Nov 1, 2016

* Critical points are points in the domain

that $\begin{cases} f'(x) = 0 \text{ or} \\ f'(x) = \text{not defined.} \end{cases}$

* a local Extremum has to be a critical Point

Question: How we can classify critical Points to local maximum, local minimum, neighter?

* definition: function $f(x)$ is increasing on interval I if for $a \leq b$ in I,

$f(a) \leq f(b)$

" " decreasing

" " $a \leq b$ in I

$f(a) \geq f(b)$

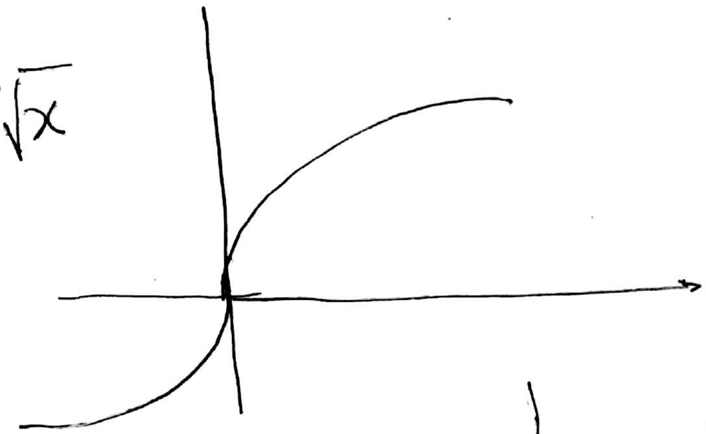
Theory: if $f'(x)$ exists and it is positive, $f(x)$ is increasing

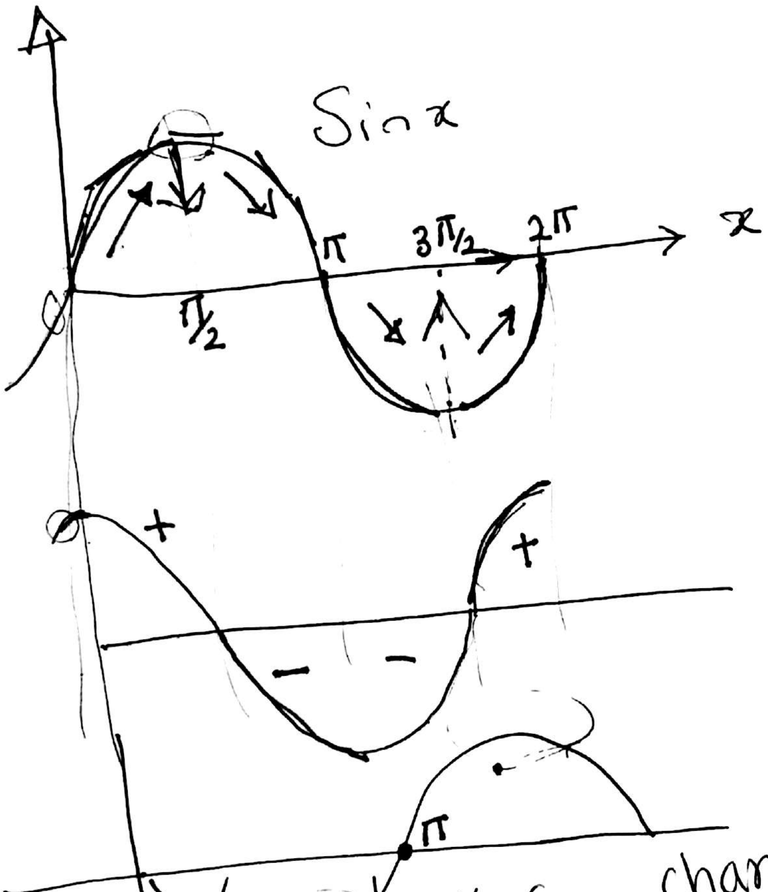
if $f'(x)$ exists and it is negative $f(x)$ is decreasing

