Online Appendix

Equilibrium Uniqueness in Entry Games with Private Information

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C Equilibrium Exists and is in Cutoff Strategies

An entry strategy for firm i is a mapping from the firm's type v_i to a probability of entering in the market $\tau_i : [a, b] \to [0, 1]$. We assume that the strategy of firm i is an integrable function with respect to its own type v_i. We study the Bayesian Equilibria of the entry game. Denote by $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ the vector of entry strategies. Given a strategy profile τ, the expected profit of firm i *after* drawing the type v_i but *before* entry decisions are realized is

$$\Pi_i(v_i; \tau) = \tau_i(v_i) \left[\sum_{e \in E_i} \left\{ \int_{[a,b]^{n-1}} \pi_i(v_e) \Pr[e|\tau_{-i}, v_{-i}] \phi(v_{-i}) d^{n-1} v_{-i} \right\} \right] \tag{C.1}$$

where $\Pr[e|\tau_{-i}, v_{-i}]$ is the probability of observing market structure e, given the vector of strategies τ_{-i} and the realizations of types v_{-i}. The integral is over each of the $n-1$ dimensions of firm i's competitors types, v_{-i}. Conditional on i's entry, which occurs with probability $\tau_i(v_i)$, the expected profit of firm i consists of the expected sum of profit that firm i would get under each feasible market structure, which is induced by the vector of strategies τ and the realization of types v_{-i}, integrated over all possible realizations of the competitors' types, $\phi(v_{-i})$.

Definition (Cutoff Strategy)**.** A strategy $\tau_i(v_i)$ is called *cutoff* if there exists a threshold $x > 0$ such that

$$\tau_i(v_i) = \begin{cases} 1 & \text{if } v_i \geq x \\ 0 & \text{if } v_i < x \end{cases} .$$

A cutoff strategy specifies whether a firm enters a market with certainty depending on whether its type is above or below some given threshold. In any best response, there exists a type, v_i, that makes a firm indifferent to enter the market. We break this indifference by assuming that firms enter. For a cutoff strategy, this means that a firm enters when its type is greater or equal to its cutoff. Given a vector τ_{-i}, a best response is given by the strategy $\hat{\tau}_i$ that maximizes (C.1) at every value of v_i.

A Bayesian Nash equilibrium is defined by a vector of strategies τ in which every firm best respond to each other. The next proposition establishes the existence of an equilibrium and that, without loss of generality, we can restrict our analysis to cutoff strategies.

Lemma C.1. *For any game $(\pi_i, F_i)_{i=1}^n$ satisfying assumptions A1-A3, there exists an equilibrium. For any vector τ_{-i}, firm i's best response is a cutoff strategy. Therefore, every equilibrium of the game is in cutoff strategies.*

Proof of Lemma C.1.
best responses are cutoff strategies: Fix any firm i and vector of strategies τ. By assumptions A3 and A2, we know that in equilibrium no firm will enter if they draw $v_j < \underline{v}_j$. For relevance, impose that τ satisfies the restriction $\tau_j(v_j) = 0$ in that range. Because firm i's profit is linear in τ_i, firm i's best response is to participate with probability one whenever there is a positive payoff of doing so. Suppose firm i enters the market with certainty ($\tau_i(v_i) = 1$). The restriction above implies that there is positive probability that firm i is the sole entrant to the market and, consequently, by A1, profits are strictly increasing in v_i. By A3, $\Pi_i(\underline{v}_i; \tau) < 0$, and $\Pi_i(\overline{v}_i; \tau) > 0$. Thus, $\Pi_i(v_i; \tau)$ single crosses zero and i's best response to τ_{-i} is the cutoff strategy defined by the value x_i that satisfies $\Pi_i(x_i; \tau_i = 1, \tau_{-i}) = 0$.

Existence: We check the conditions of Brouwer's fixed-point theorem. Because F_i is atomless and has full support and $\pi_i(v_e)$ being continuous and differentiable in v_i, firm i's best response lies in the compact and convex set $[\underline{v}_i, \bar{v}_i]$. Thus the n-dimensional function of best responses is a continuous mapping from $\times_{i=1}^n [\underline{v}_i, \bar{v}_i]$ to itself and the conditions for the proposition are met. ■

Existence follows from Brouwer's fixed-point theorem. The restriction to cutoff strategies is quite intuitive: regardless of which strategy competitors are playing, assumption A1 implies that firm i's expected profit is strictly increasing in its type. Because i's expected profit is linear in its entry probability (see eq. (C.1)), i either prefers to enter with certainty, when it is profitable to do so, or to stay out. The next Lemma characterizes all cutoff equilibria.

Lemma C.2. *The vector* $\mathbf{x}$ *of cutoff strategies constitutes an equilibrium if and only if* $\Pi_i(\mathbf{x}) = 0$ *for every firm* i.

Proof of Lemma C.2. By the previous proof a cutoff strategy is defined as the value x_i satisfying $\Pi_i(x_i; \tau_i = 1, \tau_{-i}) = 0$. Because in a cutoff equilibrium $\Pr[e|\tau, v_i]$ is either zero or one. Integrating (C.1) over payoff-irrelevant firms delivers (6). ■

Lemma C.2 characterizes every equilibrium of the entry game. Firm i's best response to $\mathbf{x}_{-i}$ is defined by a cutoff x_i equal to the value of v_i that satisfies $\Pi_i(v_i; \mathbf{x}_{-i}) = 0$. A profile of equilibrium cutoffs $\mathbf{x}$ is, thus, constructed by the collection of functions $\Pi_i(\mathbf{x})$ evaluated at a point in which every firm i is indifferent between entering the market when drawing type x_i.

D Second Price Auction

D.1 Alternative notions for Strength

In this section, we explore alternative notions for strength. In particular, we study the relationship between: (i) the cutoff strategies, x_i; (ii) the *ex-ante* probability of participating in the auction, $1 - F_i(x_i)$; and (iii) the *ex-ante* expected payoff of each bidder; which, for a given vector of cutoffs strategies $\mathbf{x} = (x_1, x_2)$, is equal to:

$$U_i(\mathbf{x}) = \int_{x_i}^{\infty} \left(vF_j(\max\{v, x_j\}) - \int_{x_j}^{\max\{v,x_j\}} s dF_j(s) - K_i \right) dF_i(v). \tag{D.1}$$

That is, for each valuation v_i under which bidder i participates (i.e., for each $v_i > x_i$), the expected payoff of participating in the auction, weighted by the probability that v_i occurs.

We explore the relation between the previous objects by means of an example. Consider two asymmetric bidders whose distribution of valuations follows a Generalized Pareto distribution (GPD) with shape parameter κ and scale parameter σ.[28] The choice of GPD yields a simple concave distribution with positive support that is flexible enough to change its mean and variance. Suppose both bidders have a symmetric participation cost K, but bidder 1 is characterized by $(\kappa_1, \sigma_1) = (0, 1)$ and bidder 2 by $(\kappa_2, \sigma_2) = (0.25, 0.75)$. Both distributions have the same mean but the second distribution has twice the variance. That is, the second distribution is a mean-preserving spread of the first. Because the CDFs cross, distributions are *not* ordered by FOSD. Consequently, the game is not *ordered* and it is not self-evident which bidder is stronger.

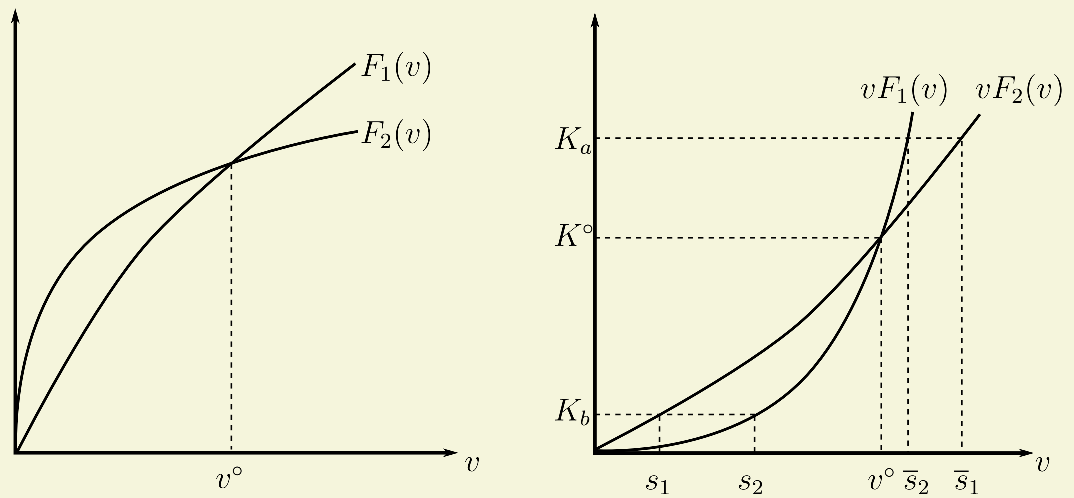
Intuitively, the stronger bidder would be the one whose distribution of valuations has more mass to the right of the equilibrium cutoffs strategies, as this implies the bidder is more likely to obtain higher valuations. If the equilibrium cutoff strategies are high, then bidder 2 would have more mass to the right of the cutoffs, and thus bidder 2 would be the stronger bidder. High equilibrium cutoff strategies are likely to occur when participation costs are high. Conversely, if the cutoff strategies are low, then bidder 1 would have more probability mass to the right of the cutoffs, and thus bidder 1 would be the stronger bidder. Low equilibrium cutoff strategies are likely to occur when participation costs are low.

This situation is illustrated in Figure 5. Panel (a) shows that both distributions are concave, thus Lemma 2 implies that the participation game has a unique equilibrium for any participation costs $K > 0$. Panel (a) also shows that both distributions cross at $v^\circ = 2.2007$. Panel (b) depicts the bidders' strength. It shows that bidders are equally strong when $K^\circ = 1.957$. For participation costs above K°, bidder 2 is stronger ($s_2 < s_1$) and, in the unique equilibrium, bidder 2 plays a lower cutoff strategy ($x_2 < x_1$). For instance, if $K_a = 2 > K^\circ$, then the vector of equilibrium cutoffs is $\mathbf{x} = (2.241, 2.238)$. Alternatively, when $K < K^\circ$, bidder 1 is stronger ($s_1 < s_2$) and plays a lower equilibrium cutoff strategy ($x_1 < x_2$). For

[28] For $\kappa \in \mathbb{R}$ and $\sigma \in (0, \infty)$, the Generalized Pareto CDF is defined over $\mathbb{R}_+$ and given by

$$F(x|\kappa, \sigma) = \begin{cases} 1 - \left(1 + \frac{\kappa x}{\sigma}\right)^{-\frac{1}{\kappa}} & \kappa \neq 0 \\ 1 - e^{-\frac{x}{\sigma}} & \kappa = 0 \end{cases}.$$

The CDF is concave whenever $\kappa > -1$, its mean is well defined for $\kappa < 1$ and given by $\sigma/(1-\kappa)$, whereas its variance is defined for $\kappa < 1/2$ and given by $\sigma^2/(1-\kappa)^2(1-2\kappa)$.



(a) Distributions are concave **(b)** Strength under different participation costs

Figure 5: Strength under second-order stochastic dominance. Distributions are Generalized Pareto with parameters $(\kappa_1, \sigma_1) = (0, 1)$ and $(\kappa_2, \sigma_2) = (0.25, 0.75)$ respectively. Panel (a) shows that distributions cross at $v^\circ = 2.2007$. Panel (b) shows that, depending on the entry cost, either bidder can be stronger.

example, if $K_b = 1 < K^\circ$, then the equilibrium is $\mathbf{x} = (1.281, 1.383)$.

When the participation cost is equal to K°, bidders are equally strong ($s_i = v^\circ$). Because the CDFs are concave, the unique equilibrium is given by the symmetric cutoffs equal to the bidders' strength ($x_i = v^\circ$). The expected payoff of bidder 2, however, is greater than the expected payoff of bidder 1. Using equation (D.1), we obtain $(U_1, U_2) = (0.103, 0.185)$. This means that although bidders' cutoffs are not ranked, their expected profits are. The intuition in this scenario follows from $F_2(v) < F_1(v)$ for every $v > v^\circ$. Relative to bidder 1, bidder 2's valuations (distributed according to $F_2(v)$) are skewed to the right tail of the distribution, whereas their expected payment price (distributed according to $F_1(v)$) is skewed towards the left (see Figure 5a). In other words, for valuations greater than v°, bidder 2's conditional distribution of valuations FOSD the bidder 1's conditional distribution.

