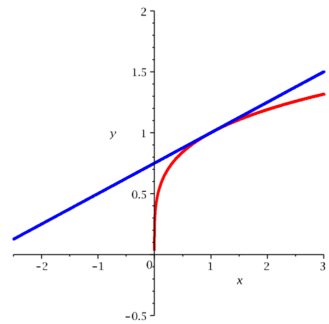
Linear Approximations and Differentials

PROBLEM: Approximate the number $\sqrt[4]{1.1}$.



IDEA: We have seen that a curve lies very close to its tangent line near the point of tangency. This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function (in our case $\sqrt[4]{1}$), but difficult (or even impossible) to compute nearby values of f (in our case $\sqrt[4]{1.1}$). So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$.

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a.

Solution: The point-slope equation of the tangent line is

$$y = f(a) + f'(a)(x - a)$$

therefore

$$f(x) \approx f(a) + f'(a)(x - a) \quad \text{if } x \text{ is close to } a$$

In particular, if $f(x) = \sqrt[4]{x}$, then

$$\sqrt[4]{x} \approx \sqrt[4]{a} + \frac{1}{4a^{3/4}}(x - a)$$

since $f'(a) = \dfrac{1}{4a^{3/4}}$. Plugging in $x = 1.1$ and $a = 1$, we get

$$\sqrt[4]{1.1} \approx \sqrt[4]{1} + \frac{1}{4 \cdot 1^{3/4}}(1.1 - 1) = 1 + \frac{1}{4}(0.1) = 1.025 \quad \text{(the true value is 1.024113689...)}$$

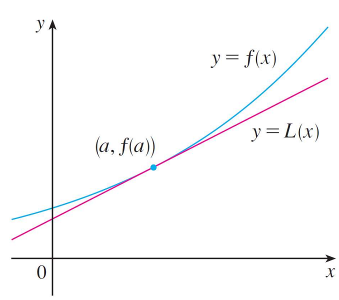
The approximation

$$\boxed{f(x) \approx f(a) + f'(a)(x - a)} \tag{1}$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is,

$$\boxed{L(x) = f(a) + f'(a)(x - a)} \tag{2}$$

is called the **linearization** of f at a.



EXAMPLE: Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

Solution: The derivative of $f(x) = \sqrt{x+3}$ is

$$f'(x) = \left((x+3)^{1/2}\right)' = \frac{1}{2}(x+3)^{1/2-1} \cdot (x+3)' = \frac{1}{2}(x+3)^{-1/2} \cdot 1 = \frac{1}{2\sqrt{x+3}}$$

and so we have $f(1) = 2$ and $f'(1) = \frac{1}{4}$. Putting these values into (2), we see that the linearization is

$$L(x) = f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1) = \frac{7}{4} + \frac{x}{4}$$

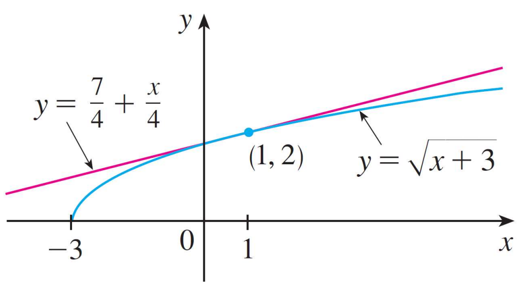
The corresponding linear approximation (1) is

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4} \quad \text{(when } x \text{ is near 1)}$$

In particular, we have

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995 \quad \text{and} \quad \sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125 \tag{3}$$

The linear approximation is illustrated in the figure below.



	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176 . . .
$\sqrt{3.98}$	0.98	1.995	1.99499373 . . .
$\sqrt{4}$	1	2	2.00000000 . . .
$\sqrt{4.05}$	1.05	2.0125	2.01246117 . . .
$\sqrt{4.1}$	1.1	2.025	2.02484567 . . .
$\sqrt{5}$	2	2.25	2.23606797 . . .
$\sqrt{6}$	3	2.5	2.44948974 . . .

We see that, indeed, the tangent line approximation is a good approximation to the given function when x is near 1. We also see that our approximations are overestimates because the tangent line lies above the curve.

