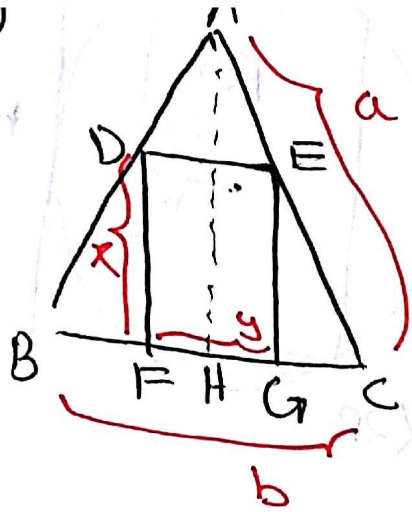
1)



QUIZ 5

$AB = AC = a$ $\quad$ $FG = y$

$BC = b$ $\quad$ $DF = x$

Find x and y such that the rectangle FGED has the largest area

$$\text{Area} = x \cdot y$$

we should eliminate one of variables.

We use similar triangles.

$$CGE \sim CHA \rightarrow \frac{CG}{CH} = \frac{EG}{AH}$$

$$CG = \frac{b}{2} - \frac{y}{2}, \quad CH = \frac{b}{2}, \quad EG = x, \quad AH = a^2 - \frac{b^2}{4}$$

$$\frac{\frac{b}{2} - \frac{y}{2}}{\frac{b}{2}} = \frac{x}{a^2 - \frac{b^2}{4}} \rightarrow \left(1 - \frac{y}{b}\right)\left(a^2 - \frac{b^2}{4}\right) = x$$

$$A = xy = y\left(1 - \frac{y}{b}\right)\left(a^2 - \frac{b^2}{4}\right)$$

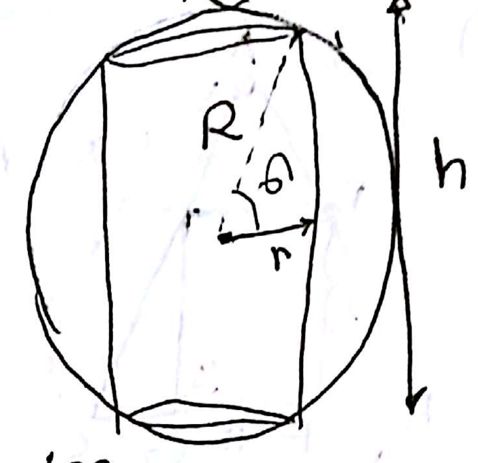
$$\frac{dA}{dy} = \left[\left(1 - y/b\right) - \frac{1}{b}y\right]\left(a^2 - \frac{b^2}{4}\right) = 0 \rightarrow 1 - \frac{2y}{b} = 0 \rightarrow y = \frac{b}{2}$$

$$\frac{d^2A}{dy^2} = -\frac{2}{b}\left(a^2 - b^2/4\right) < 0 \rightarrow \text{we have a maximum}$$

$$A_{max} = \frac{b}{2}\left(1 - \frac{1}{2}\right)\left(a^2 - \frac{b^2}{4}\right) = \frac{b}{4}\left(a^2 - \frac{b^2}{4}\right)$$

Scanned by CamScanner

Solution 1



$V = \pi r^2 h$

$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \rightarrow r^2 = R^2 - \frac{h^2}{4}$

$V = \pi\left(R^2 - \frac{h^2}{4}\right)h = \pi R^2 h - \frac{\pi}{4} h^3 \qquad 0 < h < 2R$

$\frac{dV}{dh} = \pi R^2 - \frac{3\pi}{4} h^2$

$\frac{dV}{dh} = 0 \rightarrow h^2 = \frac{4R^2}{3} \Rightarrow h = \frac{2R}{\sqrt{3}}$ → maximum happens when $h = 2R/\sqrt{3}$