Beginning from the previous example, we construct an equilibrium in which bidder 1 receives a lower expected payoff than bidder 2, despite playing a lower participation cutoff and having a higher participation probability. By decreasing bidder 1's participation cost, bidder 1 becomes stronger than bidder 2 and will play a lower cutoff in the unique equilibrium of the game. By continuity, if the decrease in bidder 1's cost is small, we can construct an equilibrium with said characteristics. Take for example $(K_1, K_2) = (1.9, K^\circ)$, then bidder 1 is stronger and plays a lower cutoff—in this case $\mathbf{x} = (2.1327, 2.2196)$—but also receives lower expected payoffs $(U_1, U_2) = (1.11, 1.83)$. At a cutoff equal to v°, both bidders are equally likely to enter. Thus, $x_1 < v^\circ < x_2$ implies that bidder 1 is simultaneously more likely to participate and receive a lower expected payoff.

Finally, to show that cutoff order need not coincide with entry-probability order, modify the participation costs to $(K_1, K_2) = (1.1, 1)$. In this scenario, bidder 1 plays a higher entry cutoff $x_1 = 1.434 > 1.313 = x_2$ while also participating more frequently $1 - F_1(x_1) = .238 > .234 = 1 - F_2(x_2)$.

D.2 Example of Non-Existence of a Herculean Equilibrium when the Game is not Ordered

We provide an example of a non-ordered game with three entrants which does not have a herculean equilibrium. Suppose the three bidders have identical entry costs, $K = 1$, and the distributions of valuations are given by

$$F_1(v) = 1 - e^{-\frac{v}{2}} \qquad F_2(v) = 1 - (1 + \frac{v}{0.3322})^{-1} \qquad F_3(v) = 1 - e^{-v}.$$

These distributions are concave. In this game, $s_1 = 1.545$ and $s_2 = s_3 = 1.909$. Thus, bidder one is strongest, and bidders 2 and 3 are equally strong. A herculean equilibrium prescribes that bidders 2 and 3 play the same strategy in equilibrium. However, there is no equilibrium with such property. In fact, the unique equilibrium of the game is given by: $x_1 = 1.2938$, $x_2 = 2.1718$, and $x_3 = 2.2180$.

D.3 Derivation of Equation (A.4)

Recall equation (1)

$$\Pi_i(x_i; \mathbf{x}_{-i}) = A_i^n R_i(x_i; \mathbf{x}_{i-1}) - K_i.$$

where

$$R_i(x_i; \mathbf{x}_{i-1}) = x_i B_i(x_i) - r A_0^{i-1} - \sum_{j=1}^{i-1} \left(A_j^{i-1} \int_{x_j}^{x_{j+1}} \max\{r, s\} dB_{j+1}(s) \right),$$

Before differentiating its is worth noticing that

$$\frac{dB_i(v)}{dv} = B_i(v) \sum_{s=1}^{i-1} h_s(v) \text{ and } \frac{dA_i^n}{dx_j} = A_i^n h_j(x_j) \text{ for } j > i.$$

For a given vector $\mathbf{x} = \left(\chi_1(\mathbf{x}^{k+1}), \ldots, \chi_k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}\right)$, the derivative of $\Pi_i(\mathbf{x})$ with respect to x_j for $j > k$ is

$$\frac{d\Pi_i(\mathbf{x})}{dx_j} = A_i^n R_i(\mathbf{x}_i) \sum_{s=i+1}^{k} h_s(x_s) \frac{d\chi_s}{dx_j} + A_i^n R_i(\mathbf{x}_i) h_j(x_j) + A_i^n \frac{dR_i(\mathbf{x}_i)}{dx_j}$$

where

$$
\begin{aligned}
\frac{dR_i(\mathbf{x}_i)}{dx_j} =& B_i(x_i)\frac{d\chi_i}{dx_j} + x_i B_i(x_i)\left(\sum_{s=1}^{i-1} h_s\left(x_i\right)\right)\frac{d\chi_i}{dx_j} - rA_0^{i-1}\left(\sum_{\ell=1}^{i-1} h_\ell\left(x_\ell\right)\frac{d\chi_\ell}{dx_j}\right) \\
& - \sum_{\ell=1}^{i-1}\left(A_\ell^{i-1}\left(\sum_{s=\ell+1}^{i-1} h_s\left(x_s\right)\frac{d\chi_s}{dx_j}\right)\left(\int_{x_\ell}^{x_{\ell+1}} v dB_{\ell+1}\left(v\right)\right)\right. \\
& \quad + A_\ell^{i-1} x_{\ell+1} B_{\ell+1}\left(x_{\ell+1}\right)\left(\sum_{s=1}^{\ell} h_s\left(x_{\ell+1}\right)\right)\frac{d\chi_{\ell+1}}{dx_j} \\
& \quad \left. - A_\ell^{i-1} x_\ell B_{\ell+1}\left(x_\ell\right)\left(\sum_{s=1}^{\ell} h_s\left(x_\ell\right)\right)\frac{d\chi_\ell}{dx_j}\right) \qquad \text{(D.2)}
\end{aligned}
$$

But observe

$$
\begin{aligned}
& A_{\ell-1}^{i-1} x_\ell B_\ell\left(x_\ell\right)\left(\sum_{s=1}^{\ell-1} h_s\left(x_\ell\right)\right)\frac{d\chi_\ell}{dx_j} - A_\ell^{i-1} x_\ell B_{\ell+1}\left(x_\ell\right)\left(\sum_{s=1}^{\ell} h_s\left(x_\ell\right)\right)\frac{d\chi_\ell}{dx_j} \\
&= A_{\ell-1}^{i-1} x_\ell\left(B_\ell\left(x_\ell\right)\sum_{s=1}^{\ell-1} h_s\left(x_\ell\right) - \frac{B_{\ell+1}\left(x_\ell\right)}{F_\ell\left(x_\ell\right)}\sum_{s=1}^{\ell} h_s\left(x_\ell\right)\right)\frac{d\chi_\ell}{dx_j} \\
&= A_{\ell-1}^{i-1} x_\ell\left(B_\ell\left(x_\ell\right)\sum_{s=1}^{\ell-1} h_s\left(x_\ell\right) - B_\ell\left(x_\ell\right)\sum_{s=1}^{\ell} h_s\left(x_\ell\right)\right)\frac{d\chi_\ell}{dx_j} \\
&= -A_{\ell-1}^{i-1} x_\ell B_\ell\left(x_\ell\right) h_\ell\left(x_\ell\right)\frac{d\chi_\ell}{dx_j}
\end{aligned}
$$

substituting in, the substracting summation in (D.2) becomes

$$
\begin{aligned}
& \sum_{\ell=1}^{i-1}\left(A_\ell^{i-1}\left(\int_{x_\ell}^{x_{\ell+1}} v dB_{\ell+1}\left(v\right)\right)\left(\sum_{s=\ell+1}^{i-1} h_s\left(x_s\right)\frac{d\chi_s}{dx_j}\right)\right. \\
& \qquad \left. - A_{\ell-1}^{i-1} x_\ell B_\ell\left(x_\ell\right) h_\ell\left(x_\ell\right)\frac{d\chi_\ell}{dx_j} + x_i B_i(x_i)\left(\sum_{s=1}^{i-1} h_s\left(x_i\right)\right)\frac{d\chi_i}{dx_j}\right)
\end{aligned}
$$

Then, the derivative of $R_i(\mathbf{x}_i)$ becomes

$$
\begin{aligned}
\frac{dR_i(\mathbf{x}_i)}{dx_j} =& \sum_{\ell=1}^{i-1}\left(A_{\ell-1}^{i-1}\left(x_\ell B_\ell\left(x_\ell\right) - rA_0^{\ell-1}\right) h_\ell\left(x_\ell\right)\frac{d\chi_\ell}{dx_j}\right) + B_i(x_i)\frac{d\chi_i}{dx_j} \\
& - \sum_{\ell=1}^{i-1}\left(A_\ell^{i-1}\left(\sum_{s=\ell+1}^{i-1} h_s\left(x_s\right)\right)\frac{d\chi_s}{dx_j}\int_{x_\ell}^{x_{\ell+1}} v dB_{\ell+1}\left(v\right)\right)
\end{aligned}
$$

The last term can be rewritten as:

$$\sum_{\ell=1}^{i-1}\left(A_{\ell}^{i-1}\left(\sum_{s=\ell+1}^{i-1}h_s\left(x_s\right)\right)\frac{d\chi_s}{dx_j}\int_{x_\ell}^{x_{\ell+1}}vdB_{\ell+1}\left(v\right)\right)$$
$$=\sum_{\ell=1}^{i-1}\left(\sum_{s=1}^{\ell-1}A_s^{i-1}\left(\int_{x_s}^{x_{s+1}}vdB_{s+1}\left(v\right)\right)h_\ell\left(x_\ell\right)\frac{d\chi_\ell}{dx_j}\right)$$

Re arranging and using $A_{\ell-1}^{i-1}h_\ell\left(x_\ell\right)=A_\ell^{i-1}f_\ell\left(x_\ell\right)$ we obtain

$$\frac{dR_i(\mathbf{x}_i)}{dx_j}=\sum_{\ell=1}^{i-1}\left(A_{\ell-1}^{i-1}R_\ell(\mathbf{x}_\ell)f_\ell(x_\ell)\frac{d\chi_\ell}{dx_j}\right)+B_i(x_i)\frac{d\chi_i}{dx_j}$$

and equation (A.4) follows.

E A Weaker Sufficient Condition for Uniqueness

In this section we show that, in the two-group model, if the expected profit gain (see equation (7) in the main text) satisfies a condition that is analogous to (but stronger than) supermodularity, we can weaken the sufficient conditions for uniqueness in Proposition 4.

Proposition 5. *Let $\Delta_{i,j}(\mathbf{x})=F_j(x_j)\hat{\Delta}_{i,j}(\mathbf{x})$. Suppose that for every vector of group-symmetric cutoff strategies $\mathbf{x}$, the expected profit gain satisfies the following property*[29]

$$\hat{\Delta}_{1,1}(\mathbf{x})\hat{\Delta}_{2,2}(\mathbf{x})\geq\hat{\Delta}_{1,2}(\mathbf{x})\hat{\Delta}_{2,1}(\mathbf{x}) \tag{E.1}$$

Then, the game has a unique equilibrium if for every firm i and each opponent j, the following condition holds

$$\frac{f_i(x_i)}{F_i(x_i)}\frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})}<1 \tag{E.2}$$

hold for every vector $\mathbf{x}$ such that each dimension k satisfies $x_k\in[\underline{v}_{g(k)},\overline{v}_{g(k)}]$.

Before proving the result we note that, when $n_g=1$, $\Delta_{i,i}(\mathbf{x})$ is not defined; i.e., firm i's profit gain when a firm of group $g(i)$ exits when there only is one firm in group $g(i)$. This, however, can be corrected if in property (E.1) we substitute $\hat{\Delta}_{i,i}(\mathbf{x})$ for $\Pi_i'(\mathbf{x})f_i(x_i)$. Under sufficient condition (E.2) this substitution is a bit more demanding than (E.1), as condition (E.2) implies $\hat{\Delta}_{i,i}(\mathbf{x})<\Pi_i'(\mathbf{x})/f_i(x_i)$. Below, we show that both the SPA and the linear model satisfy condition (E.1) for any $n_g\geq 1$.

Proof. By the proof of Proposition 4 we know that a herculean equilibrium exists. We need to prove that it is unique. By Lemma B.3 we know that firms will play group symmetric strategies. As in the proof of Proposition 4, let $\hat{\mathbf{x}}=$

[29] Condition (E.1) is equivalent to $\Delta_{1,1}(\mathbf{x})\Delta_{2,2}(\mathbf{x})\geq\Delta_{1,2}(\mathbf{x})\Delta_{2,1}(\mathbf{x})$.

$(x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2)$ be a vector of group-symmetric cutoff strategies. Pick any firm in group $i \in \{1, 2\}$ and let $\Pi_i^{gs}(x_1, x_2) = \Pi_i(\hat{\mathbf{x}})$ —where gs stands for group-symmetric— represent the expected profit of a firm belonging to group i entering with a valuation x_i, when opponents play group-symmetric strategies x_1 and x_2. Observe that the function $\Pi_i^{gs}(x_1, x_2)$ has a two-dimensional domain, taking as input the group-symmetric strategy of each group.

