Efficiency in Second-Price Auctions with Participation Costs*

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Abstract

We study equilibrium efficiency in second-price auctions with participation costs. Costly participation generates three inefficiencies: insufficient entry, excessive entry, and misallocation of the good. Even though all equilibria are ex-post inefficient, an ex-ante efficient equilibrium always exists. We explain why asymmetric equilibria are more efficient than the symmetric equilibrium in symmetric games.

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1 Introduction

An important question in competition policy and market design is whether efficient entry occurs when firms follow their private incentives. Mankiw and Whinston (1986) showed that excessive entry happens when decisions are driven by business stealing (see also Amir *et al.*, 2014). Their analysis focuses on markets with an ample supply of fully-informed entrants. However, with a limited number of firms privately-informed about how much they value entering, their conclusions might not carry through.

We revisit the question of entry efficiency in the context of a second-price auction where n privately-informed potential bidders deciding *when* (as opposed to *whether*) to enter. Despite every equilibrium being ex-post inefficient, we show that the auction has an ex-ante efficient equilibrium. More generally, if the seller sets a reservation price equal to its valuation, every equilibrium matches a critical point of the social welfare function. The marginal social contribution of entry at a given valuation is the gap between the first and the second-highest valuation (minus the entry cost). In a second-price auction, this gap corresponds to the difference between the valuation and the price paid by a successful entrant, inducing private and social incentives to coincide.

This article contributes to the literature on entry to auctions, which is divided into two broad classes of informational assumptions. Levin and Smith (1994) study entry in environments with fully-informed bidders (see also McAfee and McMillan, 1987; Tan, 1992; Jehiel and Lamy, 2015). We build upon Samuelson (1985), who studied costly entry into a symmetric auction with bidders that are privately informed about their valuation. Within this framework, Campbell (1998) studies coordinated entry, whereas Tan and Yilankaya (2007) examine collusive outcomes, and Menezes and Monteiro (2000) study optimal auction design. Tan and Yilankaya (2006), Cao and Tian (2013) and Espín-Sánchez *et al.* (2021) identify conditions for when the auction has a unique entry equilibrium.

Our welfare analysis expands the early work of Stegeman (1996) (see also Lu, 2009), who —following a mechanism design approach— indirectly shows the existence of an efficient equilibrium. We build upon them by providing direct proof of the feasibility of efficiency. By making the tradeoffs faced by a planer explicit, our method allow us to link welfare outcomes with the game's equilibrium structure, providing better intuitions and a deeper understanding of the connection between welfare and market outcomes. We organized the article as follows. Section 2 introduces the model, Section 3 the main result, and Section 4 discusses the findings, relating them to the literature. We relegate proofs to the Appendix.

2 Model, Payoffs, and Strategies

Set up. Consider a sealed-bid second-price auction with independent private values. The auction consists of one seller, n potential bidders, and one indivisible good. The seller values the good in $v_0 \geq 0$ and sets a reserve price $r \geq 0$. Each bidder $i \in \{1, 2, \ldots, n\}$ independently observes her valuation for the object, v_i, and then chooses to participate in the auction by paying a participation cost of K_i. The valuation v_i is drawn from a finite expectation, atomless, continuously differentiable distribution function F_i with full support on $[0, \infty)$. The distribution of valuations, participation costs, and the number of potential bidders are commonly known by every player.

Strategies, payoffs, and equilibrium. The post-entry game matches a traditional second-price auction. In equilibrium, bidders submit a bid equal to their valuation v_i. We focus the rest of our analysis on participation decisions. Define a *cutoff* strategy as a threshold x_i such that bidder i enters the auction whenever she values the good by at least x_i (i.e., when $v_i \geq x_i$) and stays out otherwise.