Of course, a calculator could give us approximations for $\sqrt{3.98}$ and $\sqrt{4.05}$, but the linear approximation gives an approximation *over an entire interval.*

In the table above we compare estimates from the obtained linear approximation with the true values. Notice from this table, and also from the figure above, that the tangent line approximation gives good estimates when x is close to 1 but the accuracy of the approximation deteriorates when x is farther away from 1.

EXAMPLE: Find the linearization of the function $f(x) = \sqrt{x}$ at $a = 4$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

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Solution: The derivative of $f(x) = \sqrt{x}$ is

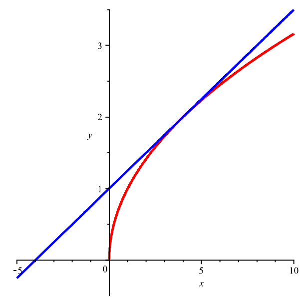
$$f'(x) = \left(x^{1/2}\right)' = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

and so we have $f(4) = 2$ and $f'(4) = \frac{1}{4}$. Putting these values into (2), we see that the linearization is

$$L(x) = f(4) + f'(4)(x-4) = 2 + \frac{1}{4}(x-4) = 1 + \frac{x}{4}$$

The corresponding linear approximation (1) is

$$\sqrt{x} \approx 1 + \frac{x}{4} \quad \text{(when } x \text{ is near 4)}$$



In particular, we have

$$\sqrt{3.98} \approx 1 + \frac{3.98}{4} = 1.995 \quad \text{and} \quad \sqrt{4.05} \approx 1 + \frac{4.05}{4} = 2.0125$$

Note that we got the same results as in (3) since

$$1 + \frac{3.98}{4} = 1 + \frac{3 + 0.98}{4} = 1 + \frac{3}{4} + \frac{0.98}{4} = \frac{7}{4} + \frac{0.98}{4}$$

and

$$1 + \frac{4.05}{4} = 1 + \frac{3 + 1.05}{4} = 1 + \frac{3}{4} + \frac{1.05}{4} = \frac{7}{4} + \frac{1.05}{4}$$

or, in general,

$$1 + \frac{x}{4} = 1 + \frac{3 + x - 3}{4} = 1 + \frac{3}{4} + \frac{x-3}{4} = \frac{7}{4} + \frac{x-3}{4}$$

Our approximations are overestimates because the tangent line lies above the curve.

EXAMPLE: Find the linearization of the function $f(x) = \sqrt[3]{1+x}$ at $a = 0$ and use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$. Are these approximations overestimates or underestimates?

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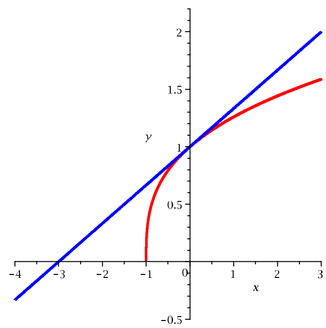
Solution: The derivative of $f(x) = \sqrt[3]{1+x}$ is

$$f'(x) = \left((1+x)^{1/3}\right)' = \frac{1}{3}(1+x)^{1/3-1} \cdot (1+x)' = \frac{1}{3}(1+x)^{-2/3} \cdot 1 = \frac{1}{3\sqrt[3]{(1+x)^2}}$$

and so we have $f(0) = 1$ and $f'(0) = \frac{1}{3}$. Putting these values into (2), we see that the linearization is

$$L(x) = f(0) + f'(0)(x-0) = 1 + \frac{x}{3}$$



The corresponding linear approximation (1) is

$$\sqrt[3]{1+x} \approx 1 + \frac{x}{3} \quad \text{(when } x \text{ is near 0)}$$

In particular, we have

$$\sqrt[3]{0.95} \approx 1 + \frac{-0.05}{3} = 0.9833... \quad \text{(the true value is 0.9830475725...)}$$

and

$$\sqrt[3]{1.1} \approx 1 + \frac{0.1}{3} = 1.0333... \quad \text{(the true value is 1.032280115...)}$$

Our approximations are overestimates because the tangent line lies above the curve.