$\frac{d^2V}{dh^2} = -\frac{6\pi}{4} h \qquad \frac{d^2V}{dh^2} < 0 \rightarrow$

$\rightarrow V_{max} = \pi R^2 \frac{2R}{\sqrt{3}} - \frac{\pi}{4}\frac{8R^3}{3\sqrt{3}} = \frac{\pi R^3}{\sqrt{3}}\left(2 - \frac{2}{3}\right) = \frac{4}{3\sqrt{3}}\pi R^3$

Solution 2:

$r = R\cos\theta, \quad 0 < \theta < \frac{\pi}{2}$

$h = 2R\sin\theta \quad \longrightarrow \quad V = \pi(2R\sin\theta)(R\cos\theta)^2 = 2\pi R^3 \cos^2\theta \cdot \sin\theta$

$\frac{dV}{d\theta} = 0 \rightarrow 2\pi R^3\left(-2\cos\theta\sin^2\theta + \cos^3\theta\right) = 0$

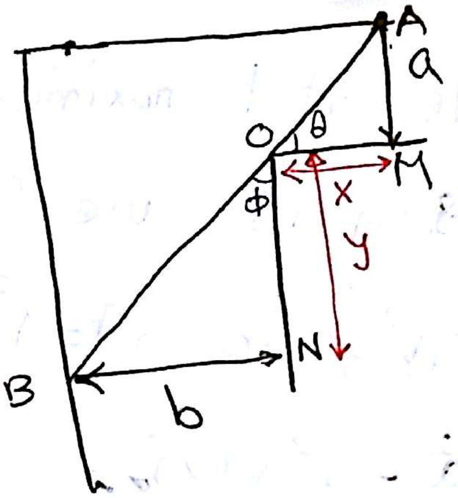
$\rightarrow +\cos\theta\left(-2\sin^2\theta + \cos^2\theta\right) = 0$

$\rightarrow \cos\theta = 0 \rightarrow \theta = \frac{\pi}{2}$ ← Not what we want, bc $V = 0$

$-2\sin^2\theta + \cos^2\theta = 0 \rightarrow \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{2} \rightarrow \tan^2\theta = \frac{1}{2}$ & $\theta = \left(\arctan\frac{1}{\sqrt{2}}\right)$

$h/r = \frac{2\sin\theta}{\cos\theta} = 2\frac{1}{\sqrt{2}} = \sqrt{2} \rightarrow \boxed{h = \sqrt{2}\, r}$

We want to minimise AOB,



$$l = AOB = AO + OB = \sqrt{a^2+x^2} + \sqrt{b^2+y^2}$$

We need a relation between x, y

~~[illegible]~~ : Similar triangles.

$$\overset{\Delta}{AMO} \sim \overset{\Delta}{ONB} \Rightarrow$$

$$\frac{AM}{ON} = \frac{MO}{BN} \Rightarrow \frac{a}{x} = \frac{y}{b} \Rightarrow$$

$$ab = xy \rightarrow y = \frac{ab}{x}$$

$$l = \sqrt{a^2+x^2} + \sqrt{b^2 + \frac{a^2b^2}{x^2}} = \sqrt{a^2+x^2} + b\sqrt{1+\frac{a^2}{x^2}}$$

$$\frac{dl}{dx} = 0 \Rightarrow \frac{2x}{2\sqrt{a^2+x^2}} + b\frac{\frac{-2a^2}{x^3}}{2\sqrt{1+\frac{a^2}{x^2}}} = 0 \rightarrow$$

$$\frac{x}{\sqrt{a^2+x^2}} + \frac{\frac{-\cancel{2}a^2b}{x^{\cancel{3}}}}{\frac{\cancel{2}}{\cancel{x}}\sqrt{x^2+a^2}} = \frac{x}{\sqrt{a^2+x^2}} - \frac{\frac{a^2b}{x^2}}{\sqrt{a^2+x^2}} = 0$$

$$\frac{x - \frac{a^2b}{x^2}}{\sqrt{a^2+x^2}} = 0 \rightarrow x^3 = a^2b \rightarrow x = \sqrt[3]{a^2b}$$