Without loss of generality, we simplify the equilibrium characterization by ordering the bidders' identities according to their participation cutoffs. Bidder 1 has the lowest participation cutoff, and bidder n has the highest. Given vector of cutoff strategies $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, define: i) $A_i^k = \prod_{j>i}^{k} F_j(x_j)$ to be the probability that bidders playing cutoffs greater than bidder i, up to bidder k, do not enter the auction; and, ii) $B_i(v) = \prod_{j<i} F_j(v)$, the probability that bidders playing cutoffs lower than bidder i obtain valuations lower than v.

Denote by $\mathbf{x}_i = (x_1, x_2, \ldots, x_i)$ the vector containing the cutoff strategies of bidder 1 up to bidder i. Given $\mathbf{x}_i$, bidder i's expected *revenue* of participating in an auction with i potential participants, having a valuation v_i equal to x_i, and the other $i-1$ bidders play cutoffs lower than x_i is:[1]

$$R_i(x_i; \mathbf{x}_{i-1}) = x_i B_i(x_i) - r A_0^{i-1} - \sum_{j=1}^{i-1} \left(A_j^{i-1} \int_{x_j}^{x_{j+1}} \max\{r, s\} dB_{j+1}(s) \right).$$

Bidder i's revenue consists of its value, x_i, times the probability of obtaining the highest valuation (and winning the good), $B_i(x_i)$, minus the expected price paid. With probability A_0^{i-1} bidder i is the sole participant, paying the reserve price r. When bidder i faces competition, it pays the maximum between the reserve price and the highest competitors' bid. The maximum competitors' bid falls in the interval $[x_j, x_{j+1})$

[1] We use the following notation throughout the article: $\sum_{\emptyset} = 0$ and $\prod_{\emptyset} = 1$.

when competitors playing cutoffs higher or equal to x_{j+1} stay out of the auction, event occurring with probability A_j^{i-1}. Consequently, the price paid by bidder i distributes according to B_{j+1} in such interval.

A Bayesian equilibrium is a vector of cutoff strategies $\mathbf{x}$ such that each bidder i satisfies the following condition:

$$A_i^n R_i(x_i, \mathbf{x}_{i-1}) = K_i \tag{1}$$

In equilibrium, each bidder must be indifferent to participating in the auction when drawing a valuation equal to its participation cutoff x_i. At that valuation, bidder i loses the object when any bidder playing a higher cutoff participates in the auction. This event occurs with probability $1 - A_i^n$ and leaves bidder i with zero revenue. Consequently, bidder i only makes revenue with probability A_i^n. In this scenario, bidder i is the participating bidder with the highest participation cutoff, receiving $R_i(x_i, \mathbf{x}_{i-1})$. Bidder i is indifferent when her expected revenue equals her participation costs K_i.

3 Welfare

When participating in the auction is costly, ex-ante and ex-post efficiency are not equivalent. After participation decisions are made, the auction delivers an efficient outcome as long as the seller sets a reservation price equal to its valuation (i.e., $r = v_0$). From an ex-ante perspective, however, a social planner trades off the cost of inducing participation with the benefits of a better ex-post allocation. This tradeoff generates *ex-post* misallocation with positive probability.

Figure 1 illustrates the previous point by depicting an equilibrium with two bidders with equal participation costs ($K_i = K$), no reserve price ($r = 0$), and different cutoff equilibrium strategies ($x_1 < x_2$). An allocation is *ex-post* efficient when the bidder with the highest valuation participates in the auction. Three types of inefficiencies arise: (i) *insufficient participation* (dark-shaded area): at least one bidder values the good more than its participation costs, but every bidder stays out of the auction; (ii) *excessive participation* (lightly-shaded area): represents situations in which both bidders enter the auction, paying excessive participation costs, and; (iii) *misallocation* (dotted area): only the low valuation bidder participates in the auction.

