First Name: ____________________ Last Name: ____________________

Student-No: ____________________ Section: ____________________

1. 3 marks Differentiate the function $f(x) = (\arcsin x)^{\arccos x}$. Assume $0 < x < 1$.

Solution: We use logarithmic differentiation:

$$y = (\arcsin x)^{\arccos x} \Rightarrow$$
$$\ln(y) = \ln((\arcsin x)^{\arccos x}) = \arccos x \ln(\arcsin x) \Rightarrow$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln(y)) = \frac{\mathrm{d}}{\mathrm{d}x}(\arccos x \ln(\arcsin x)) \Rightarrow$$
$$\frac{y'}{y} = \frac{-1}{\sqrt{1-x^2}}.\ln(\arcsin x) + \arccos(x).\frac{1}{\sqrt{1-x^2}}\frac{1}{\arcsin x} \Rightarrow$$
$$y' = y\left(\frac{-1}{\sqrt{1-x^2}}.\ln(\arcsin x) + \arccos(x).\frac{1}{\sqrt{1-x^2}}\frac{1}{\arcsin x}\right) =$$
$$(\arcsin x)^{\arccos x}\left(\frac{-1}{\sqrt{1-x^2}}.\ln(\arcsin x) + \arccos(x).\frac{1}{\sqrt{1-x^2}}\frac{1}{\arcsin x}\right)$$

Marking scheme: 1pt for taking ln, 1pt for taking derivative correctly, 1pt for a complete solution

2. $f(x) = \arctan x$ is the inverse of function $g(x) = \tan x$.

(a) 2 marks Using properties of inverse functions, show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2}.$$

You may need to use this identity $\sec^2 x = 1 + \tan^2 x$.

Solution: $\tan x$ is the inverse of $\arctan x$ therefore we can write:

$$y = \arctan x \Rightarrow \tan y = \tan(\arctan x) = x \Rightarrow$$
$$y' \sec^2 y = 1 \Rightarrow y' = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Marking scheme: 1pt for each line of proof.

(b) 1 mark Find the derivative of

$$k(x) = 2^{\arctan x}$$

Solution:

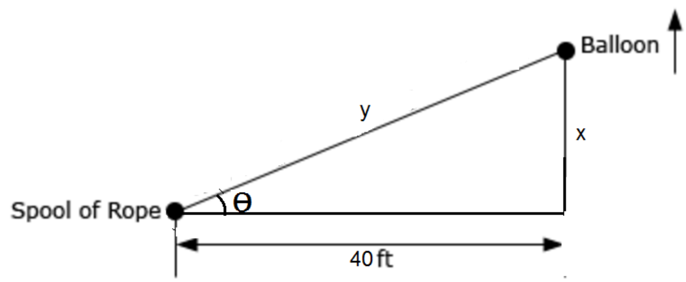
$$k'(x) = \frac{2^{\arctan x}(\ln 2)}{1 + x^2}$$

Marking scheme: 1pt for correct derivative

3. A hot air balloon is attached to a spool of rope that is 40 meters away from the balloon when it is on the ground. The hot air balloon rises straight up in such a way that the length of rope increases at a rate of 5 meters/sec.

(a) [4 marks] How fast is the balloon rising when the length of rope is $50m$. Do not forget to write its unit. Sketch a graph that explains the problem.

Solution: Graph below is a schematic view of the problem. The balloon rises up and



therefore it stretches the rope.
Parameters we have are: The height of balloon (x), Rate of change of height of balloon ($\frac{dx}{dt}$, the length of rope (y) and the rate of change of length of rope $\frac{dy}{dt}$. We have y and $\frac{dy}{dt}$ and we want to find $\frac{dx}{dt}$ The third length of the right angle triangle is constant and equal to 40 m . We need to relate x and y, we do so by writing the Pythagoras equation:

$$x^2 + 40^2 = y^2 \Rightarrow 2x\frac{dx}{dt} = 2y\frac{dy}{dt} \Rightarrow \frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$$

we also know

$$y = 50, \text{ then } x = \sqrt{50^2 - 40^2} = 30 \text{ therefore}$$

$$\frac{dx}{dt} = \frac{50}{30}5 = \frac{25}{3}\ m/s$$

Marking scheme: 1pt for a correct graph, 1pt for listing the parameters correctly, 1pt for writing the equations and 1 pt for finding the answer.

(b) [bonus 2 marks] What is the rate at which the angle that the rope makes with the ground changes? You may need to use results of problem 2.

Solution:

$$\theta = \arctan\left(\frac{x}{40}\right) \Rightarrow \frac{d\theta}{dt} = \frac{1}{40}\frac{1}{1+\left(\frac{x}{40}\right)^2}\frac{dx}{dt} = \frac{1}{40}\frac{40^2}{40^2+x^2}\frac{dx}{dt} = \frac{40}{1600+x^2}\frac{dx}{dt} \Rightarrow$$

$$\frac{d\theta}{dt} = \frac{40}{1600+900}(25/3) = 0.133\frac{rad}{s} = 7.6^\circ/s$$

Marking scheme: 1pt for for writing the relation equation and 1 pt for finding the answer.