Define $\chi_1(x)$ to be the function that solves $\Pi_1^{gs}(\chi_1(x), x) = 0$. Thus, $\chi_1(x)$ corresponds to group 1's symmetric best response to group 2 playing the group-symmetric cutoff x. By Lemma H.1, the value $\chi_1(x)$ exists and is unique; i.e., $\chi_1(x)$ is well defined.

Using implicit differentiation, the chain rule, that groups members are symmetric, and equation (B.2)

$$\chi_1'(x) = -\frac{\frac{\partial \Pi_1^{gs}(\chi_1(x),x)}{\partial x_2}}{\frac{\partial \Pi_1^{gs}(\chi_1(x),x)}{\partial x_1}} = -\frac{\sum_{j\in G_2} \frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_j}}{\sum_{j\in G_1} \frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_j}} \tag{E.3}$$

$$= \frac{-n_2 h_2(x_2)\Delta_{1,2}(\hat{\mathbf{x}})}{\Pi_1'(\hat{\mathbf{x}}) + (n_1 - 1)h_1(x_1)\Delta_{1,1}(\hat{\mathbf{x}})} > -\frac{n_2 h_2(x_2)\Delta_{1,2}(\hat{\mathbf{x}})}{n_1 h_1(x_1)\Delta_{1,1}(\hat{\mathbf{x}})} \tag{E.4}$$

where $x_1 = \chi(x_2)$ and $h_i(v) = f_i(v)/F_i(v)$ is the reversed hazard rate. The inequality in (E.4) follows from substituting sufficient condition (E.2) in the denominator.

To prove uniqueness $\hat{\Pi}_2'(x) > 0$ so that $\hat{\Pi}_2(x)$ single crosses zero from below. Recall $\hat{\mathbf{x}} = (\chi_1(x), \ldots, \chi_1(x), x, \ldots, x)$. Differentiating $\hat{\Pi}_2(x)$, using the chain rule, and that firms play group-symmetric strategies, we obtain[30]

$$\begin{aligned}
\hat{\Pi}_2'(x) &= \sum_{j\in G_2} \frac{\partial \Pi_2(\hat{\mathbf{x}})}{\partial x_j} + \chi_1'(x) \sum_{j\in G_1} \frac{\partial \Pi_2(\hat{\mathbf{x}})}{\partial x_j} \\
&= \Pi_2'(\hat{\mathbf{x}}) + (n_2 - 1)h_2(x)\Delta_{2,2}(\hat{\mathbf{x}}) + \chi_1'(x) n_1 h_1(\chi_1(x))\Delta_{2,1}(\hat{\mathbf{x}}) \\
&> n_2 h_2(x) \left[\Delta_{2,2}(\hat{\mathbf{x}}) - \frac{\Delta_{2,1}(\hat{\mathbf{x}})\Delta_{1,2}(\hat{\mathbf{x}})}{\Delta_{1,1}(\hat{\mathbf{x}})}\right] > 0.
\end{aligned}$$

The second equality follows from using Lemma B.2. The first inequality follows from using condition (E.2) in $\Pi_2'(\mathbf{x})$ and using the lower bound (E.4) for $\chi'(x)$. The last inequality follows from property (E.1). Proving that the derivative is always positive and equilibrium uniqueness. ■

Example. We show that property (E.1) holds in the linear model and in SPAs.

[30] If $n_1 = 1$ simply use $\chi_1'(x) = -n_2 h_2(x_2)\Delta_{1,2}(\hat{\mathbf{x}})/\Pi_1'(\hat{\mathbf{x}})$. Then,

$$\hat{\Pi}_2'(x) > n_2 h_2(x) \left[\Delta_{2,2}(\hat{\mathbf{x}}) - h_1(x_1)\frac{\Delta_{2,1}(\hat{\mathbf{x}})\Delta_{1,2}(\hat{\mathbf{x}})}{\Pi_1'(\hat{\mathbf{x}})}\right] > 0.$$

The last inequality follows from the property (E.1) modified. Similar steps can be applied if $n_2 = 1$.

(i) **Linear model:** Consider the linear model

$$\pi_i(v_e) = \eta_i - \delta_i(n_e - 1) + v_i,$$

where n_e is the number of entrants in market structure e. In this context, for a given vector of cutoff strategies $\mathbf{x}$, equation (6) is given by

$$\Pi_i(v_i; \mathbf{x}_{-i}) = \eta_i + v_i - \delta_i \mathbb{I}_{n_e>1} \sum_{e \in E_i} \left\{ \left(\prod_{j \in O_i(e)} F_j(x_j) \right) \left(\prod_{\ell \in I_i(e)} (1 - F_\ell(x_\ell)) \right) \right\}$$

and $\Pi'(\mathbf{x}) = 1$. Similarly, noticing that $\pi(v_i, v_{e\setminus i}) - \pi(v_i, v_j, v_{e\setminus i}) = \delta_i$ we obtain $\hat{\Delta}_{i,j}(\mathbf{x}) = \delta_i$. Then, the sufficient conditions for uniqueness, which are independent of n_g, become

$$\frac{f_i(x_i)}{F_i(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} = \begin{cases} \delta_i f_i(x_i) < 1 & \text{if } j \in g(i), \\ \delta_i F_j(x_j) f_i(x_i)/F_i(x_i) < 1 & \text{if } j \notin g(i) \end{cases}.$$

Finally, property (E.1) when $n_g > 1$ holds as

$$\hat{\Delta}_{1,1}(\mathbf{x})\hat{\Delta}_{2,2}(\mathbf{x}) - \hat{\Delta}_{1,2}(\mathbf{x})\hat{\Delta}_{2,1}(\mathbf{x}) = \delta_1\delta_2 - \delta_1\delta_2 = 0.$$

When $n_1 = 1$ (similarly for n_2)

$$\frac{\Pi_1'(\mathbf{x})}{f_1(x_1)}\hat{\Delta}_{2,2}(\mathbf{x}) - \hat{\Delta}_{1,2}(\mathbf{x})\hat{\Delta}_{2,1}(\mathbf{x}) = \frac{1}{f_1(x_1)}\delta_2 - \delta_1\delta_2 = \delta_2 \left[\frac{1}{f_1(x_1)} - \delta_1 \right] > 0$$

where the inequality follows from sufficient condition (E.2).

(ii) **Second Price Auction** In a SPA, we already know that $vf_i(v)/F_i(v)$ is a sufficient condition for uniqueness. We show that the framework satisfies condition (E.1). When $x_1 \leq x_2$ we have (the proof when $x_2 \leq x_1$ is analogous)

$$\begin{aligned} \Delta_{1,1}(\mathbf{x}) &= \Pi_1(\mathbf{x}) & \Delta_{1,2}(\mathbf{x}) &= \Pi_1(\mathbf{x}) \\ \Delta_{2,1}(\mathbf{x}) &= (x_1 - r)F_1(x_1)^{n_1} F_2(x_2)^{n_2-1} & \Delta_{2,2}(\mathbf{x}) &= \Pi_2(\mathbf{x}) \\ \Pi_1'(\mathbf{x}) &= F_1(x_1)^{n_1-1} F_2(x_2)^{n_2} & \Pi_2'(\mathbf{x}) &= F_1(x_1)^{n_1} F_2(x_2)^{n_2-1} \end{aligned}$$

where $\Pi_1(\mathbf{x}) = (x_1 - r)F_1(x_1)^{n_1-1}F_2(x_2)^{n_2}$ and $\Pi_2(\mathbf{x}) = F_2(x_2)^{n_2-1}R_2(x_1, x_2)$. Then,

$$\begin{aligned} &\Delta_{1,1}(\mathbf{x})\Delta_{2,2}(\mathbf{x}) - \Delta_{1,2}(\mathbf{x})\Delta_{2,1}(\mathbf{x}) \\ &= \Pi_1(\mathbf{x})F_2(x_2)^{n_2-1}\left[R_2(x_1, x_2) - (x_1 - r)F_1(x_1)^{n_1}\right] \\ &= \Pi_1(\mathbf{x})F_2(x_2)^{n_2-1}\left[x_2 F_1(x_2)^{n_1} - rF_1(x_1) - \int_{x_1}^{x_2} v dF_1(v)^{n_1} - (x_1 - r)F_1(x_1)^{n_1}\right]. \end{aligned}$$

$$= \Pi_1(\mathbf{x}) F_2(x_2)^{n_2-1} \left[\int_{x_1}^{x_2} F_1(v)^{n_1} dv \right] > 0$$

where the last equality follows from integrating by parts. When $n_1 = 1$, sufficient condition for uniqueness implies $\Pi_1'(\mathbf{x})/h_1(x) > \Pi_1(\mathbf{x}) = \Delta_{1,1}(\mathbf{x})$, which implies condition (E.1). A similar argument applies for $n_2 = 1$.

F Uniqueness with Partially Informed Bidders

In this section we show that, when $n_1 = n_2 = 1$, $x_i f_i(x_i)/F_i(x_i) < 1$ implies

$$\frac{f_i(x_i)}{F_i(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1$$

Thus, sufficient condition (3) implies sufficient conditions (8) and (9). Start by observing that, by construction, $\Delta_{i,j}(\mathbf{x}) \leq \Pi_i(\mathbf{x})$. Then, it is sufficient to show

$$\frac{f_i(x_i)}{F_i(x_i)} \Pi_i(\mathbf{x}) < \Pi_i'(\mathbf{x}).$$

Recall

$$\pi_i(x_e) = \int_{r/x_i}^{\infty} \left(\int_0^{x_i \varepsilon} (x_i \varepsilon - \max\{r, s\}) d\Psi_i(s, x_e) \right) dG(\varepsilon) - K_i,$$

where $\Psi_i(s, x_e) = \prod_{j \in e \setminus i} G(s/x_j)$. Then,

$$\pi_i'(x_e) = \int_{r/x_i}^{\infty} \varepsilon \Psi_i(\varepsilon x_i, v_e) \, dG(\varepsilon)$$

we show that $\frac{f_i(x_i)}{F_i(x_i)} \pi_i(x_e) < \pi_i'(x_e)$, which implies the result

$$\begin{aligned}
\frac{f_i(x_i)}{F_i(x_i)} \pi_i(x_e) &= \int_{r/x_i}^{\infty} \left(\int_0^{x_i \varepsilon} \left(\frac{f_i(x_i)}{F_i(x_i)} x_i \varepsilon - \max\{r, s\} \right) d\Psi_i(s, x_e) \right) dG(\varepsilon) - K_i \\
&< \int_{r/x_i}^{\infty} \left(\int_0^{x_i \varepsilon} \left(\frac{f_i(x_i)}{F_i(x_i)} x_i \varepsilon \right) d\Psi_i(s, x_e) \right) dG(\varepsilon) \\
&= \int_{r/x_i}^{\infty} \frac{f_i(x_i)}{F_i(x_i)} x_i \varepsilon \Psi_i(x_i \varepsilon, x_e) \, dG(\varepsilon) \\
&< \int_{r/x_i}^{\infty} \varepsilon \Psi_i(x_i \varepsilon, x_e) \, dG(\varepsilon) = \pi_i'(x_e)
\end{aligned}$$

where in the first inequality we took all the subtracting terms to zero, the second equality integrated the inner integral, and the second inequality used $x_i f_i(x_i)/F_i(x_i) < 1$. This proves the result.

G Uniqueness in Ordered Games

In an entry game, there are two elements that determine payoffs: the distribution of types $F_i(v_i)$ and the profit function $\pi_i(v_e)$. A game is called *ordered* when firms are symmetric in one of these two dimensions and are ordered in the other. In this section, we extend our results to ordered environments.

Definition (Ordered games)**.** A game is *ordered by profit* when firms have symmetric distributions of types, $F_i(v_i) = F(v_i)$ for every i, and anonymous profit functions that, for any realization v_e, satisfy $\pi_i(v_i, \mathbf{v}_{n_e-1}) \geq \pi_{i+1}(v_i, \mathbf{v}_{n_e-1})$, where n_e is the number of entrants in e and $\mathbf{v}_{n_e-1}$ is an (n_e-1)-dimensional vector containing the types of i's competitors.

An entry game is called *ordered by distributions* when firms have symmetric and anonymous profit functions, $\pi_i(v_e) = \pi(v_i, \mathbf{v}_{n_e-1})$ for every i, and their distributions of types, $F_i(v_i)$, are ordered in terms of first-order stochastic dominance.[31]

Without loss of generality, we order firms' identities so they satisfy $F_i(v) \leq F_{i+1}(v)$ for all v when ordered by distributions, or $\pi_i(v, \mathbf{v}_{n_e-1}) \geq \pi_{i+1}(v, \mathbf{v}_{n_e-1})$ when ordered by profit.

