1. Find all local extrema of the function

$$f(x) = \frac{x}{x^2+4}$$

The derivative of f is

$$\begin{aligned} f'(x) &= \frac{(x^2+4)(x)' - x(x^2+4)'}{(x^2+4)^2} \\ &= \frac{x^2+4-2x^2}{(x^2+4)^2} \\ &= \frac{4-x^2}{(x^2+4)^2} \end{aligned}$$

Solution:
The expression for f' shows that $f'(x)$ exists everywhere and so the critical numbers and hence the local extrema occur where $f'(x) = 0$. From the derivative this happens when $4 - x^2 = 0$, that is, at $x = -2$ and $x = 2$.
To determine which of these are local maxs or local mins, we need to do either of the following calculations:
First Derivative Test: From the expression for $f'(x)$ we see that $f'(x) > 0$ on $(-2, 2)$ and $f'(x) < 0$ on $(-\infty, -2)$ and $(2, \infty)$. This implies that f is increasing on $(-2, 2)$ and decreasing on $(-\infty, -2)$ and $(2, \infty)$. By the first derivative test, $x = -2$ is a local min and $x = 2$ is a local max.
Second Derivative Test: The second derivative of f is

$$\begin{aligned} f''(x) &= \frac{(x^2+4)^2(4-x^2)' - (4-x^2)((x^2+4)^2)'}{(x^2+4)^4} \\ &= \frac{(x^2+4)^2(-2x) - (4-x^2)2(x^2+4)(2x)}{(x^2+4)^4} \\ &= \frac{(x^2+4)(-2x) - (4-x^2)2(2x)}{(x^2+4)^3} \\ &= \frac{-2x^3-8x-16x+4x^3}{(x^2+4)^3} \\ &= \frac{2x^3-24x}{(x^2+4)^3} \\ &= \frac{2x(x^2-12)}{(x^2+4)^3} \end{aligned}$$

Thus $f''(2) = 2 \cdot 2 \cdot (-8)/(8)^3 < 0$ and so $x = 2$ is a local max. Similarly $f''(-2) = 2 \cdot (-2) \cdot (-8)/(8)^3 > 0$ and so $x = -2$ is a local min.

2. Determine the intervals where the graph of

$$g(x) = x^3 - 9x^2 + 2x - 1$$

is concave up and concave down. Express your answer in interval notation. Is there an inflection point?
Solution: The derivative of g is

$$g'(x) = 3x^2 - 18x + 2$$

Therefore the second derivative of g is

$$\begin{aligned} g''(x) &= 6x - 18 \\ &= 6(x-3) \end{aligned}$$

We see that $g''(x) > 0$ on $(3, \infty)$ and $g''(x) < 0$ on $(-\infty, 3)$. This implies that g is concave up on $(3, \infty)$ and concave down on $(-\infty, 3)$. This also shows that $(3, g(3))$ is an inflection point.

Here are a few examples of this method.

Example 4.5. Find the absolute extrema of $f(x) = x^{2/5}(x^{1/5} - 1)$ on $[0, 32]$.

Solution. First, compute the derivative with the power rule as follows.

$$\begin{aligned} f(x) &= x^{3/5} - x^{2/5} \\ f'(x) &= \frac{3}{5}x^{-2/5} - \frac{2}{5}x^{-3/5} \end{aligned}$$

Note that this is defined everywhere except at $x = 0$, so 0 is one critical number. To find the others, solve $f'(x) = 0$ as follows.

$$\begin{aligned} \frac{3}{5}x^{-2/5} - \frac{2}{5}x^{-3/5} &= 0 \\ \frac{3}{5}x^{-2/5} &= \frac{2}{5}x^{-3/5} \\ \frac{x^{-2/5}}{x^{-3/5}} &= \frac{2/5}{3/5} \\ x^{1/5} &= \frac{2}{3} \\ x &= \left(\frac{2}{3}\right)^5 \end{aligned}$$

So there are two critical numbers: $x = 0$ and $x = \left(\frac{2}{3}\right)^5$.

Now, the two endpoints are 0 and 32. So evaluate $f(x)$ on the critical values and the endpoints (note that 0 is both a critical number and an endpoint).

$$\begin{aligned} f(0) &= 0^{2/5}(0^{1/5} - 1) \\ &= 0 \\ f\left(\left(\frac{2}{3}\right)^5\right) &= \left(\left(\frac{2}{3}\right)^5\right)^{2/5}\left(\left(\left(\frac{2}{3}\right)^5\right)^{1/5} - 1\right) \\ &= \left(\frac{2}{3}\right)^2\left(\frac{2}{3} - 1\right) \\ &= -\frac{4}{27} \\ f(32) &= 32^{2/5}(32^{1/5} - 1) \\ &= 4(2 - 1) \\ &= 4 \end{aligned}$$

So the three values we compute in step 2 of the closed interval method are $0, -\frac{4}{27}$, and 4. The maximum of these is the absolute maximum: $f(32) = 4$. The smallest is the absolute minimum: $f(\left(\frac{2}{3}\right)^5) = -4/27$.

5 Sign diagrams

In this section, I just explore a bit exactly how much you can know about a function if you just know where it is increasing and decreasing. For example, suppose that you have a function $f(x)$ on $[1, 4]$, and all you know about it is that it has critical numbers 2 and 3, it is increasing on $[1, 2]$, decreasing on $[2, 3]$, and increasing on $[3, 4]$. This information can be denoted by a "sign diagram."