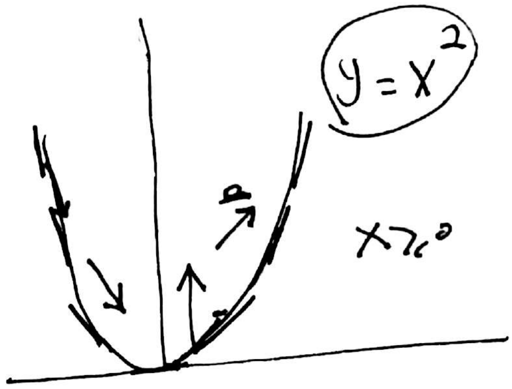
*

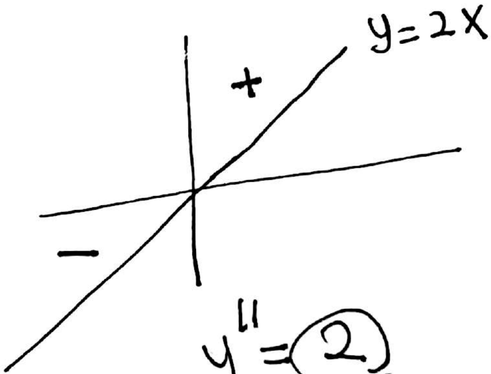
$y = \sqrt[3]{x}$

$y' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$







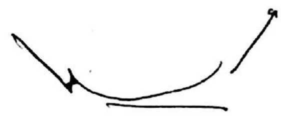


$y'' = 2$

if $f'(x)$ at $X = C$ changes sign from + to −

$X = C$ is to position of a local maximum.

if $f'(x)$ at $X = C$ changes sign from − to +



$X = C$ is to Position of a local minimum.

Find local min & local max of

$f(x) = e^x \cdot x^{-2}$ in [1,4]

$f(x) = \frac{e^x}{x^2}$ $\mathbb{R} - \{0\}$

$f'(x) = \frac{e^x \cdot \cancel{x^2}x^2 - 2x \cdot e^x}{(x^2)^2} = \frac{e^x \cdot x(x-2)}{x^4}$

$= \frac{e^x(x-2)}{x^3} = 0$

$e^x(x-2) = 0 \rightarrow e^x = 0 \leftarrow$ No Solution

or

$x - 2 = 0 \longrightarrow x = 2$

$f'(x) =$ Not defined.

$\Downarrow$

$x^3 = 0 \Rightarrow x = 0$

$\Downarrow$

Not in domain.

	1		2		4
e^x		+		+	
$(x-2)$		—		+	
x^3		+		+	
f'		—		+	
		$\searrow$		$\nearrow$	

$\boxed{x = 2}$ is the position of a local minimum.

$f(x) = \dfrac{\ln x}{\sqrt{x}}$, determine the interval over which the function is increasing/decreasing

Domain: $x > 0$

$$f'(x) = \frac{\frac{1}{x}\sqrt{x} - \frac{1}{2\sqrt{x}} \cdot \ln x}{(\sqrt{x})^2} = \frac{\frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{x}} \ln x}{x}$$

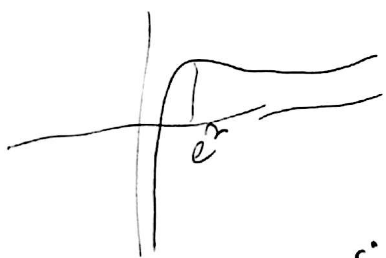
$$= \frac{\frac{1}{\sqrt{x}}\left(1 - \frac{\ln x}{2}\right)}{x} = \frac{1 - \frac{\ln x}{2}}{x\sqrt{x}}$$

critical points

→ $f'(x) = 0 \rightarrow 1 - \frac{\ln x}{2} = 0 \Rightarrow \ln x = 2 \rightarrow x = \boxed{e^2}$

→ $f'(x)$ not defined

if denom $= 0 \rightarrow x\sqrt{x} = 0 \Rightarrow \cancel{\boxed{x=0}}$ Not in domain



$(0, e^2) \rightarrow$ increasing

$(e^2, \infty) \rightarrow$ decreasing

$x = e^2$ is position of a local maximum.

	0	1	e^2	e^4	∞
$1 - \frac{\ln x}{2}$		$+$		$-$	
$x\sqrt{x}$		$+$		$+$	
f'		$+$		$-$	

show $x=0$ is a critical point for

$$f(x) = x\sin x$$

and then classify it.

$$f'(x) = \sin x + x\cos x \rightarrow f'(0) \stackrel{\checkmark}{=} 0$$

$$\sin(0) + 0 \cdot \cos(0) = 0 + 0 \stackrel{\checkmark}{=} 0$$

$\sin x + x\cos x$

0^+ + + +

0^- − − +

	$-\infty$	0	$+\infty$
f'	−		+
	↘		↗

$x=0$ is a position of local min.