Lemma G.1. *Suppose an entry game in which firms are ordered (either by profit or distributions). Then, the firms are also ordered by strength, with $s_i < s_{i+1}$; i.e., firm 1 is the strongest and firm n the weakest.*

Proof of Lemma G.1. We start by showing the order in the context of ordered by profit. Let s_i be the strength of firm i, using $\sigma_i(s) \equiv \Pi_i(s; s, \ldots, s)$ we obtain

$$0 = \sigma_i(s_i) = \sum_{e \in \mathcal{E}_i} \left\{ \left(\prod_{j \in e^c} F(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{b} \pi_i\left(s_i, v_{e\setminus i}\right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\}$$
$$> \sum_{e \in \mathcal{E}_i} \left\{ \left(\prod_{j \in e^c} F(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{b} \pi_{i+1}\left(s_i, v_{e\setminus i}\right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} = \sigma_{i+1}(s_i),$$

where in the inequality we used $\pi_i(v, \mathbf{v}_{n_e-1}) > \pi_{i+1}(v, \mathbf{v}_{n_e-1})$. In the last equality, after changing the firm's identity, we used $\mathcal{E}_i = \mathcal{E}_{i+1}$. Then, by Lemma 5, $\sigma_{i+1}(s)$ is increasing in s and $s_{i+1} > s_i$.

For games ordered by distributions, rewriting $\sigma_i(s_i)$ we obtain

$$0 = \sigma_i(s_i) = \sum_{e \in \mathcal{E}_i \cap \mathcal{E}_{i+1}} \left\{ \left(\prod_{j \in e^c} F_j(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{b} \pi\left(s_i, v_{e\setminus i}\right) \phi(v_{e\setminus \{i,i+1\}}) f_{i+1}(v_{i+1}) d^{n_e-1} v_{e\setminus i} \right\} +$$
$$\sum_{e \in \mathcal{E}_i \setminus \mathcal{E}_{i+1}} \left\{ \left(F_{i+1}(s_i) \prod_{j \in e^c \setminus i+1} F_j(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{b} \pi\left(s_i, v_{e\setminus i}\right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\}$$

[31] Our results below also extend to environments in which firms are ranked consistently across both dimensions; i.e., $F_i(v) \leq F_{i+1}(v)$ for all v and $\pi_i(v_i, \mathbf{v}_{n_e-1}) \geq \pi_{i+1}(v_i, \mathbf{v}_{n_e-1})$ for all v_e.

$$> \sum_{e \in \mathcal{E}_i \cap \mathcal{E}_{i+1}} \left\{ \left(\prod_{j \in e^c} F_j(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{b} \pi\left(s_i, v_{e \setminus i}\right) \phi(v_{e \setminus \{i,i+1\}}) f_i(v_{i+1}) d^{n_e - 1} v_{e \setminus i} \right\} +$$

$$\sum_{e \in \mathcal{E}_i \setminus \mathcal{E}_{i+1}} \left\{ \left(F_i(s_i) \prod_{j \in e^c \setminus i+1} F_j(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{b} \pi\left(s_i, v_{e \setminus i}\right) \phi(v_{e \setminus i}) d^{n_e - 1} v_{e \setminus i} \right\} = \sigma_{i+1}(s_i),$$

where the inequality uses two properties of FOSD. In the second term we used $F_i(v) < F_{i+1}(v)$. In the first term, we used $\int_{s_i}^{b} \varphi(v) f_i(v) dv \leq \int_{s_i}^{b} \varphi(v) f_{i+1}(v) dv$ for any non-increasing function $\varphi(x)$. Then, by Lemma 5, $\sigma_{i+1}(s)$ is increasing in s and $s_{i+1} > s_i$. ■

The previous lemma shows that the firms' ranking provided by strength coincides with the order of the game. In ordered games, the firm ranking provided by strength is robust to adding competitors. That is, if we add a new firm to the game, the existing strength order between the firms remains unchanged (as illustrated in Figure 4a). Recall $\underline{v} = \min\{\underline{v}_i\}_{i=1}^{n}$ and $\overline{v} = \max\{\overline{v}_i\}_{i=1}^{n}$.

Proposition 6. *In ordered games, there always exists a* herculean *equilibrium. Moreover, the entry game has a unique equilibrium if the following condition holds*

$$(n-1) \frac{f_i(x_i)}{F_i(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1 \tag{G.1}$$

for every pair of firms $\{i, j\}$ and every vector $\mathbf{x}$ such that each dimension satisfies $x_k \in [\underline{v}, \overline{v}]$, and the game is: i) ordered by profit or, ii) ordered by distributions and the profit gain does not depend on the type of competitors; i.e., $\delta_{i,j}(x_i, x_j, v_{e \setminus i}) = \delta_i(x_i, n_e)$.

We postpone the proof to the next section. Observe that Proposition 6 is not a particular case nor a generalization of Proposition 4 in the main text. While the former can handle more than two groups of asymmetric firms, the latter allows for a larger degree of firm heterogeneity between the two groups. There are also differences in the sufficient condition for uniqueness.

Although Proposition 6 says that condition (G.1) needs to hold for every pair of potential entrants, for certain ordered structures it is sufficient to check the sufficient condition for a single pair of firms.

Lemma G.2. *1) If firms are ordered by distribution and belong to the exponentiated family, i.e., $F_i(v_i) = F(v_i)^{\theta_i}$, then condition (G.1) is satisfied for every firm if it holds for the strongest firm (that is, the firm with the highest θ_i).*
2) If firms are ordered by profits and satisfy $\pi_i(v_i, \mathbf{v}_{n_e-1}) = \pi(v_i, \mathbf{v}_{n_e-1}) + K_i$, then condition (G.1) is satisfied for every firm if it holds for any firm.

Proof of Lemma G.2. In condition (G.1), when firms are ordered by distribution, the term $\Delta_{i,j}(\mathbf{x})/\Pi_i'(\mathbf{x})$ is common across firms and the term $f_i(x_i)/F_i(x_i) = \theta_i x_i f(x_i)/F(x_i)$ can be ordered using θ_i. For the second claim, $f(x_i)/F(x_i)$ is common across firms, and the term K_i cancels out from $\Delta_{i,j}(\mathbf{x})$ and $\Pi_i'(\mathbf{x})$. Thus, the same restriction applies to every firm. ■

G.1 Proof of Proposition 6

We present the proof when firms are ordered by distributions. The proof when firms are ordered by profit is, basically, identical but we can drop the subindices from the distribution functions. Using Lemma G.1 we order firms using stochastic dominance, from stronger (firm 1) to weakest (firm n).

Existence of an herculean equilibrium. We prove existence by construction. For any vector of cutoff strategies $\mathbf{x}$ and $k \in \{2, \ldots, n\}$ let $\mathbf{x}^k = (x_k, x_{k+1}, \ldots, x_n)$. Construct the equilibrium vector sequentially, as follows:

- **Firm** 1: Define $\chi_1^1(\mathbf{x}^2)$ to be firm's 1 best response to $\mathbf{x}^2$; i.e., $\chi_1^1(\mathbf{x}^2)$ satisfies

  $$\Pi_1(\chi_1^1(\mathbf{x}^2); \mathbf{x}^2) = 0.$$

  where $\Pi_1(\mathbf{x})$ is defined in (6). By Lemma H.1 in the Auxiliary Result section, $\chi_1^1(\mathbf{x}^2)$ exists and (the best response) is unique and continuous.

- **Firm** 2: Let $\hat{\Pi}_2(\mathbf{x}^2) = \Pi_2(\chi_1^1(\mathbf{x}^2); \mathbf{x}^2)$; that is, $\hat{\Pi}_2(\mathbf{x}^2)$ represents firm's 2 profit after incorporating that firm 1 is best responding to $\mathbf{x}^2$. Define $\chi_2^2(\mathbf{x}^3)$ to be a solution to $\hat{\Pi}_2(\chi_2^2(\mathbf{x}^3), \mathbf{x}^3) = 0$. By Lemma H.1, $\chi_2^2(\mathbf{x}^3)$ exists and is continuous in each dimension of $\mathbf{x}^3$. This function represents firm's 2 best response when firms 1 and 2 are mutually best responding to each other and to $\mathbf{x}^3$. For ease in notation, denote firm's 1 best response after incorporating firm's 2 best response as $\chi_1^2(\mathbf{x}^3) = \chi_1^1(\chi_2^2(\mathbf{x}^3), \mathbf{x}^3)$.[32] This function is also continuous in each dimension of $\mathbf{x}^3$.

  Claim 9. For any $\mathbf{x}^3$, $\chi_2^2(\mathbf{x}^3) > \chi_1^2(\mathbf{x}^3)$.

  Proof. Fix $\mathbf{x}^3$ and find the value $\hat{x}$ that satisfies $\hat{x} = \chi_1^1(\hat{x}, \mathbf{x}^3)$. The value $\hat{x}$ exists by continuity of $\chi_1^1(\mathbf{x}^2)$ and by $\chi_1^1(\mathbf{x}^2)$ being bounded below and above by $\underline{v}_1$ and $\overline{v}_1$ respectively (by assumption A3). Then by Lemma H.2 in the auxiliary results section we have $\Pi_2(\hat{x}; \hat{x}, \mathbf{x}^3) < \Pi_1(\hat{x}; \hat{x}, \mathbf{x}^3) = 0$. Define a pair of sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ satisfying: (i) $y_1 = z_1 = \hat{x}$; (ii) y_{m+1} is the unique (by Lemma H.1) value that solves $\Pi_2(z_m; y_{m+1}, \mathbf{x}^3) = 0$ (i.e., y_{m+1} is firm's 2 best response to the cutoffs $(z_m, \mathbf{x}^3)$) and; (iii) $z_{m+1} = \chi_1^1(y_{m+1}, \mathbf{x}^3)$. By definition, z_{m+1} solves $\Pi_1(z_{m+1}; y_{m+1}, \mathbf{x}^3) = 0$ and, by Lemma H.1, the value z_{m+1} is also unique. We show that $\{y_m\}_{m\in\mathbb{N}}$ is increasing and $\{z_m\}_{m\in\mathbb{N}}$ decreasing. Because $\Pi_2(\hat{x}; \hat{x}, \mathbf{x}^3) < 0$ and $\Pi_2(\mathbf{x})$ being strictly increasing in the 2nd dimension, $y_2 > y_1 = \hat{x}$. Similarly, because (by Lemma B.2) $\Pi_1(\mathbf{x})$ is also increasing in the 2nd dimension, we have $\Pi_1(z_1; y_2, \mathbf{x}^3) > 0$, which implies $z_2 = \chi_1^1(y_2, \mathbf{x}^3) < z_1 = \chi_1^1(y_1, \mathbf{x}^3)$. This, in turn, implies (by Lemma H.2)

  $$\Pi_2(z_2; y_2, \mathbf{x}^3) < \Pi_1(z_2; y_2, \mathbf{x}^3) = 0;$$

  which implies $y_3 > y_2$. By induction, the argument generalizes to an arbitrary step m and the sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ are monotonically increasing and

[32] More generally, for $j < k$ the notation $\chi_j^k(\mathbf{x}^k)$ represents firm's j best response to $\mathbf{x}^{j+1}$ (i.e., $\chi(\mathbf{x}^{j+1})$) after substituting subsequent best responses from firm $j+1$ up to firm k.

decreasing respectively. By assumption A3, $\{y_m\}_{m\in\mathbb{N}}$ is bounded above by $\overline{v}_2$ and $\{z_m\}_{m\in\mathbb{N}}$ is bounded below by $\underline{v}_1$. Thus, the sequences converge to y_∞ and z_∞, respectively. By convergence, we have: (i) $z_\infty = \chi_1^1(y_\infty, \mathbf{x}^3)$ and; (ii) $\Pi_2(z_\infty; y_\infty, \mathbf{x}^3) = \hat{\Pi}_2(y_\infty, \mathbf{x}^3) = 0$ (i.e., $y_\infty = \chi_2^2(\mathbf{x}^3)$). Thus, $\chi_1^1(y_\infty, \mathbf{x}^3) = \chi_1^2(\mathbf{x}^3)$ and, as $z_\infty < \hat{x} < y_\infty$, we have $\chi_2^2(\mathbf{x}^3) > \chi_1^2(\mathbf{x}^3)$. □

- **Firm** $k \leq n$: Suppose we have shown the existence of $\chi_\ell^\ell(\mathbf{x}^{\ell+1})$ for every $\ell \in \{1, \ldots, k-1\}$, have recursively defined $\chi_j^\ell(\mathbf{x}^{\ell+1}) = \chi_j^{\ell-1}(\chi_\ell^\ell(\mathbf{x}^{\ell+1}), \mathbf{x}^{\ell+1})$ for $j \in \{1, \ldots, \ell\}$, and that both constructions are continuous. Let $\hat{\Pi}_k(\mathbf{x}^k) = \Pi_k(\chi_1^{k-1}(\mathbf{x}^k), \ldots, \chi_{k-1}^{k-1}(\mathbf{x}^k), \mathbf{x}^k)$ represent firm's k profit after incorporating that every firm $j \in \{1, \ldots, k-1\}$ is mutually best responding to each other and to $\mathbf{x}^k$. Define $\chi_k^k(\mathbf{x}^{k+1})$ (observe that χ_n^n is a number, not a function, as $\mathbf{x}^{k+1}$ is empty when $k = n$) to be a solution to $\hat{\Pi}_k(\chi_k^k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}) = 0$. By Lemma H.1, $\chi_k^k(\mathbf{x}^k)$ exists and is continuous in each dimension of $\mathbf{x}^k$. This function represents firm's k best response to $\mathbf{x}^{k+1}$ when every firm $j \in \{1, \ldots, k-1\}$ is mutually best responding to each other and to $\mathbf{x}^k$.

