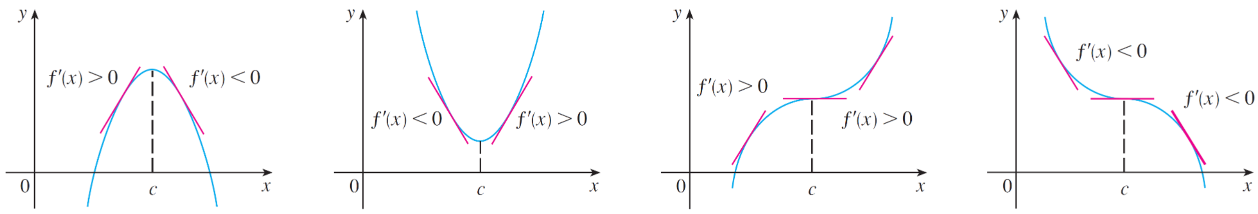
THE FIRST DERIVATIVE TEST: Suppose c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.



(a) Local maximum (b) Local minimum (c) No maximum or minimum (d) No maximum or minimum

EXAMPLE: Find where

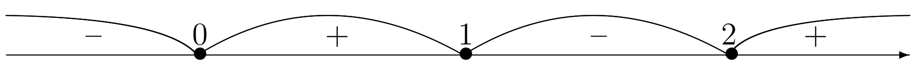
$$f(x) = x^4 - 4x^3 + 4x^2$$

is increasing and where it is decreasing. Find the local maximum and minimum values of f.

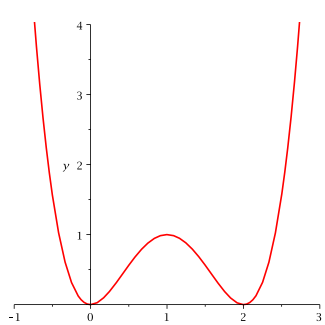
Solution: Since

$$\begin{aligned} f'(x) = (x^4 - 4x^3 + 4x^2)' &= (x^4)' - (4x^3)' + (4x^2)' \\ &= (x^4)' - 4(x^3)' + 4(x^2)' \\ &= (4x^3) - 4(3x^2) + 4(2x) \\ &= 4x^3 - 12x^2 + 8x \\ &= 4x(x^2 - 3x + 2) \\ &= 4x(x-1)(x-2) \end{aligned}$$

the critical numbers are $x = 0, 1$ and 2. We have



Therefore f is increasing on $(0, 1)$ and $(2, \infty)$; it is decreasing on $(-\infty, 0)$ and $(1, 2)$. Because $f'(x)$ changes from negative to positive at 0 and 2, the First Derivative Test tells us that $f(0) = 0$ and $f(2) = 0$ are local minimum values. Similarly, since $f'(x)$ changes from positive to negative at 1, $f(1) = 1$ is a local maximum value.



EXAMPLE: Let $f(x) = 2x + 3\sqrt[3]{x^2}$.

(a) Find the critical numbers of f, if any.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f, if any.

Solution:

(a) We have

$$\begin{aligned} f'(x) &= (2x + 3x^{2/3})' \\ &= (2x)' + (3x^{2/3})' \\ &= 2(x)' + 3(x^{2/3})' \\ &= 2(1) + 3 \cdot \frac{2}{3} x^{2/3-1} \\ &= 2 + 2x^{-1/3} \end{aligned}$$

This can be rewritten as $\dfrac{2(\sqrt[3]{x}+1)}{\sqrt[3]{x}}$ in two different ways. Either

$$2 + 2x^{-1/3} = \frac{2}{1} + \frac{2}{\sqrt[3]{x}} = \frac{2\sqrt[3]{x}}{\sqrt[3]{x}} + \frac{2}{\sqrt[3]{x}} = \frac{2\sqrt[3]{x}+2}{\sqrt[3]{x}} = \frac{2\sqrt[3]{x}+2\cdot 1}{\sqrt[3]{x}} = \frac{2(\sqrt[3]{x}+1)}{\sqrt[3]{x}}$$

or

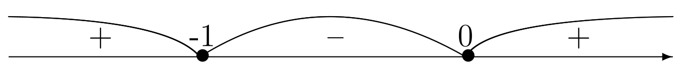
$$2 + 2x^{-1/3} = 2x^{-1/3} \cdot x^{1/3} + 2x^{-1/3} \cdot 1 = 2x^{-1/3}(x^{1/3}+1) = \frac{2(\sqrt[3]{x}+1)}{\sqrt[3]{x}}$$

So,

$$f'(x) = \frac{2(\sqrt[3]{x}+1)}{\sqrt[3]{x}}$$

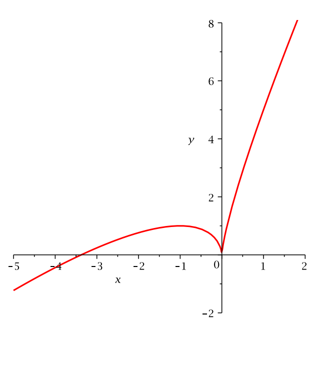
therefore the critical numbers are $x = -1$ (where the top is equal to 0) and 0 (where the bottom is equal to 0).