From an ex-ante perspective, however, there is an efficient equilibrium. Consider a planner prescribing each bidder an entry strategy that depends on the bidder's private

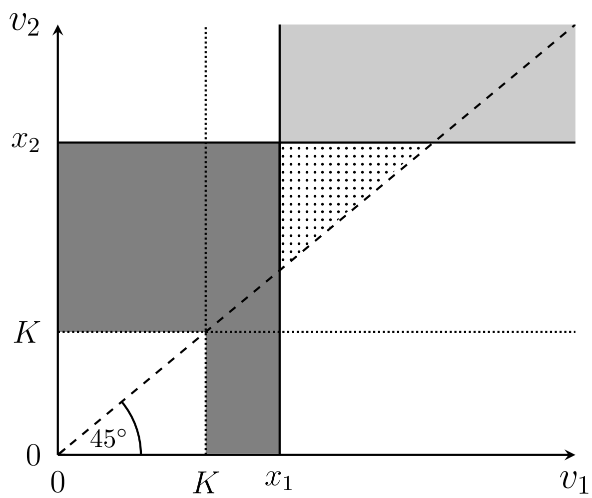

Figure 1: Ex-post inefficiency. The dark-shaded area represents realization of valuations in which there is insufficient participation by both firms; the dotted represents realization with misallocated entry, and; the light-shaded realizations with excessive participation.

information. The planner chooses a vector of cutoffs $\mathbf{x}$ that maximizes

$$W(\mathbf{x}) = v_0 A_0^n + \sum_{i=1}^{n} \left\{ \int_{x_i}^{\infty} [v_i \Omega_i(v_i, x_{-i}) - K_i] \, dF_i(v_i) \right\} \tag{2}$$

where $\Omega_i(v_i, x_{-i}) = \prod_{k \neq i} F_k(\max\{v_i, x_k\})$ is the probability that bidder i obtains the object when her valuation is v_i and opponents play according to $\mathbf{x}_{-i}$. Notice that transfers between the winning bidder and the seller are irrelevant from the point of view of welfare. The total welfare consists of the seller's value v_0 when nobody obtains the good, which occurs with probability A_0^n, plus the expected welfare obtained from each bidder. With probability $dF_i(v_i)$, bidder i draws valuation v_i and enters the auction whenever $v_i \geq x_i$, paying the participation cost K_i and winning the object with probability $\Omega_i(v_i, x_{-i})$. Bidder i's contribution to welfare is integrated over the values for which the bidder enters.

Proposition. *If the seller sets a reservation price equal to his valuation, an ex-ante efficient equilibrium exists. Furthermore, every equilibrium* $\mathbf{x}$ *is a critical point of the welfare function and is either a (possibly local) maximum or saddle point.*

The necessity of the seller setting a reserve price equal to its valuation mirrors the classical auction result without entry costs. Suppose the seller sets a different reserve price. In that case, there is the risk of allocating the good to a bidder with a lower valuation than the seller (when $r < v_0$) or the seller keeping the good when a participating bidder values it more ($r > v_0$).

To see the intuition for a critical point in the welfare function being an equilibrium, take the case where $v_0 = r = 0$. Consider the social contribution of a marginal decrease in bidder i's entry cutoff, x_i. By decreasing her cutoff, bidder i enters the auction on a larger range of values, paying the entry cost K_i more often but increasing her expected revenue by $x_i \Omega_i(\mathbf{x})$. The decrease in x_i also decreases the opponents' probability of obtaining the good for those bidders playing entry cutoffs lower than x_i, as bidder i can outbid them. In these cases, bidder i's social contribution is the gap between bidder i's valuation and the now second-highest valuation. Because in a second-price auction the price paid is the second-highest valuation, the social contribution of the gap in valuations matches precisely bidder's i private gain. That is, social and private tradeoffs coincide. At each inflection point these tradeoffs equilibrate, generating an equilibrium.

4 Discussion

The previous efficiency result stands in contrast to Levin and Smith (1994), which shows that every equilibrium in a symmetric complete-information auction with costly participation is ex-ante efficient. Under equilibrium multiplicity, the different equilibria weight the three ex-post inefficiencies differently, leading to different welfare outcomes. Different expectations may lead bidders to coordinate in inefficient equilibria. In symmetric games, where the symmetric equilibrium might be a natural focal point, the outcome might be inefficient, as asymmetric equilibria might be more efficient.