local min

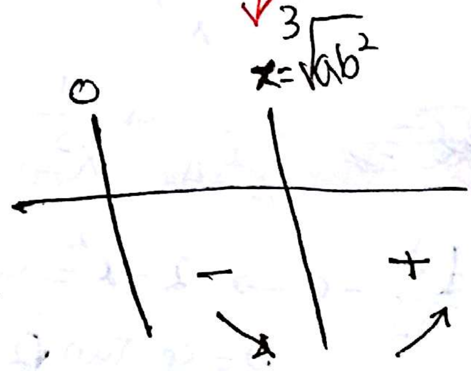
$$y = \frac{ab}{\sqrt[3]{a^2b}} = \sqrt[3]{ab^2}$$

$$l_{min} = \sqrt{a^2 + (\sqrt[3]{a^2b})^2} + \sqrt{b^2 + (\sqrt[3]{ab^2})^2}$$

~~l_{min}~~

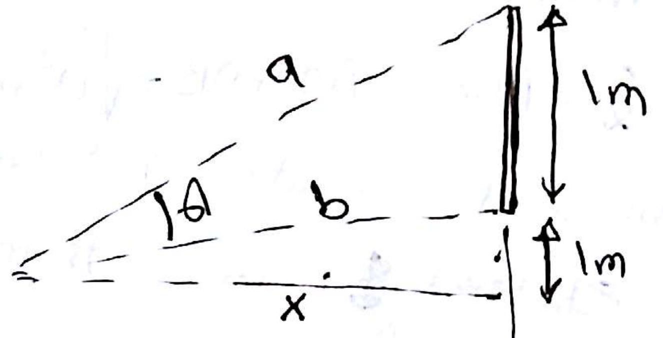
=

$$\frac{x - \frac{a^2b}{x^2}}{\sqrt{a^2+x^2}}$$



We want to maximise θ

Solution I, use Cosine law.



$$\left.\begin{array}{l} a^2+b^2-2ab\cos\theta=1^2 \\ a^2=x^2+(2)^2 \\ b^2=x^2+1^2 \end{array}\right\} \longrightarrow x^2+4+x^2+1-2\sqrt{x^2+4}\sqrt{x^2+1}\cos\theta=1$$

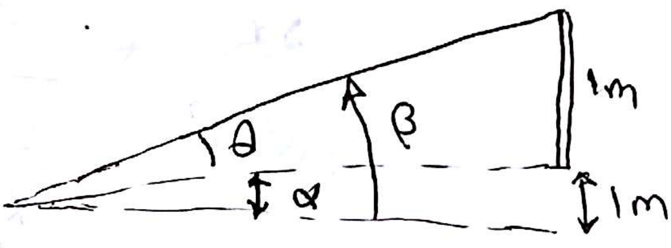
$$\cos\theta=\frac{2x^2+4}{2\sqrt{x^2+4}\sqrt{x^2+1}}=\frac{x^2+2}{\sqrt{x^2+4}\sqrt{x^2+1}}$$

Instead of maximizing θ, I can minimise $\cos\theta$, so

$$\frac{d(\cos\theta)}{dx}=0 \Rightarrow \text{find } \theta_{\text{optimum}}.$$

Solution 2:

maximise θ.



$$\theta=\beta-\alpha=\arctan\frac{2}{x}-\arctan\frac{1}{x} \qquad 0<x<\infty$$

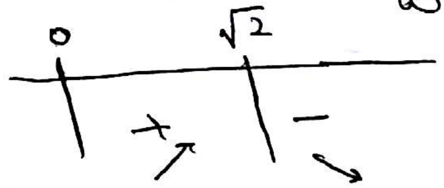
ready to differentiate: (Remember $(\arctan x)'=\frac{1}{x^2+1}$)

$$\frac{d\theta}{dx}=\frac{-\frac{2}{x^2}}{1+\left(\frac{2}{x}\right)^2}-\frac{-\frac{1}{x^2}}{1+\left(\frac{1}{x}\right)^2}=\frac{-\frac{2}{\cancel{x^2}}}{\frac{x^2+4}{\cancel{x^2}}}+\frac{\frac{1}{\cancel{x^2}}}{\frac{x^2+1}{\cancel{x^2}}}=$$