(b) We have



Therefore f is increasing on $(-\infty, -1)$ and $(0, \infty)$; it is decreasing on $(-1, 0)$.

(c) Because $f'(x)$ changes from positive to negative at -1, the First Derivative Test tells us that $f(-1) = 1$ is a local maximum value. Similarly, since $f'(x)$ changes from negative to positive at 0, $f(0) = 0$ is a local minimum value.



EXAMPLE: Let $f(x) = (x-2)e^{-x}$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f.

Solution:

(a) We have

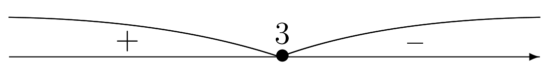
$$\begin{aligned} f'(x) = \Big((x-2)e^{-x}\Big)' &= (x-2)'e^{-x} + (x-2)\left(e^{-x}\right)' \\ &= (x'-2')e^{-x} + (x-2)e^{-x}\cdot(-x)' \\ &= (1-0)e^{-x} + (x-2)e^{-x}\cdot(-1) \\ &= 1\cdot e^{-x} - (x-2)e^{-x} \\ &= e^{-x}\Big(1-(x-2)\Big) \\ &= e^{-x}(1-x+2) \\ &= e^{-x}(3-x) \end{aligned}$$

Since

$$e^{-x}(3-x) = 0 \quad \iff \quad 3-x=0 \quad \iff \quad x=3$$

it follows that $f' = 0$ at $x = 3$. Therefore the critical number of f is $x = 3$.

(b) We have



Therefore f is increasing on $(-\infty, 3)$; it is decreasing on $(3, \infty)$.

(c) Because $f'(x)$ changes from positive to negative at 3, the First Derivative Test tells us that

$$f(3) = (3-2)e^{-3} = 1\cdot e^{-3} = e^{-3}$$

is a local maximum value.

Applications

EXAMPLE: Sales (in thousands of dollars) for Huttig Building Products, Inc. can be approximated by

$$S(x) = 8x^3 - 180x^2 + 1170x - 1260 \quad (3 \le x \le 11)$$

where $x = 3$ corresponds to the year 2003. Determine when sales were increasing and when sales were decreasing. Also find all local extrema of the sales function. (Data from: www.morningstar.com.)

Solution: To determine the critical numbers, we find the derivative

$$\begin{aligned} S'(x) &= (8x^3 - 180x^2 + 1170x - 1260)' \\ &= (8x^3)' - (180x^2)' + (1170x)' - (1260)' \\ &= 8(x^3)' - 180(x^2)' + 1170(x)' - (1260)' \\ &= 8(3x^2) - 180(2x) + 1170(1) - 0 \\ &= 24x^2 - 360x + 1170 \end{aligned}$$

Since the derivative exists for every x, the only critical numbers occur where the derivative is zero. Solve $S'(x) = 0$ using the quadratic formula:

$$24x^2 - 360x + 1170 = 0$$

hence

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-360) \pm \sqrt{(-360)^2 - 4(24)(1170)}}{2(24)} \approx 4.8, 10.2$$

Since the domain of the sales function is $[3, 11]$, both critical numbers lie within the domain. We choose the interval $[4, 5]$ to test the critical number 4.8, and the interval $[5, 11]$ to test the critical number 10.2:

$$S'(4) = 114 > 0$$

$$S'(5) = -30 < 0$$

$$S'(11) = 114 > 0$$

Therefore, sales are increasing on the intervals $(3, 4.8)$ and $(10.2, 11)$ and decreasing on the interval $(4.8, 10.2)$. By the first-derivative test, there is a local maximum at $x = 4.8$ and a local minimum at $x = 10.2$. In other words, sales reach a local maximum toward the end of 2004 and a local minimum near the beginning of 2010. The local maximum value is $S(4.8) \approx 1093.5$ (sales of approximately \$1,093,500) and the local minimum value is $S(10.2) \approx 436.5$ (sales of approximately \$436,500), as can be seen in the Figure below.