Claim 10. For any $\mathbf{x}^{k+1}$, if firm $k-1$ is stronger than k the solution $\chi_k^k(\mathbf{x}^{k+1})$ satisfies $\chi_k^k(\mathbf{x}^{k+1}) > \chi_{k-1}^k(\mathbf{x}^{k+1})$.

Proof. Fix any $\mathbf{x}^{k+1}$ and let $\chi_k^k(\mathbf{x}^{k+1})$ be one of the solutions found in the previous step. Then define the vector of cutoffs $\mathbf{x} = \big(\chi_1^k(\mathbf{x}^{k+1}), \ldots, \chi_k^k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}\big)$. Throughout the proof, the vector of strategies for every firm except firm k and $k-1$, $\mathbf{x}_{E\setminus\{k,k-1\}}$, remains fixed (i.e., they are numbers not functions). Define $\hat{x}$ to be a value satisfying $\hat{x} = \chi_{k-1}^{k-1}(\hat{x}, \mathbf{x}^{k+1})$. The value $\hat{x}$ exists by continuity of $\chi_{k-1}^{k-1}(\mathbf{x}^k)$ and by $\chi_{k-1}^{k-1}(\mathbf{x}^k)$ being bounded below and above by $\underline{v}_{k-1}$ and $\overline{v}_{k-1}$ respectively (by assumption A3). By definition of best response $\hat{x}$ satisfies $\Pi_{k-1}(\hat{x}; \hat{x}, \mathbf{x}_{E\setminus\{k,k-1\}}) = 0$. Then, by Lemma H.2, we have

$$\Pi_k(\hat{x}; \hat{x}, \mathbf{x}_{E\setminus\{k,k-1\}}) < \Pi_{k-1}(\hat{x}; \hat{x}, \mathbf{x}_{E\setminus\{k,k-1\}}) = 0.$$

Define a pair of sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ satisfying: (i) $y_1 = z_1 = \hat{x}$; (ii) y_{m+1} is the unique (by Lemma H.1) value that solves $\Pi_k(z_m; y_{m+1}, \mathbf{x}_{E\setminus\{k,k-1\}}) = 0$ (i.e., y_{m+1} is firm's k best response to the cutoffs $(z_m, \mathbf{x}_{E\setminus\{k,k-1\}})$) and; (iii) $z_{m+1} = \chi_{k-1}^{k-1}(y_{m+1}, \mathbf{x}^{k+1})$. By definition, z_{m+1} solves $\Pi_{k-1}(z_{m+1}; y_{m+1}, \mathbf{x}_{E\setminus\{k,k-1\}}) = 0$ and, Lemma H.1, the value z_{m+1} is also unique. We show that $\{y_m\}_{m\in\mathbb{N}}$ is increasing and $\{z_m\}_{m\in\mathbb{N}}$ decreasing. Because $\Pi_k(\hat{x}; \hat{x}, \mathbf{x}_{E\setminus\{k,k-1\}}) < 0$ and $\Pi_k(\mathbf{x})$ being strictly increasing in the kth dimension, $y_2 > y_1 = \hat{x}$. Similarly, because (by Lemma B.2) $\Pi_{k-1}(\mathbf{x})$ is also increasing in the kth dimension, we have $\Pi_{k-1}(\hat{x}; y_2, \mathbf{x}_{E\setminus\{k,k-1\}}) > 0$, which implies $z_2 = \chi_{k-1}^{k-1}(y_2, \mathbf{x}^{k+1}) < \chi_{k-1}^{k-1}(y_1, \mathbf{x}^{k+1}) = z_1$. This, in turn, implies (by Lemma H.2)

$$\Pi_k(z_2; y_2, \mathbf{x}_{E\setminus\{k,k-1\}}) < \Pi_{k-1}(z_2; y_2, \mathbf{x}_{E\setminus\{k,k-1\}}) = 0,$$

which, in turns, implies $y_3 > y_2$. By induction, the argument generalizes to an arbitrary step m and the sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ are monotonically increasing and decreasing respectively. By assumption A3, $\{y_m\}_{m\in\mathbb{N}}$ is bounded above by

$\overline{v}_k$ and $\{z_m\}_{m\in\mathbb{N}}$ is bounded below by $\underline{v}_{k-1}$. Thus, the sequences converge to y_∞ and z_∞, respectively. By convergence, we have: (i) $z_\infty = \chi^{k-1}_{k-1}(y_\infty, \mathbf{x}^{k+1})$ and; (ii) $\Pi_k(z_\infty; y_\infty, \mathbf{x}_{E\setminus\{k,k-1\}}) = \hat{\Pi}_k(y_\infty; \mathbf{x}_{E\setminus\{k,k-1\}}) = 0$ (i.e., $y_\infty = \chi^k_k(\mathbf{x}^{k+1})$). Thus, $\chi^{k-1}_{k-1}(y_\infty, \mathbf{x}^{k+1}) = \chi^k_{k-1}(\mathbf{x}^{k+1})$ and, as $z_\infty < \hat{x} < y_\infty$, we have $\chi^k_k(\mathbf{x}^{k+1}) > \chi^k_{k-1}(\mathbf{x}^{k+1})$. □

Thus, we have constructed an equilibrium vector $\mathbf{x} = (\chi^n_1(x_n), \ldots, \chi^n_{n-1}(x_n), x_n)$ with the property that $x_i < x_{i+1}$; i.e., a Herculean equilibrium.

Uniqueness within the herculean-equilibrium class: We show that at each step k of the previous construction there is a unique best response $x_k = \chi^k_k(\mathbf{x}^{k+1})$ to $\mathbf{x}^{k+1}$.

- **Firm** 1: The uniqueness of $\chi^1_1(\mathbf{x}^2)$ follows from Lemma H.1. Let $h_i(x) = f_i(x)/F_i(x)$ be the reversed hazard rate of firm i's distribution of private information. The next result is needed for subsequent steps.

  Claim 11. Under condition (G.1), for every $j \in \{2, \ldots, n\}$, $\partial\chi^1_1(\mathbf{x}^2)/\partial x_j$ satisfies:

$$0 > \frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_j} = -h_j(x_j)\frac{\Delta_{1,j}(\mathbf{x})}{\Pi'_1(\mathbf{x})} > -\frac{h_j(x_j)}{h_1(x_1)}\frac{1}{n-1}. \tag{G.2}$$

$$\frac{1}{h_j(x_j)}\frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_j} < \frac{1}{h_q(x_q)}\frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_q}\frac{1}{n-1} \quad \text{for } q \in \{2, \ldots, j-1\} \tag{G.3}$$

  Proof. Let $\mathbf{x} = (\chi^1_1(\mathbf{x}^2), \mathbf{x}^2)$; using implicit differentiation and Lemma B.2 we obtain

$$\frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_j} = -\frac{\partial\Pi_1(\mathbf{x})/\partial x_j}{\partial\Pi_1(\mathbf{x})/\partial x_1} = -h_j(x_j)\frac{\Delta_{1,j}(\mathbf{x})}{\Pi'_1(\mathbf{x})}, \tag{G.4}$$

  which is negative as, $\Delta_{1,j}(\mathbf{x}) > 0$ and $\Pi_{1,j}(\mathbf{x}) > 0$ for every $\mathbf{x}$. The lower bound in equation (G.2) follows from applying condition (G.1) into equation (G.4). Property (G.3) follows from observing

$$\frac{1}{h_q(x_q)}\frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_q}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_j} = \frac{1}{\Pi'_1(\mathbf{x})}\left(\Delta_{1,j}(\mathbf{x}) - \frac{\Delta_{1,q}(\mathbf{x})}{n-1}\right) > 0,$$

  where the equality follows from substituting in equation (G.4), and the inequality follows from Lemma H.3 and the fact that $q \in \{2, \ldots, j-1\}$. □

- **Firm** 2: Fix $\mathbf{x}^3$ and let $\mathbf{x} = (\chi^1_1(\mathbf{x}^2), \mathbf{x}^2)$, we need to show that the best response $\chi^2_2(\mathbf{x}^3)$ is unique. We do this by showing that $\hat{\Pi}_2(\mathbf{x}^2) = \Pi_2(\chi^1_1(\mathbf{x}^2); \mathbf{x}^2)$ is strictly increasing in x_2; so that, $\hat{\Pi}_2(x_2, \mathbf{x}^3)$ single crosses zero and there is a unique value $\chi^2_2(\mathbf{x}^3)$ satisfying $\hat{\Pi}_2(\chi^2_2(\mathbf{x}^3), \mathbf{x}^3) = 0$. Using the chain rule and equation (B.2)

$$\begin{aligned}\hat{\Pi}'_2(\mathbf{x}^2) = \Pi'_2(\mathbf{x}) + \frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_2}\frac{\partial\Pi_2}{\partial x_1} &= \Pi'_2(\mathbf{x}) + \frac{\partial\chi^1_1(\mathbf{x}^2)}{\partial x_2}h_1(\chi^1_1(\mathbf{x}^2))\Delta_{2,1}(\mathbf{x}) \\ &> \Pi'_2(\mathbf{x}) - h_2(x_2)\frac{\Delta_{2,1}(\mathbf{x})}{n-1} > \Pi'_2(\mathbf{x})\left[1 - \frac{1}{(n-1)^2}\right] > 0,\end{aligned} \tag{G.5}$$

where in the first inequality follows from the lower bound in equation (G.2) and the second inequality follows from sufficient condition (G.1). This proves uniqueness of the best response. The next result is needed for the induction argument in the proof.