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find local min and local max on $[-1,1]$

$$\boxed{f(x) = e^x - e^{-x}} =$$

$$f'(x) = \underbrace{e^x}_{>0} + \underbrace{e^{-x}}_{>0} \leftarrow$$

$$f'(x) = 0 \rightarrow e^x + e^{-x} = 0$$

$$e^x = -e^{-x} \Rightarrow e^x = -\frac{1}{e^x} \rightarrow e^x \cdot e^x = e^{2x} = -1$$

Never has a solution

$\cancel{x = \frac{1}{2}\ln(-1)}$ incorrect

Domain of ln $x > 0$

$x=0 \rightarrow f'(x) = 1+1 = 2 \rightarrow f'(x) > 0 \rightarrow$ No max No Min.

$$f(x) = x^3 + x^2 - x + 1$$

determine interval(s) where function is increasing.

$$f'(x) = 0 \rightarrow 3x^2 + 2x - 1 = 0 \Rightarrow (3x-1)(x+1) = 0$$

$$\rightarrow 3x - 1 = 0 \Rightarrow x = \tfrac{1}{3}$$

$$\rightarrow x + 1 = 0 \Rightarrow x = -1$$

	$-\infty$ to -1	-1 to $\frac{1}{3}$	$\frac{1}{3}$ to ∞
$(3x-1)$	$-$	$-$	$+$
$(x+1)$	$-$	$+$	$+$
f'	$+$	$-$	$+$
	$\nearrow$	$\searrow$	$\nearrow$

$x = -1$: local max

$x = \frac{1}{3}$: local min

$(-\infty, -1) \cup (\frac{1}{3}, \infty) \rightarrow$ increasing.

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* $f(x)$ is called concave upward (CU) if tangent lines of $f(x)$ lie under the function

* $f(x)$ is called concave downward (CD) if tangent lines of $f(x)$ lie above the function

if $f'' > 0 \longrightarrow$ function CU

$f'' < 0 \longrightarrow$ " CD

if $x = c$ is in the domain of the function & concavity at $x = c$ changes from CU $\to$ CD or CD $\to$ CU, $x = c$ is an inflection point.

Find inflection points of $f(x) = \frac{x^5}{20} + \frac{5x^3}{6} + 500x + 1000$

* to find inflection point, you should look for critical points of $f'(x)$. i.e. $f''(x) = 0$ or $f''(x)$ not defined

$$f' = \frac{5}{20}x^4 + \frac{15}{6}x^2 + 500$$

$$f'' = \frac{20}{20}x^3 + \frac{30}{6}x = x^3 + 5x = x(x^2+5)$$

$$f'' = 0 \rightarrow x(x^2+5) = 0$$

$\boxed{x = 0}$

$x^2 + 5 = 0$ ✗

~~$x = \sqrt{-5}$~~

	$-\infty$ to 0	0 to ∞
x	$-$	$+$
x^2+5	$+$	$+$
f''	$-$	$+$

$(-\infty, 0) \to$ CD $\cap$

$(0, \infty) \to$ CU

$x=0$ inflection point.

$$f(x)=\begin{cases} x^2-2x & x<0 \\ -x^2+2x & x\geq 0\end{cases}$$

$x(x-2)$

Find local max, local min & inflection points.

$$f'(x)=\begin{cases} \underline{2x-2} & x<0 \\ -2x+2 & x>0\end{cases}$$

$f'(0^+) = 2$

$f'(0^-) = -2$

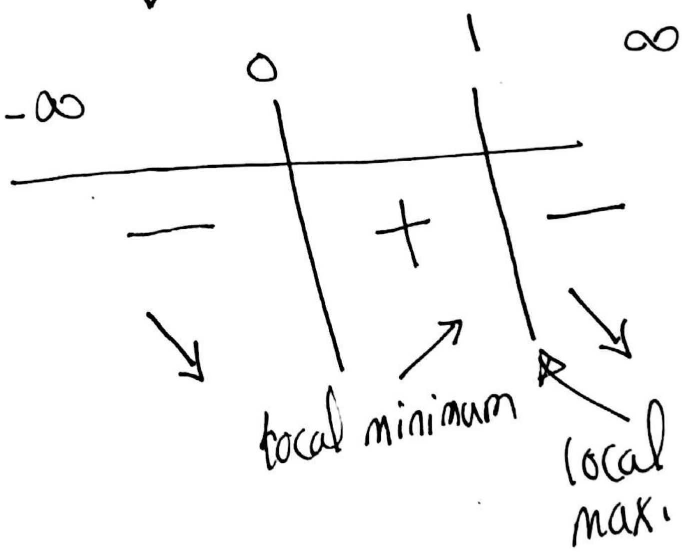
$\longrightarrow$ $f'(x)$ at $x=0$ is not defined

$\Downarrow$

$x=0$ is critical point.