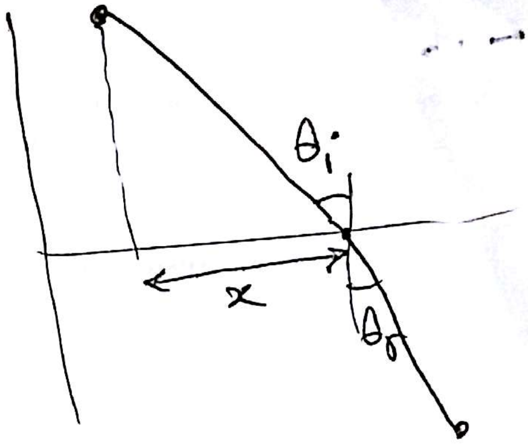
$$\frac{-2}{x^2+4}+\frac{1}{x^2+1}=\frac{-2(x^2+1)+x^2+4}{(x^2+1)(x^2+4)}=\frac{-x^2+2}{(x^2+1)(x^2+4)}$$

$$\frac{d\theta}{dx}=0 \rightarrow 2-x^2=0 \rightarrow x=\sqrt{2} \hookrightarrow \text{maximizer.}$$

$$\theta=\arctan\sqrt{2}-\arctan\frac{1}{\sqrt{2}}$$



minimise t_{total}



$$t_{total} = t_{PO} + t_{OQ}$$

$$t_{PO} = \frac{PO}{V_W} = \frac{\sqrt{(x-x_1)^2+y_1^2}}{V_W}$$

$$t_{OQ} = \frac{OQ}{V_a} = \frac{\sqrt{(x_2-x)^2+y_2^2}}{V_a}$$

$$t_{tot} = \frac{1}{V_W}\sqrt{(x-x_1)^2+y_1^2} + \frac{1}{V_a}\sqrt{(x_2-x)^2+y_2^2}$$

$$\frac{dt_{tot}}{dx} = 0 \rightarrow \frac{\frac{1}{V_W}\,2(x-x_1)}{2\sqrt{(x-x_1)^2+y_1^2}} + \frac{1}{V_a}\frac{-2(x_2-x)}{2\sqrt{(x_2-x)^2+y_2^2}} = 0$$

$$\rightarrow \frac{1}{V_W}\frac{x-x_1}{\sqrt{(x-x_1)^2+y_1^2}} = \frac{1}{V_a}\frac{x_2-x}{\sqrt{(x_2-x)^2+y_2^2}}$$

$$\Downarrow$$

$$\frac{1}{V_W}\sin\theta_i = \frac{1}{V_a}\sin\theta_r \rightarrow \boxed{\frac{\sin\theta_r}{\sin\theta_i} = \frac{V_a}{V_W}}$$

Snell's law.

$$\frac{\sqrt{(x_2-x)^2+y_2^2}}{\sqrt{(x-x_1)^2+y_1^2}} = \frac{V_W}{V_a}\frac{(x_2-x)}{x-x_1} \rightarrow \frac{(x_2-x)^2+y_2^2}{(x-x_1)^2+y_1^2} = \left(\frac{V_W}{V_a}\right)^2\frac{(x_2-x)^2}{(x-x_1)^2}$$

$$\rightarrow \left((x_2-x)^2+y_2^2\right)(x-x_1)^2 = \left(\frac{V_W}{V_a}\right)^2(x_2-x)^2\left((x-x_1)^2+y_1^2\right)$$

$$\rightarrow (x^2-2xx_2+x_2^2+y_2^2)(x^2-2xx_1+x_1^2) = \left(\frac{V_W}{V_a}\right)^2(x^2-2xx_2+x_2^2)(x^2-2xx_1+x_1^2+y_1^2)$$

Solve for $\boxed{x}$