Claim 12. Let $\chi_1^2(\mathbf{x}^3) = \chi_1^1(\chi_2^2(\mathbf{x}^3), \mathbf{x}^3)$. Under condition (G.1), for every $j \in \{3, \ldots, n\}$ and $\ell \in \{1, 2\}$, $\partial\chi_\ell^2(\mathbf{x}^3)/\partial x_j$ satisfies:

$$\frac{\partial\chi_2^2(\mathbf{x}^3)}{\partial x_j} = -\frac{h_j(x_j)\Delta_{2,j}(\mathbf{x}) + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_j} h_1(x_1)\Delta_{2,1}(\mathbf{x})}{\Pi_2'(\mathbf{x}) + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_2} h_1(x_1)\Delta_{2,1}(\mathbf{x})} \tag{G.6}$$

$$0 > \frac{\partial\chi_\ell^2(\mathbf{x}^3)}{\partial x_j} > -\frac{h_j(x_j)}{h_\ell(x_\ell)}\frac{1}{n-1} \quad \text{and,} \tag{G.7}$$

$$\frac{1}{h_j(x_j)}\frac{\partial\chi_2^2(\mathbf{x}^3)}{\partial x_j} < \frac{1}{h_q(x_q)}\frac{\partial\chi_2^2(\mathbf{x}^3)}{\partial x_q}\frac{1}{n-1} \quad \text{for } q \in \{3, \ldots, j-1\} \tag{G.8}$$

Proof. To show equation (G.6) use implicit differentiation, the chain rule, and equation (B.2) to obtain

$$-\frac{\partial\chi_2^2(\mathbf{x}^3)}{\partial x_j} = \frac{\frac{\partial\hat{\Pi}_2}{\partial x_j}}{\frac{\partial\hat{\Pi}_2}{\partial x_2}} = \frac{\frac{\partial\Pi_2}{\partial x_j} + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_j}\frac{\partial\Pi_2}{\partial x_1}}{\Pi_2'(\mathbf{x}) + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_2}\frac{\partial\Pi_2}{\partial x_1}} = \frac{h_j(x_j)\Delta_{2,j}(\mathbf{x}) + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_j}h_1(x_1)\Delta_{2,1}(\mathbf{x})}{\Pi_2'(\mathbf{x}) + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_2}h_1(x_1)\Delta_{2,1}(\mathbf{x})}$$

Observe, by equation (G.5), that the denominator is positive. Using lower bound (G.2) and Lemma H.3 we can see that the numerator is also positive, implying that $\partial\chi_2^2(\mathbf{x}^3)/\partial x_j$ is negative; which proves the upper bound of (G.7) when $\ell = 2$. For the lower bound of equation (G.7) when $\ell = 2$, using equation (G.6), observe that equation (G.7) holds if and only if the following expression is positive (replace (G.6) into (G.7) and work out the inequality):

$$h_j(x_j)\left[\left(\frac{1}{h_2(x_2)}\frac{\Pi_2'(\mathbf{x})}{n-1} - \Delta_{2,j}(\mathbf{x})\right) + h_1(x_1)\left(\frac{1}{h_2(x_2)}\frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_2}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_j}\right)\Delta_{2,1}(\mathbf{x})\right].$$

The first round bracket is positive by sufficient condition (G.1). The second round bracket is positive by property (G.3). Thus, the expression is indeed positive and the lower bound in equation (G.7) holds.

We now prove the bounds of (G.7) when $\ell = 1$. Using $\chi_1^2(\mathbf{x}^3) = \chi_1^1(\chi_2^2(\mathbf{x}^3), \mathbf{x}^3)$, observe

$$\frac{\partial\chi_1^2(\mathbf{x}^3)}{\partial x_j} = \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_j} + \frac{\partial\chi_1^1(\mathbf{x}^2)}{\partial x_2}\frac{\partial\chi_2^2(\mathbf{x}^3)}{\partial x_j}. \tag{G.9}$$

Using (G.4) to substitute for $\partial\chi_1^1(\mathbf{x}^2)/\partial x_\ell$ with $\ell \in \{2, j\}$ and using the lower bound in equation (G.7) when $\ell = 2$, we obtain the following upper bound:

$$\frac{\partial \chi_1^2(\mathbf{x}^3)}{\partial x_j} < \frac{h_j(x_j)}{\Pi_1'(\mathbf{x})}\left[\frac{\Delta_{1,2}(\mathbf{x})}{n-1} - \Delta_{1,j}(\mathbf{x})\right] < 0,$$

the inequality follows from Lemma H.3; proving the upper bound. The lower bound in equation (G.7) follows from using equation (G.9) and observing

$$\frac{\partial \chi_1^2(\mathbf{x}^3)}{\partial x_j} > \frac{\partial \chi_1^1(\mathbf{x}^2)}{\partial x_j} > -\frac{h_j(x_j)}{h_1(x_1)}\frac{1}{n-1},$$

where the inequalities follow from $\partial \chi_2^2(\mathbf{x}^3)/\partial x_j \cdot \partial \chi_1^1(\mathbf{x}^2)/\partial x_2 > 0$ and equation (G.2), respectively.

Finally, to prove property (G.8) use equation (G.6) to write

$$\frac{1}{h_q(x_q)}\frac{\partial \chi_2^2(\mathbf{x}^3)}{\partial x_q}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial \chi_2^2(\mathbf{x}^3)}{\partial x_j} = \frac{1}{D_2}\left[\Delta_{2,j}(\mathbf{x}) - \frac{\Delta_{2,q}(\mathbf{x})}{n-1} + \right.$$
$$\left. h_1(x_1)\left(\frac{1}{h_j(x_j)}\frac{\partial \chi_1^1(\mathbf{x}^2)}{\partial x_j} - \frac{1}{h_q(x_q)}\frac{\partial \chi_1^1(\mathbf{x}^2)}{\partial x_q}\frac{1}{n-1}\right)\Delta_{2,1}(\mathbf{x})\right],$$

where $D_2 = \Pi_2'(\mathbf{x}) + \frac{\partial \chi_1^1(\mathbf{x}^2)}{\partial x_2} h_1(x_1)\Delta_{2,1}(\mathbf{x}) > 0$. We show that a lower bound of this expression is positive. Taking $-\partial \chi_1^1(\mathbf{x}^2)/\partial x_q > 0$ to zero, we obtain

$$\frac{1}{D_2}\left[\Delta_{2,j}(\mathbf{x}) - \frac{\Delta_{2,q}(\mathbf{x})}{n-1} + \frac{h_1(x_1)}{h_j(x_j)}\frac{\partial \chi_1^1(\mathbf{x}^2)}{\partial x_j}\Delta_{2,1}(\mathbf{x})\right]$$
$$> \frac{1}{D_2}\left[\Delta_{2,j}(\mathbf{x}) - \frac{\Delta_{2,q}(\mathbf{x})}{n-1} - \frac{\Delta_{2,1}(\mathbf{x})}{n-1}\right] > \frac{1}{D_2}\left[\Delta_{2,j}(\mathbf{x}) - \frac{2\Delta_{2,q}(\mathbf{x})}{n-1}\right] > 0.$$

The first inequality follows from using lower bound (G.2). The other two inequalities follow from Lemma H.3 and the fact that $q \in \{2, \ldots, j-1\}$. □

- **Firm** $k \in \{3, \ldots, n\}$: Suppose that, for every $p \in \{1, \ldots, k-1\}$ and $j \in \{p+1, \ldots, n\}$, we have proven that: $\chi_p^p(\mathbf{x}^{p+1})$ is unique;

$$0 > \frac{\partial \chi_p^p(\mathbf{x}^k)}{\partial x_j} = -\frac{h_j(x_j)\Delta_{p,j}(\mathbf{x}) + \sum_{\ell=1}^{p-1}\frac{\partial \chi_\ell^{p-1}(\mathbf{x}^p)}{\partial x_j}h_\ell(x_\ell)\Delta_{p,\ell}(\mathbf{x})}{\Pi_p'(\mathbf{x}) + \sum_{\ell=1}^{p-1}\frac{\partial \chi_\ell^{p-1}(\mathbf{x}^p)}{\partial x_p}h_\ell(x_\ell)\Delta_{p,\ell}(\mathbf{x})}; \tag{G.10}$$

$$0 > \frac{\partial \chi_q^p(\mathbf{x}^k)}{\partial x_j} > -\frac{h_j(x_j)}{h_q(x_q)}\frac{1}{n-1} \quad \text{for } q \in \{1, \ldots, p\} \text{ and;} \tag{G.11}$$

$$\frac{1}{h_j(x_j)}\frac{\partial \chi_p^p(\mathbf{x}^{p+1})}{\partial x_j} < \frac{1}{h_q(x_q)}\frac{\partial \chi_p^p(\mathbf{x}^{p+1})}{\partial x_q}\frac{1}{n-1} \quad \text{for } q \in \{p+1, \ldots, j-1\}. \tag{G.12}$$

Fix $\mathbf{x}^{k+1}$ and let $\mathbf{x} = (\chi_1^{k-1}(\mathbf{x}^k), \ldots, \chi_{k-1}^{k-1}(\mathbf{x}^k), \mathbf{x}^k)$. We show that the best response $\chi_k^k(\mathbf{x}^{k+1})$ is unique by showing that $\hat{\Pi}_k(\mathbf{x}^k)$ is strictly increasing in x_k. Differentiating,

$$\begin{aligned}\hat{\Pi}_k'(\mathbf{x}^k) &= \Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1} \frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k} h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x}) \\ &> \Pi_k'(\mathbf{x}) - h_k(x_k)\sum_{\ell=1}^{k-1}\frac{\Delta_{k,\ell}(\mathbf{x})}{n-1} > \Pi_k'(\mathbf{x}) - h_k(x_k)\frac{(k-1)\Delta_{k,k-1}(\mathbf{x})}{n-1} > 0,\end{aligned}$$

where the inequalities follow from lower bound (G.11), Lemma H.3, and sufficient condition (G.1), respectively. This proves uniqueness of the best response. The next result completes the induction argument.

Claim 13. Under condition (G.1), for every $j \in \{k+1,\dots,m\}$ and $p \in \{1,\dots,k\}$, $\partial \chi_p^k(\mathbf{x}^{k+1})/\partial x_j$ satisfies

$$\frac{\partial \chi_k^k(\mathbf{x}^{k+1})}{\partial x_j} = -\frac{h_j(x_j)\Delta_{k,j}(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x})}{\Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x})} \tag{G.13}$$

$$0 > \frac{\partial \chi_p^k(\mathbf{x}^{k+1})}{\partial x_j} > -\frac{h_j(x_j)}{h_p(x_p)}\frac{1}{n-1} \quad \text{and,} \tag{G.14}$$

$$\frac{1}{h_j(x_j)}\frac{\partial \chi_k^k(\mathbf{x}^{k+1})}{\partial x_j} < \frac{1}{h_q(x_q)}\frac{\partial \chi_k^k(\mathbf{x}^{k+1})}{\partial x_q}\frac{1}{n-1} \quad \text{for } q \in \{k+1,\dots,j-1\} \tag{G.15}$$

Proof. To show equation (G.13) use the implicit differentiation, the chain rule, and equation (B.2) to obtain

$$\begin{aligned}\frac{\partial \chi_k^k(\mathbf{x}^{k+1})}{\partial x_j} &= -\frac{\partial \hat{\Pi}_k(\mathbf{x})/\partial x_j}{\partial \hat{\Pi}_k(\mathbf{x})/\partial x_k} = -\frac{\frac{\partial \Pi_k(\mathbf{x})}{\partial x_j} + \sum_{\ell=1}^{k-1}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}\frac{\partial \Pi_k(\mathbf{x})}{\partial x_\ell}}{\Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{\partial \Pi_k(\mathbf{x})}{\partial x_\ell}}. \\ &= -\frac{h_j(x_j)\Delta_{k,j}(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x})}{\Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x})}.\end{aligned}$$

We already showed that the denominator is positive. We show that a lower bound of the numerator is positive, which immediately implies the upper bound in equation (G.14) for the case when $p = k$. Using equation (G.11) a lower bound for the numerator is

$$h_j(x_j)\left[\Delta_{k,j}(\mathbf{x}) - \sum_{\ell=1}^{k-1}\frac{\Delta_{k,\ell}(\mathbf{x})}{n-1}\right] > h_j(x_j)\left[\Delta_{k,j}(\mathbf{x}) - \frac{(k-1)\Delta_{k,k-1}(\mathbf{x})}{n-1}\right] > 0,$$

where both inequalities follows from Lemma H.3. Thus, the numerator is positive.