Figure 2 illustrates the previous point. The symmetric equilibrium induces no misallocation at the cost of paying excessive participation costs. In terms of welfare, the symmetric equilibrium is a saddle point of the welfare function. On the other hand, asymmetric equilibria minimizes the cost of excessive entry by choosing a bidder to enter almost every time her value is above the participation costs and the other bidder only participates when her value is high. The asymmetric equilibiria, however, does induce some expost misallocation but it is, over all, more efficient.

There is deeper connection between equilibrium uniqueness and efficient outcomes. To illustrate this connection consider an scenario with $n = 2$ potential bidders. The Hessian of the planner's problem, evaluated at a critical point, is equal to:

$$H(\mathbf{x}) = -\begin{pmatrix} f_1(x_1)F_2(x_2) & x_1 f_1(x_1) f_2(x_2) \\ x_1 f_1(x_1) f_2(x_2) & f_2(x_2)F_1(x_2) \end{pmatrix}.$$

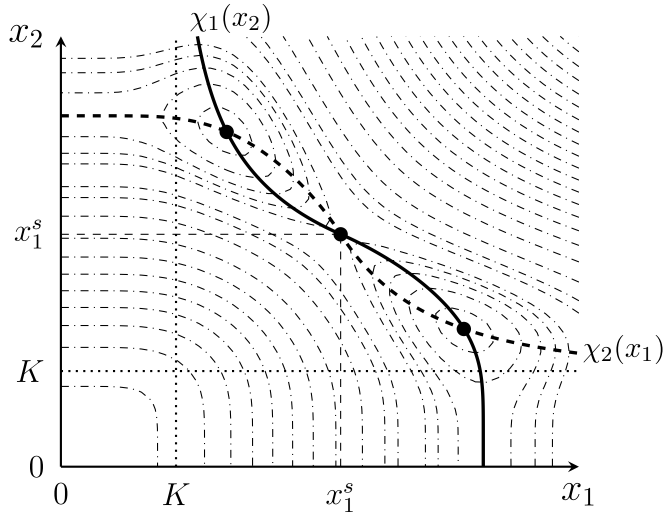

Figure 2: Welfare and multiple equilibria. Depiction of iso-welfare curves and best response functions $\chi_i(x_j)$ of a symmetric entry game with valuations distributing standard log-normal, $K = 1$, and $r = v_0 = 0$. Asymmetric equilibria maximize welfare. The symmetric equilibrium is a saddle point.

Espín-Sánchez *et al.* (2021) show that there is a unique equilibrium in the entry game when the distribution of valuations, F_i, are concave (i.e., $F_i(x) \geq x f_i(x)$ for every $x > 0$). We can show that concavity also implies that the second-order condition for a maximum is satisfied at *every* critical point. In turn, this implies that at most one equilibrium exists, delivering equilibrium uniqueness. In other words, concavity of the CDF implies concavity of the welfare function. To see this, notice that at every equilibrium $x_1 < x_2$, the first minor of $H(\mathbf{x})$ is always negative and

$$\det(H(\mathbf{x})) = f_1(x_1) f_2(x_2) \left(F_1(x_1) F_2(x_2) - (x_1)^2 f_1(x_1) f_2(x_2) \right) > 0.$$

Therefore, only one critical point exists and the game has a unique, efficient equilibrium. This result implies concavity of the CDF is sufficient to guarantee both uniqueness and efficient outcomes.