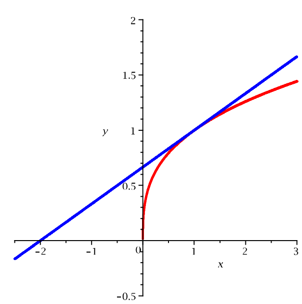
EXAMPLE: Find the linearization of the function $f(x) = \sqrt[3]{x}$ at $a = 1$ and use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$. Are these approximations overestimates or underestimates?

Solution: The derivative of $f(x) = \sqrt[3]{x}$ is

$$f'(x) = \left(x^{1/3}\right)' = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

and so we have $f(1) = 1$ and $f'(1) = \frac{1}{3}$. Putting these values into (2), we see that the linearization is

$$L(x) = f(1) + f'(1)(x-1) = 1 + \frac{x-1}{3} = \frac{2}{3} + \frac{x}{3}$$



The corresponding linear approximation (1) is

$$\sqrt[3]{x} \approx \frac{2}{3} + \frac{x}{3} \quad \text{(when } x \text{ is near 1)}$$

In particular, we have

$$\sqrt[3]{0.95} \approx \frac{2}{3} + \frac{0.95}{3} = 0.9833... \quad \text{(the true value is 0.9830475725...)}$$

and

$$\sqrt[3]{1.1} \approx \frac{2}{3} + \frac{1.1}{3} = 1.0333... \quad \text{(the true value is 1.032280115...)}$$

EXAMPLE: Find the linearization of the function $f(x) = \sin x$ at $a = 0$ and use it to approximate the numbers $\sin(-0.1)$ and $\sin(0.1)$. Are these approximations overestimates or underestimates?

EXAMPLE: Find the linearization of the function $f(x)=\sin x$ at $a=0$ and use it to approximate the numbers $\sin(-0.1)$ and $\sin(0.1)$. Are these approximations overestimates or underestimates?

Solution: The derivative of $f(x)=\sin x$ is

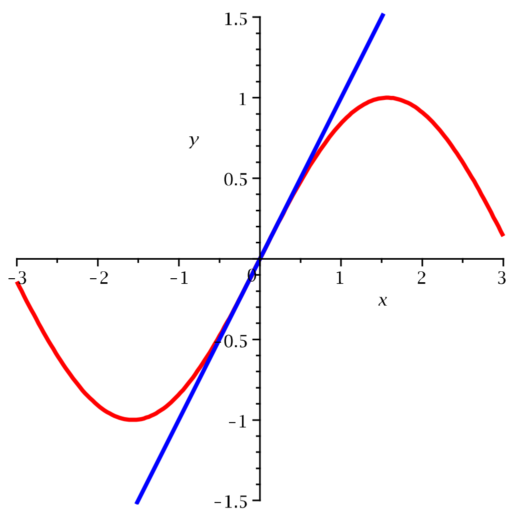
$$f'(x)=(\sin x)'=\cos x$$

and so we have $f(0)=0$ and $f'(0)=1$. Putting these values into (2), we see that the linearization is

$$L(x)=f(0)+f'(0)(x-0)=0+1\cdot(x-0)=x$$

The corresponding linear approximation (1) is

$$\sin x\approx x \quad \text{(when } x \text{ is near 0)}$$



In particular, we have

$$\sin(-0.1)\approx -0.1 \quad \text{(the true value is -0.09983341665...)}$$

and

$$\sin(0.1)\approx 0.1 \quad \text{(the true value is 0.09983341665...)}$$

The first approximation is an underestimate because the tangent line lies below the curve when x is near 0 from the left. The second approximation is an overestimate because the tangent line lies above the curve when x is near 0 from the right.

EXAMPLE: For what values of x is the linear approximation $\sin x\approx x$ accurate to within 0.1?

Solution: Accuracy to within 0.1 means that the functions should differ by less than 0.1:

$$|\sin x-x|<0.1 \quad \Longleftrightarrow \quad -0.1<\sin x-x<0.1$$

Using a graphing calculator we can conclude that the approximation $\sin x\approx x$ is accurate to within 0.1 when $-0.86<x<0.86$.