For the lower bound in equation (G.14) in the case $p = k$, replace (G.13) into (G.14) and observe that the inequality holds if and only if the following expression is positive

$$h_j(x_j)\left[\left(\frac{1}{h_k(x_k)}\frac{\Pi'_k(\mathbf{x})}{n-1} - \Delta_{k,j}(\mathbf{x})\right) + \sum_{\ell=1}^{k-1} h_\ell(x_\ell)\left(\frac{1}{h_k(x_k)}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}\right)\Delta_{k,\ell}(\mathbf{x})\right]. \quad \text{(G.16)}$$

The first term in round brackets is positive due to sufficient condition (G.1). We now work with the summation and show that it is also positive. Before doing this, observe that, by definition, for every $\ell \in \{1, \ldots, k-1\}$

$$\chi_\ell^k(\mathbf{x}^{k+1}) = \chi_\ell^\ell(\chi_{\ell+1}^k(\mathbf{x}^{k+1}), \chi_{\ell+2}^k(\mathbf{x}^{k+1}), \ldots, \chi_k^k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}).$$

Then, for any $j \in \{k+1, \ldots, m\}$

$$\frac{\partial \chi_\ell^k(\mathbf{x}^{k+1})}{\partial x_j} = \frac{\partial \chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_j} + \sum_{q=\ell+1}^{k}\frac{\partial \chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_q}\frac{\partial \chi_q^k(\mathbf{x}^{k+1})}{\partial x_j}. \quad \text{(G.17)}$$

For a given ℓ in the summation in equation (G.16), we use equation (G.17) to write the round bracket as

$$\left(\frac{1}{h_k(x_k)}\frac{\partial \chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_k}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial \chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_j}\right) + \sum_{q=\ell+1}^{k-1}\frac{\partial \chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_q}\left(\frac{1}{h_k(x_k)}\frac{\partial \chi_q^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial \chi_q^{k-1}(\mathbf{x}^k)}{\partial x_j}\right). \quad \text{(G.18)}$$

Substitute equation (G.18) when $\ell = 1$ into the summation in equation (G.16) to obtain

$$\sum_{\ell=2}^{k-1}\left(h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x}) + \frac{\partial \chi_1^1(\mathbf{x}^2)}{\partial x_2}a_1\right)\left(\frac{1}{h_k(x_k)}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial \chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}\right) + a_1\left(\frac{1}{h_k(x_k)}\frac{\partial \chi_1^1(\mathbf{x}^k)}{\partial x_k}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial \chi_1^1(\mathbf{x}^k)}{\partial x_j}\right), \quad \text{(G.19)}$$

where $a_1 = \Delta_{k,1}(\mathbf{x})h_1(x_1) > 0$. Then, substituting (in increasing order) into equation (G.19) the expression (G.18) for $\ell = 2$, $\ell = 3$ until $\ell = k-1$, we obtain that the summation in equation (G.16) is equal to

$$\sum_{\ell=1}^{k-1} a_\ell\left(\frac{1}{h_k(x_k)}\frac{\partial \chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_k}\frac{1}{n-1} - \frac{1}{h_j(x_j)}\frac{\partial \chi_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_j}\right) > 0, \quad \text{(G.20)}$$

where

$$a_\ell = h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x}) + \sum_{p=1}^{\ell-1} \frac{\partial \chi_p^p(\mathbf{x}^{p+1})}{\partial x_\ell} a_p \tag{G.21}$$

is defined recursively. The parenthesis in equation (G.20) is positive by equation (G.12). We show that each a_ℓ is positive, which proves the lower bound in equation (G.14) when $p = k$. By induction, suppose that for every $p \in \{1, \ldots, \ell-1\}$ we have shown that $0 < h_p(x_p)\Delta_{k,p}(\mathbf{x}) \leq a_p$ (we already showed this for a_1). We need to show that the same inequalities hold for equation (G.21). First, because $\partial\chi_p^p(\mathbf{x}^{p+1})/\partial x_\ell < 0$ and $a_p > 0$ (by the induction hypothesis) it is easy to see that $a_\ell < h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x})$. Using the lower bound in equation (G.11) and the upper bound for a_p we obtain the following lower bound for equation (G.21)

$$a_\ell > h_\ell(x_\ell)\left[\Delta_{k,\ell}(\mathbf{x}) - \sum_{p=1}^{\ell-1} \frac{\Delta_{k,p}(\mathbf{x})}{n-1}\right] > h_\ell(x_\ell)\left[1 - \frac{(\ell-1)}{n-1}\right]\Delta_{k,\ell}(\mathbf{x}) > 0,$$

where the second inequality follows from Lemma H.3; which proves the result.

To prove the upper bound in equation (G.14) for $p \in \{1, \ldots, k-1\}$ we proceed by induction downwards. Suppose that for every firm $\ell \in \{p+1, \ldots, k\}$ we have proven

$$0 > \frac{\partial \chi_\ell^k(\mathbf{x}^{k+1})}{\partial x_j} > -\frac{h_j(x_j)}{h_\ell(x_\ell)} \frac{1}{n-1} \tag{G.22}$$

we prove that equation (G.14) holds for p. Observing that, in equation (G.17), $\partial\chi_p^p(\mathbf{x}^{p+1})/\partial x_\ell < 0$, we can construct an upper bound for $\partial\chi_p^k(\mathbf{x}^{k+1})/\partial x_j$ using the induction hypothesis (G.22)

$$\frac{\partial \chi_p^k(\mathbf{x}^{k+1})}{\partial x_j} < \frac{\partial \chi_p^p(\mathbf{x}^{p+1})}{\partial x_j} - \sum_{\ell=p+1}^{k} \frac{\partial \chi_p^p(\mathbf{x}^{p+1})}{\partial x_\ell} \frac{h_j(x_j)}{h_\ell(x_\ell)} \frac{1}{n-1}$$

Using equation (G.10), the upper bound for $\partial\chi_p^k(\mathbf{x}^{k+1})/\partial x_j$ is equal to

$$\frac{h_j(x_j)}{D_p} \sum_{\ell=p+1}^{k} \left(h_\ell(x_\ell)\Delta_{p,\ell}(\mathbf{x}) + \sum_{q=1}^{p-1} \frac{\partial \chi_q^{p-1}(\mathbf{x}^p)}{\partial x_\ell} h_q(x_q)\Delta_{p,q}(\mathbf{x}) \right) \frac{1}{h_\ell(x_\ell)} \frac{1}{n-1}$$
$$- \frac{h_j(x_j)}{D_p} \left(\Delta_{p,j}(\mathbf{x}) + \frac{1}{h_j(x_j)} \sum_{q=1}^{p-1} \frac{\partial \chi_q^{p-1}(\mathbf{x}^p)}{\partial x_j} \frac{\Delta_{p,q}(\mathbf{x})}{h_q(x_q)} \right)$$

where $D_p = \Pi_p'(\mathbf{x}) + \sum_{q=1}^{p-1} \frac{d\chi_q^{p-1}(\mathbf{x}^p)}{dx_p} h_q(x_q)\Delta_{p,q}(\mathbf{x}) > 0$. Taking $\partial\chi_q^{p-1}(\mathbf{x}^p)/\partial x_\ell < 0$ equal to zero and $\partial\chi_q^{p-1}(\mathbf{x}^p)/\partial x_j < 0$ to the lower bound in equation (G.11), we build the following upper bound for the previous expression (and omitting D_p, as it does not affect the sign)

$$h_j(x_j)\left[\sum_{\ell=p+1}^{k}\frac{\Delta_{p,\ell}(\mathbf{x})}{n-1}+\sum_{q=1}^{p-1}\frac{\Delta_{p,q}(\mathbf{x})}{n-1}-\Delta_{p,j}(\mathbf{x})\right]<h_j(x_j)\left[\frac{k-1}{n-1}-1\right]\Delta_{p,j}(\mathbf{x})$$
$$\leq 0.$$

The inequality follows from equation Lemma H.3; proving $\partial\chi_p^k(\mathbf{x}^{k+1})/\partial x_j<0$. The lower bound for $\partial\chi_p^k(\mathbf{x}^{k+1})/\partial x_j$ follows from equation (G.17) and observing

$$\frac{\partial\chi_p^k(\mathbf{x}^{k+1})}{\partial x_j}>\frac{\partial\chi_p^p(\mathbf{x}^{p+1})}{\partial x_j}>-\frac{h_j(x_j)}{h_p(x_p)}\frac{1}{n-1}$$

where the first inequality follows from $(\partial\chi_p^p(\mathbf{x}^{k+1})/\partial x_\ell)\cdot(\partial\chi_\ell^k(\mathbf{x}^{k+1})/\partial x_j)>0$ for every ℓ, and the second from the lower bound in equation (G.11).

Finally, we prove equation (G.15) using equation (G.13) to write

$$\frac{1}{h_q(x_q)}\frac{\partial\chi_k^k(\mathbf{x}^{k+1})}{\partial x_q}\frac{1}{n-1}-\frac{1}{h_j(x_j)}\frac{\partial\chi_k^k(\mathbf{x}^{k+1})}{\partial x_j}=\frac{1}{D_k}\Bigg[\Delta_{k,j}(\mathbf{x})-\frac{\Delta_{k,q}(\mathbf{x})}{n-1}+$$
$$\sum_{\ell=1}^{k-1}h_\ell(x_\ell)\left(\frac{1}{h_j(x_j)}\frac{\partial\chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}-\frac{1}{h_q(x_q)}\frac{\partial\chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_q}\frac{1}{n-1}\right)\Delta_{k,\ell}(\mathbf{x})\Bigg],$$

where $D_k=\Pi_k'(\mathbf{x})+\sum_{\ell=1}^{k-1}\frac{\partial\chi_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}h_\ell(x_\ell)\Delta_{k,\ell}(\mathbf{x})>0$. We show that a lower bound of this expression is positive. Taking $-\partial\chi_\ell^{k-1}(\mathbf{x}^k)/\partial x_q>0$ to zero and $\partial\chi_\ell^{k-1}(\mathbf{x}^k)/\partial x_j<0$ to the lower bound in equation (G.11), we obtain

$$\frac{1}{D_k}\left[\Delta_{k,j}(\mathbf{x})-\frac{\Delta_{k,q}(\mathbf{x})}{n-1}-\sum_{\ell=1}^{k-1}\frac{\Delta_{k,\ell}(\mathbf{x})}{n-1}\right]>\frac{1}{D}\left[\Delta_{k,j}(\mathbf{x})-\frac{k\Delta_{k,q}(\mathbf{x})}{n-1}\right]>0.$$

The inequalities follow from Lemma H.3 and the fact that $q\in\{k,\dots,j-1\}$. □

Because at each step best responses are unique and at $k=n$ the firm has only one best response when every firm $k<n$ is best responding to x_n and to each other, there is a unique Herculean equilibrium within the herculean class.

No non-herculean equilibria exists: By contradiction. Suppose $\mathbf{x}$ represents a non-herculean equilibrium. Order firms from smallest cutoff x_1 to largest, x_n. Let p be the first instance (smallest cutoff) that a strength reversal occurs. That is, $x_p<x_{p+1}$ but $s_p>s_{p+1}$. Because every firm $k\in\{1,\dots,p\}$ is ordered by strength, they satisfy conditions (G.13), (G.11), and (G.12). We show that x_{p+1} cannot lie above x_p (i.e, a contradiction). Fix the strategies of all the firms but p and $p+1$, i.e., $\mathbf{x}_{E\setminus\{p,p+1\}}$, and let $\hat{x}$ be the value that satisfies $\chi_p(\hat{x},\mathbf{x}_{E\setminus\{p,p+1\}})=\hat{x}$, where $\chi_p(\mathbf{x}_{-p})$ is firm's p unique best response to $\mathbf{x}_{-p}$. This best response exists (and is unique) by Lemma H.1. The value $\hat{x}$ exists because $\chi_p(\mathbf{x}_{-p})$ is continuously decreasing in x_{p+1}. This implies that, for every $x_{p+1}>\hat{x}$, $\chi_p(\mathbf{x}_{-p})<x_{p+1}$. In addition,

following analogous steps to those in Claim 11, we can show that $\partial \chi_p(\mathbf{x}_{-p})/\partial x_{p+1} > -h_{p+1}(x_{p+1})/(h_p(x_p)(n-1))$. Then, by Lemma H.2, $\Pi_p(\hat{x}, \hat{x}, \mathbf{x}_{E\setminus\{p,p+1\}}) = 0 < \Pi_{p+1}(\hat{x}, \hat{x}, \mathbf{x}_{E\setminus\{p,p+1\}})$. Also, letting $\hat{\mathbf{x}} = (\chi_p(\mathbf{x}_{-p}), \mathbf{x}_{-p})$ observe that

$$\begin{aligned} \frac{d\Pi_{p+1}(\hat{\mathbf{x}})}{dx_{p+1}} &= \Pi'_{p+1}(\hat{\mathbf{x}}) + \frac{\partial \chi_p(\mathbf{x}_{-p})}{\partial x_{p+1}} \frac{\partial \Pi_{p+1}(\hat{\mathbf{x}})}{\partial x_p} \\ &> \Pi'_{p+1}(\hat{\mathbf{x}}) - h_{p+1}(x_{p+1}) \frac{\Delta_{p+1,p}(\hat{\mathbf{x}})}{n-1} > 0 \end{aligned}$$

Thus, $\Pi_{p+1}(\hat{\mathbf{x}})$ is strictly increasing in x_{p+1} which implies that $\Pi_{p+1}(\hat{\mathbf{x}}) > 0$ for every $x_{p+1} \geq \hat{x}$, which implies that no equilibrium cutoff $x_{p+1} > \chi_p(\mathbf{x}_{-p}) = x_p$ exists. ■

