_t = \frac{e^{2\phi(T-t)} - 2\phi(T-t)e^{\phi(T-t)} - 1}{\phi(e^{\phi(T-t)} - 1)^2}. \tag{A1}$$

The online web Appendix III shows that $\psi_T = 0$ and, for all $t < T$, $\psi_t > 0$ and $\psi_t' < 0$.

Because it will be useful for the proof of Lemma 11 in the online web Appendix, I compute the derivative of $v_t(z) - z$ with respect to z, evaluated at $\hat{v}_0$, at any $t \leq T$,

$$\frac{dv_t(\hat{v}_0)}{dz} - 1 = -\frac{\lambda^2(\hat{v}_0 - v_t)^2 + \mu^2(1-b)v_t^2 + 2\pi + \psi_t k v_t^2}{2\pi + (\lambda\hat{v}_0)^2},$$

where $k = \mu^2(2\lambda^2 + \mu^2)(1-b)^2\hat{v}_0 + r(\lambda^2 + \mu^2(1-b))$ is a positive constant. Therefore, the previous derivative is negative for all t. In particular, the derivative is negative at $t = 0$, which corresponds to $f'(\hat{v}_0)$, and the result follows. ■

Proof of Theorems 5 and 6. Formally, we want to show that there exists $\hat{t} > 0$ such that for all $t < \hat{t}$, the derivative

$$\frac{dx_{l,t}}{dT} = \lambda\left(\frac{d\hat{v}_0}{dT}\left(1 - \frac{dv_t}{d\hat{v}_0}\right) - \frac{\partial v_t}{\partial T}\right) \tag{A2}$$

is negative. Making use of implicit differentiation, we can readily check that $dx_{l,0}/dT = 0$. From the proof of Lemma 11 in the online web Appendix IV, we know that $d\hat{v}_0/dT > 0$ and that $\partial v_t/\partial T > 0$ and increasing in t. Hence, a sufficient condition for the result to hold is to show that $dv_t/d\hat{v}_0$ increases with t around $t = 0$. The derivative of previous expression with respect to t at $t = 0$ is

$$\frac{d^2 v_0}{d\hat{v}_0 dt} = -\hat{v}_0 \frac{2\mu(1-b)v_0' + k\left(2v_0'\psi_0 + v_0\psi_0'\right)}{\hat{v}_0^2\lambda^2 + 2\pi},$$

where k and ψ_0 are the positive constant and positive function defined in the proof of Propostion 4. This derivative is positive as v_0' and ψ_0' are both negative, and the result follows. Finally, to show that the terminal investment increases, simply observe that $x_{l,T+dT} = \lambda\hat{v}_0$, which increases with T as proven by Lemma 11. Theorem 6 follows from the discussion in the text and the previous result. ■

Proof of Theorem 7. Followers decrease R&D at the beginning of the patent's life as:

$$\frac{dx_{f,0}}{db} = \mu\left(-\hat{v}_0 + (1-b)\frac{d\hat{v}_0}{db}\right) = -\frac{\mu\hat{v}_0\left(\psi_0\left(\mu^2(b(3-2b)-1)2\pi + \lambda^2 r\hat{v}_0\right)\hat{v}_0 + 2\pi\right)}{\mu^2(1-b)\hat{v}_0^2 + k\psi_0\hat{v}_0^2 + 2\pi},$$

[17] There may be other fixed points such that $\hat{v}_0 \leq 0$; however, those do not have an economic meaning and, consequently, are ignored.

where k is the positive constant defined in the proof of Propostion 4. This derivative is negative whenever $b \geq 1/2$. Followers increase R&D at toward the end of the patent's life as $x_{f,T} = \mu \hat{v}_0$, which increases whenever $\hat{v}_0$ increases. Similar argument can be applied for the second claim. ■

Proof of Theorem 8. Observe that (9) can be written as

$$\mathbb{E}[t] = \int_0^T x_t t e^{-z_{0,t}} dt + e^{-z_{0,T}} \left(T + \frac{1}{(\lambda^2 + \mu^2) v_0} \right).$$

Taking the limit when $T \to 0$ shows that $\mathbb{E}[t] \to \infty$ as the value of a new innovation v_0 converges to zero, precluding $T = 0$ to be optimal. I show $T^* < \infty$ by contradiction. Start by assuming that $T^* = \infty$. Then, investments become stationary and $\mathbb{E}[t] = (\mu x_f)^{-1}$, where x_f represents the followers' R&D investment under an infinitely long patent. By Lemma 12 in the online web Appendix IV, x_f is decreasing in b. Thus, the policy $(T, b) = (\infty, 0)$ is the only candidate for optimality if $T = \infty$ were to be optimal. When $b = 0$, the followers' investments are constant for any patent duration T and equal to $x_{f,t} = \mu \hat{v}_0$ for all t. This scenario is the only case where $\mathbb{E}[t]$ can be solved analytically. For $(T, b) = (\infty, 0)$ to be a minimum, we need $\mathbb{E}[t]$ to converge to $(\mu x_f)^{-1}$ from above, as T approaches infinity. The online web Appendix V shows that $\mathbb{E}[t]$ converges from below, contradicting $T^* = \infty$ and proving the result. ■

Appendix B

This Appendix derives the system of Ordinary Differential Equations (ODE) describing the equilibrium behavior of the baseline model and summarizes the numerical method used to solve for the equilibrium.

Using the quadratic cost assumption and the optimal investments (5), the system of ODE describing how v_t and w_t evolve throughout t is given by:

$$-v_t' = \pi + \frac{x_{l,t}^2}{2} - n\lambda x_{f,t}(bq_t + (1-b)v_t) - rv_t \qquad -w_t' = \frac{x_{f,t}^2}{2} - (\lambda x_{-f,t} + r)w_t. \tag{B1}$$

To describe the numeric method, first observe that the maximum value that a leader can obtain for an innovation is to receive the profit π forever. Thus, the value of being the leader is bounded above by π/r. The model is solved as follows:

1. Define V_p to be a partition of $[0, \pi/r]$. Each element of V_p will be tested as a candidate for v_0.
2. Fix $v \in V_p$. Start with $n = 0$ and define dn to be a small increase in n.
   (a) As a function of (v, n), compute the continuation value $q(v, n)$ using equation (1).
   (b) Starting from $q(v, n)$, use the system of ODEs equation (B1) backward to compute the initial values of being a leader and a follower, $v_0(v, n)$ and $w_0(v, n)$.
   (c) If $w_0(v, n) > K$, increase n in dn and go back to step 2.(a). If $w_0(v, n) \leq K$, save results as a pair $(v, n(v))$. Start step 2 with a different $v \in V_p$.[18]
3. Once all the pairs $(v, n(v))$ have been computed, the solution (v_0, n^*) corresponds to the pair $(v, n(v))$ that minimizes $\|v_0(v, n(v)) - v\| + \|w_0(v, n(v)) - K\|$.

[18] This step uses that $w_0(v, n)$ is monotonically decreasing in n.