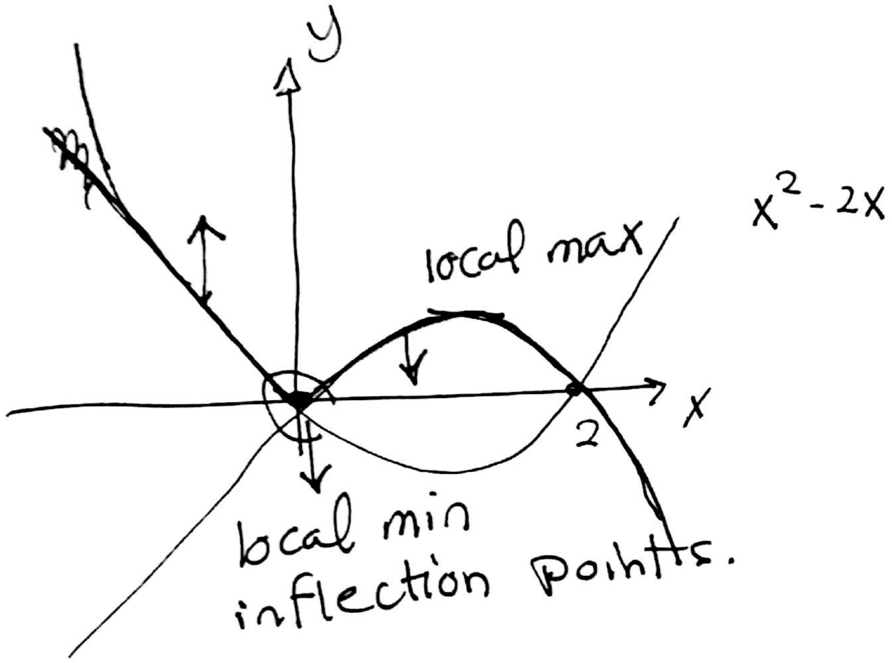
$$f'(x)=0\begin{cases} \underline{2x-2=0} \rightarrow \cancel{x=1} & x<0 \\ -2x+2=0 \rightarrow x=1 \checkmark & x>0\end{cases}$$

$x=0,\ x=1$



$$f''=\begin{cases} 2 & x<0 \\ -2 & x>0\end{cases}$$

$x=0 \longrightarrow$ inflection point



Second derivate test.

if $f'(c) = 0$, $\rightarrow$
- $f''(c) > 0 \longrightarrow$ local min.
- $f''(c) < 0 \longrightarrow$ local max
- $f''(c) = 0 \longrightarrow$ test fails

$y = xe^x$, find local min. local max, inflection points.

$y' = xe^x + e^x = (x+1)e^x$

$y' = 0 \longrightarrow (x+1)e^x = 0 \longrightarrow \boxed{x = -1}$

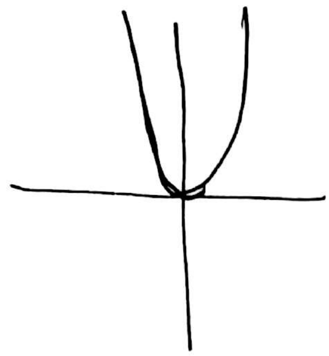
$y'' = (x+1)e^x + e^x = (x+2)e^x$

$y'' = 0 \longrightarrow (x+2)e^x = 0 \rightarrow \boxed{x = -2}$

$y''(-1) = (2-1)e^{-1} = e^{-1} > 0 \longrightarrow$ $x = -1$ is local minimum

$x = -2$ is an inflection point.

$y = x^4$



$y' = 4x^3$

$y'' = 12x^2$

$y' = 4x^3 = 0 \Downarrow x = 0$

$y''(0) = 0 \rightarrow$ second derivative test fails.

	$-\infty$	0	$+\infty$
y'	$-$		$+$
	$\searrow$		$\nearrow$

$x=0$ is local minimum.

$y = x^3$

$y' = 3x^2$

$y'' = 6x$

$y' = 0 \Downarrow 3x^2 = 0$

$x = 0$

$y''(0) = 0 \rightarrow$ second derivative fails

$-\infty$	0	∞
$+$		$+$
$\nearrow$		$\nearrow$

$x=0$ Neighter max nor min.