H Auxiliary Results

Lemma H.1. *Let Π_i be defined by (6). Let A and B be disjoint sets of k and r firms, where $k + r < n$, such that $i \in A$. Define $f : [a,b]^{k+r} \to [a,b]^{n-k-r}$ to be a continuous function and let $\mathbf{x}_B$ be any vector of cutoff strategies for firms in set B. Then, there exist a value $\tilde{x}$ such that the symmetric k-dimensional vector $\tilde{\mathbf{x}}_A = (\tilde{x})_{i\in A}$ satisfies $\Pi_i(\tilde{\mathbf{x}}_A, f(\tilde{\mathbf{x}}_A, \mathbf{x}_B), \mathbf{x}_B) = 0$. The vector $\tilde{\mathbf{x}}_A$ is continuous in each dimension of $\mathbf{x}_B$. When the function f is constant in $\tilde{x}$—i.e., when $\mathbf{x}_{E\setminus A} = (f(\tilde{\mathbf{x}}_A, \mathbf{x}_B), \mathbf{x}_B)$ does not change with $\tilde{\mathbf{x}}_A$—the value of $\tilde{x}$ is unique.*

Proof. Fix $\mathbf{x}_B$, because f is continuous, the function $\Pi_i(\mathbf{x}_A, f(\mathbf{x}_A, \mathbf{x}_B), \mathbf{x}_B)$ is continuous in the input value x of the symmetric vector $\mathbf{x}_A$. Let $\underline{\mathbf{v}}_A = (\underline{v}_i)_{i\in A}$ and $\overline{\mathbf{v}}_A = (\overline{v}_i)_{i\in A}$. Observe that assumptions A3 and A2 jointly imply

$$\Pi_i(\underline{\mathbf{v}}_A, f(\underline{\mathbf{v}}_A, \mathbf{x}_B), \mathbf{x}_B) \leq \pi_i(\underline{v}_i) < 0.$$

Similarly, assumption A3 and Lemma B.2 together imply,

$$\Pi_i(\overline{\mathbf{v}}_A, f(\overline{\mathbf{v}}_A, \mathbf{x}_B), \mathbf{x}_B) \geq \Pi_i(\overline{v}_i, a_{-i}) > 0.$$

Then, by the intermediate value theorem, there exist $\tilde{x} \in (\underline{v}_i, \bar{v}_i)$ such that

$$\Pi_i(\tilde{\mathbf{x}}_A, f(\tilde{\mathbf{x}}_A, \mathbf{x}_B), \mathbf{x}_B) = 0.$$

Because the functions Π_i and f are continuous, the value $\tilde{\mathbf{x}}_A$ is continuous in each dimension of $\mathbf{x}_B$. For uniqueness when f is constant, by the chain rule, $d\Pi_i/dx = \sum_{k\in A} \partial\Pi_i/\partial x_k > 0$ where the inequality follows from Lemma B.2. Therefore $\Pi_i(\mathbf{x}_A, f(\mathbf{x}_A, \mathbf{x}_B), \mathbf{x}_B)$, as a function of the value x for the symmetric vector $\mathbf{x}_A$, is increasing and crosses zero once. ■

Lemma H.2. *Consider an ordered game, in which the firms' identities are ordered by strength, with firm 1 being the strongest. Then, for any firm $i < j$, valuation y,* *and vector of strategies for the other firms* $\mathbf{x}_{E\setminus\{i,j\}}$*, we have*

$$\Pi_i(y; y, \mathbf{x}_{E\setminus\{i,j\}}) > \Pi_j(y; y, \mathbf{x}_{E\setminus\{i,j\}}).$$

Proof. If firms are ordered by profit, the inequality follows by definition. Recall $\phi(v_e) = \prod_{j\in e} f_j(v_j)$. For games ordered by distribution, observe

$$\begin{aligned}
\Pi_i(y; y, \mathbf{x}_{E\setminus\{i,j\}}) =& \sum_{e\in\mathcal{E}_i\setminus\mathcal{E}_j} \left\{ \left(F_j(y) \prod_{k\in e^c\setminus j} F_k(x_k) \right) \int_{(x_k)_{k\in e\setminus i}}^{b} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} + \\
& \sum_{e\in\mathcal{E}_i\cap\mathcal{E}_j} \left\{ \left(\prod_{k\in e^c} F_k(x_k) \right) \int_y^b \int_{(x_k)_{k\in e\setminus\{i,j\}}}^{b} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus\{i,j\}}) f_j(v) d^{n_e-1} v_{e\setminus i} \right\} \\
& > \sum_{e\in\mathcal{E}_i\setminus\mathcal{E}_j} \left\{ \left(F_i(y) \prod_{k\in e^c\setminus j} F_k(x_k) \right) \int_{(x_k)_{k\in e\setminus i}}^{b} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} + \\
& \sum_{e\in\mathcal{E}_i\cap\mathcal{E}_j} \left\{ \left(\prod_{k\in e^c} F_k(x_k) \right) \int_y^b \int_{(x_k)_{k\in e\setminus\{i,j\}}}^{b} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus\{i,j\}}) f_i(v) d^{n_e-1} v_{e\setminus i} \right\} \\
& = \Pi_j(y, y, x_{E\setminus\{i,j\}}),
\end{aligned}$$

where the inequality uses two properties of FOSD. The first term uses that $F_i(x) \leq F_j(x)$ for all x. The second term uses that $\int_y^b \varphi(x) f_i(x) dx \leq \int_y^b \varphi(x) f_j(x) dx$ for any non-increasing function $\varphi(x)$. ■

Lemma H.3. *Let firm* k *be stronger than firm* j*. Suppose the firms play cutoffs* $x_k < x_j$*; then, for any firm* i*,* $\Delta_{i,j}(\mathbf{x}) \geq \Delta_{i,k}(\mathbf{x})$ *if: (i) firms are ordered by profits, or; (ii) firms are ordered by distribution and the profit gain only depends on the number of entrants.*

Proof. Start by observing that, in the expression for $\Delta_{i,j}(\mathbf{x})$ (see equation (7)), the sum over market structures $\mathcal{E}_i \setminus \mathcal{E}_j$ can be divided into two disjoints sets: $(\mathcal{E}_i \cap \mathcal{E}_k) \setminus \mathcal{E}_j$ and $\mathcal{E}_i \setminus (\mathcal{E}_j \cup \mathcal{E}_k)$. Using these sets subtract $\Delta_{i,j}(\mathbf{x}) - \Delta_{i,k}(\mathbf{x})$ to obtain

$$\begin{aligned}
&\sum_{e\in\mathcal{E}_i\setminus(\mathcal{E}_j\cup\mathcal{E}_k)} \left\{ \left(\prod_{\ell\in e^c} F_\ell(x_\ell) \right) \int_{(x_\ell)_{\ell\in e\setminus i}}^{b} \left(\delta_i(x_i, x_j, v_{e\setminus i}) - \delta_i(x_i, x_k, v_{e\setminus i})\right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} \\
&\quad + \sum_{e\in(\mathcal{E}_i\cap\mathcal{E}_k)\setminus\mathcal{E}_j} \left\{ \left(\prod_{\ell\in e^c\setminus j} F_\ell(x_\ell) \right) F_j(x_j) \int_{x_k}^b \int_{(x_\ell)_{\ell\in e\setminus\{i,k\}}}^{b} \delta_i(x_i, x_j, v_{e\setminus i}) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} \\
&\quad - \sum_{e\in(\mathcal{E}_i\cap\mathcal{E}_j)\setminus\mathcal{E}_k} \left\{ \left(\prod_{\ell\in e^c\setminus k} F_\ell(x_\ell) \right) F_k(x_k) \int_{x_j}^b \int_{(x_\ell)_{\ell\in e\setminus\{i,j\}}}^{b} \delta_i(x_i, x_k, v_{e\setminus i}) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} \qquad \text{(H.1)}
\end{aligned}$$

where we used $\mathcal{E}_i \setminus (\mathcal{E}_j \cup \mathcal{E}_k) = \mathcal{E}_i \setminus (\mathcal{E}_k \cup \mathcal{E}_j)$ and, by profits being anonymous, we dropped the second sub index from the profit gain $\delta_i(x_{\{i,j\}}, v_{e\setminus i})$. Equation (H.1) has three summations. For the first one, observe that the term inside the integral is non-negative as

$$\delta_i(x_i, x_j, v_{e\setminus i}) - \delta_i(x_i, x_k, v_{e\setminus i}) = \pi_i(x_i, x_k, v_{e\setminus i}) - \pi_i(x_i, x_j, v_{e\setminus i}) \geq 0$$

where the last inequality follows from assumption A2 and $x_k < x_j$. This implies that the first summation is non-negative.

For the last two summations in (H.1), we show that a lower bound of the first term is equal to the subtracting term. Thus, the difference is non-negative. Observe that, for each market structure $e \in (\mathcal{E}_i \cap \mathcal{E}_k) \setminus \mathcal{E}_j$ in the first term, we can remove firm k and add firm j, i.e., $\hat{e} = (e \setminus j) \cup k$, and the new market structure satisfies $\hat{e} \in (\mathcal{E}_i \cap \mathcal{E}_j) \setminus \mathcal{E}_k$, which belongs to the second term. We show that a lower bound of payoffs in e is equal to those in $\hat{e}$.

(i) Ordered by profit: When the game is ordered by profit, we can drop the sub-index from the distributions of types. Bounding the expression under market structure $e \in (\mathcal{E}_i \cap \mathcal{E}_k) \setminus \mathcal{E}_j$

$$\left(\prod_{\ell \in e^c \setminus j} F(x_\ell)\right) F(x_j) \int_{x_k}^{b} \int_{(x_\ell)_{\ell \in e \setminus \{i,k\}}}^{b} \delta_i(x_i, x_j, v_{e\setminus i}) \phi(v_{e\setminus\{i,k\}}) f(v_k) d^{n_e - 1} v_{e\setminus i}$$
$$> \left(\prod_{\ell \in e^c \setminus k} F(x_\ell)\right) F(x_k) \int_{x_j}^{b} \int_{(x_\ell)_{\ell \in e \setminus \{i,j\}}}^{b} \delta_i(x_i, x_k, v_{e\setminus i}) \phi(v_{e\setminus\{i,k\}}) f(v_j) d^{n_e - 1} v_{e\setminus i}$$

where in the inequality we used $x_j > x_k$ in three places: (i) in the probability of firm j being out of the market; (ii) in the domain of integration over k's types, which jointly with $\delta_i(x_i, x_j, v_{e\setminus i}) \geq 0$ implies that we are integrating over a smaller domain, decreasing the value of the integral, and; (iii) $\delta_i(x_i, s, v_{e\setminus i})$ being increasing in s (by assumption A2). Finally, we inverted the roles of firm k and j in e using that payoffs are anonymous to re-arrange indexes. Thus, we obtain that the lower bound equals the payoffs in the third summation of (H.1) under market structure $\hat{e} \in (\mathcal{E}_i \cap \mathcal{E}_j) \setminus \mathcal{E}_k$. Because the inequality holds for every market structure $e \in (\mathcal{E}_i \cap \mathcal{E}_k) \setminus \mathcal{E}_j$, the result follows.

(ii) Ordered by distribution: When the game is ordered by distributions and the profit gain only depends on the number of entrants, the expression under market structure $e \in (\mathcal{E}_i \cap \mathcal{E}_k) \setminus \mathcal{E}_j$ becomes

$$\left(\prod_{\ell \in e^c \setminus j} F_\ell(x_\ell)\right) F_j(x_j) \left((1 - F_k(x_k)) \prod_{\ell \in e \setminus \{i,k\}} (1 - F_\ell(x_\ell))\right) \delta_i(x_i, n_e)$$
$$> \left(\prod_{\ell \in e^c \setminus k} F_\ell(x_\ell)\right) F_k(x_k) \left((1 - F_j(x_j)) \prod_{\ell \in e \setminus \{i,j\}} (1 - F_\ell(x_\ell))\right) \delta_i(x_i, n_e).$$

The inequality uses stochastic dominance, the fact that $x_k < x_j$ (so that $F_j(x_j) \geq F_j(x_k) \geq F_k(x_k)$), and re-arranges indexes. The lower bound equals the payoffs in the third summation of (H.1) under market structure $\hat{e} \in (\mathcal{E}_i \cap \mathcal{E}_j) \setminus \mathcal{E}_k$. The inequality holds for every market structure $e \in (\mathcal{E}_i \cap \mathcal{E}_k) \setminus \mathcal{E}_j$, proving the result. ■