Appendix

Proof of the Proposition. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ where, without loss of generality, we order the bidders identities from the lowest cutoff chosen by the planer, x_1, to the highest, x_n. Differentiating (2) with respect to x_i we obtain

$$W_{x_i}(\mathbf{x}) = f_i(x_i)v_0 A_0^{i-1} A_i^n + f_i(x_i)(K_i - x_i \Omega_i(\mathbf{x})) + \sum_{j \neq i} \int_{x_j}^{\infty} s \frac{d\Omega_j(s, \mathbf{x}_{-j})}{dx_i} dF_j(s).$$

Observe that $\Omega_j(v, \mathbf{x}_{-j}) = B_k(v) A_{k-1}^n / F_j(v)$ where k is the index of the smallest cutoff in $\mathbf{x}_{-j}$ satisfying $x_k > v$. The divisor $F_j(v)$ always cancels out with an element of $B_k(v)$ or A_{k-1}^n depending of the value of k. In particular, $\Omega_i(\mathbf{x}) = B_i(x_i)A_i^n$. Then, we obtain $d\Omega_j(v, \mathbf{x}_{-j})/dx_i = B_k(v) A_{k-1}^{i-1} f_i(x_i) A_i^n / F_j(v)$ if $v \leq x_i$ (and k defined as above) and zero otherwise ($v > x_i$). Using these observations, we can write

$$\int_{x_j}^{\infty} s \frac{d\Omega_j(s, \mathbf{x}_{-j})}{dx_i} dF_j(s) = f_i(x_i) A_i^n \left(A_j^{i-1} \int_{x_j}^{x_{j+1}} s B_j(s) dF_j(s) + A_{j+1}^{i-1} \int_{x_{j+1}}^{x_{j+2}} s B_j(s) F_{j+1}(s) dF_j(s) \right.$$
$$\left. + \cdots + \int_{x_{i-1}}^{x_i} s B_j(s) \prod_{\ell=j+1}^{i-1} F_\ell(s) dF_j(s) + 0 \right) = A_i^n \sum_{k=j}^{i-1} A_k^{i-1} \int_{x_k}^{x_{k+1}} s B_j(v) \prod_{\ell=j+1}^{k} F_\ell(s) dF_j(s)$$

and $W_{x_i}(\mathbf{x})$ becomes

$$- f_i(x_i) A_i^n \left[x_i B_i(x_i) - v_0 A_0^{i-1} - \sum_{j=1}^{i-1} \sum_{k=j}^{i-1} A_k^{i-1} \int_{x_k}^{x_{k+1}} s B_j(v) \prod_{\ell=j+1}^{k} F_\ell(s) dF_j(s) - K_i \right]. \quad (3)$$

Corner solutions are not welfare maximizing as $x_i = 0$ satisfies $W_{x_i}(0, \mathbf{x}_{-i}) > 0$ for all $\mathbf{x}_{-i}$, and $\lim_{x_i \to \infty} W_{x_i}(x_i, \mathbf{x}_{-i}) < 0$ due to the unboundedness of $x_i B_i(x_i)$. Therefore, an interior maximum exists and is characterized by a value of x_i satisfying $W_{x_i}(\mathbf{x}) = 0$. Using $dB_{j+1}(s) = \sum_{k=1}^{j} B_k(v) \prod_{\ell=k+1}^{j} F_\ell(s) dF_k(s)$ to re-arrange the double summation in (3) we obtain

$$W_{x_i}(\mathbf{x}) = -f_i(x_i) A_i^n \left[x_i B_i(x_i) - v_0 A_0^{i-1} - \sum_{j=1}^{i-1} \left(A_j^{i-1} \int_{x_j}^{x_{j+1}} s dB_{j+1}(s) \right) - K_i \right]. \quad (4)$$

The planner's first order condition (4) coincides with (1) if and only if $r = v_0$. In this case, every equilibrium is a critical point of W. Furthermore, because $W_{x_i, x_i}(\mathbf{x}) = -f_i(x_i)\Omega_i(\mathbf{x}) < 0$, the critical points cannot be a minimum. Thus, every equilibria is a local maximal or a saddle point. In particular, one of the equilibria is efficient. ■

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