MATH 104-104, Practice Quiz 4, Sunday Oct 30, 2016

The cost function is given by $C(q) = 2^{\sqrt{q}+5}$. Find the average cost of producing 16 units.

Solution:

$$C(q) = 2^{\sqrt{q}+5}$$
$$\overline{C}(q) = \frac{C(q)}{q} = \frac{2^{\sqrt{q}+5}}{q}$$
$$\overline{C}(16) = \frac{2^{\sqrt{16}+5}}{16} = \frac{2^{4+5}}{2^4} = 2^5 = 32$$

4 marks The demand equation is given by $p = \left(\frac{1}{2} - \frac{q}{200}\right)^2$ where p is the unit price and q is the demand quantity.

(a) Estimate the revenue made by selling the 10^{th} unit.

Note that the revenue made selling the 10th unit is $R(10) - R(9) \approx MR(9)$.

$$R(q) = q\left(\frac{1}{2} - \frac{q}{200}\right)^2$$
$$\frac{dR}{dq} = \frac{dq}{dq}\left(\frac{1}{2} - \frac{q}{200}\right)^2 + q\frac{d}{dq}\left(\frac{1}{2} - \frac{q}{200}\right)^2$$
$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)^2 + q\frac{du^2}{du}\frac{du}{dx}, \qquad u = \frac{1}{2} - \frac{q}{200}$$
$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)^2 + q2u\left(\frac{-1}{200}\right)$$
$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)^2 + q2u\left(\frac{-1}{200}\right)$$
$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)^2 + \left(\frac{1}{2} - \frac{q}{200}\right)\left(\frac{-2q}{200}\right)$$
$$MR(9) = R'(9) = \left(\frac{1}{2} - \frac{9}{200}\right)^2 + \left(\frac{1}{2} - \frac{9}{200}\right)\left(\frac{-18}{200}\right)$$

(b) Find the demand quantity at which revenue is maximized.

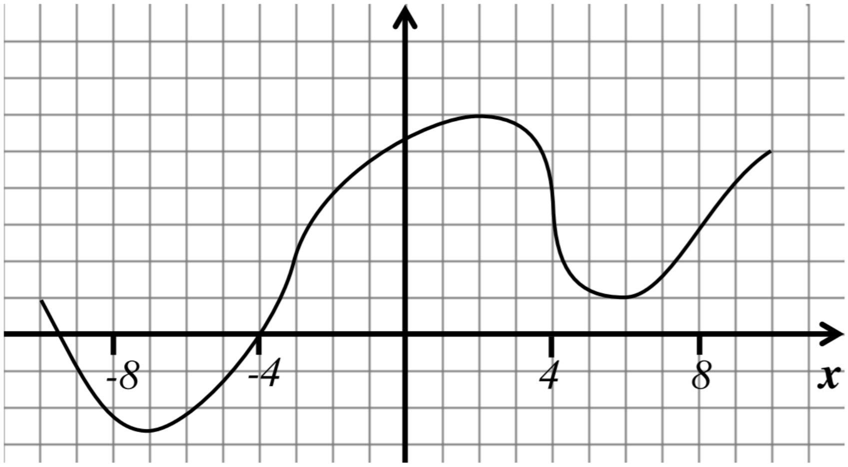
Answer: $q = 100/3$

Solution: From prevoious part we have dR/dq.

$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)^2 + \left(\frac{1}{2} - \frac{q}{200}\right)\left(\frac{-2q}{200}\right)$$
$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)\left(\frac{1}{2} - \frac{q}{200} - \frac{2q}{200}\right)$$
$$\frac{dR}{dq} = \left(\frac{1}{2} - \frac{q}{200}\right)\left(\frac{1}{2} - \frac{3q}{200}\right)$$
$$\frac{dR}{dq} = 0 \Rightarrow q = 100 \text{ or } q = \frac{100}{3}$$
$$R(100) = 0,\ R(100/3) = \frac{100}{3}\left(\frac{1}{2} - \frac{100}{600}\right)^2 > 0$$

So revenue is maximized at $q = 100/3$.

1. 8 marks Below is a graph of $f'(x)$, the derivative of $f(x)$. The domain of the function is $(-10, 10)$.



a) Determine if the function $f(x)$ has any critical points.

Critical points of $f(x)$ are where $f'(x) = 0$ (which happens twice) or $f'(x)$ is not defined.

$f'(x) = 0$ has two solutions

$f'(x)$ not defined does not have any solution, because the function derivative is defined everywhere.

So there are two critical points

b) Determine if the function $f'(x)$ has any critical points

Critical points of $f'(x)$ are where $f'(x)$ has a horizontal tangent line ($f''(x) = 0$) or where derivative of $f'(x)$ is not defined. There are three points with horizontal tangent lines. Besides at x=4, the function has a vertical tangent line which means the derivative is not defined there.

Function $f(x) = \begin{cases} x^3 - 3x^2 & 1 \le x \le 3 \\ \cos(\pi x) - 1 & 0 \le x < 1 \end{cases}$ is defiend on the interval of [0,3].

a) Determine if Extreme Value Theorem can be applied to this function.

In order for EVT to be applied we need to make sure function is continuous on [0,3]. Both branches contain ***nice*** functions which means the only point we need to check is $x = 1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \cos(\pi x) - 1 = \cos(\pi) - 1 = -2$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^3 - 3x^2 = 1 - 3 = -2$$

$$f(1) = 1 - 3 = -2$$

Therefore,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

So the function is continuous. That means function is continuous everywhere. EVT can be applied. By EVT, the function must have an absolute minimum and an absolute maximum.

b) Find the absolute minimum and absolute maximum.

EVT can be applied, so we use the method of closed interval.

- Evaluate end points

$$f(0) = 1 - 1 = 0$$

$$f(3) = 3^3 - 3(3^2) = 0$$

- Find the critical point: We first take derivative

$$f'(x) = \begin{cases} 3x^2 - 6x & 1 < x < 3 \\ \dfrac{1}{x} & 0 < x < 1 \end{cases}$$

Now solve $f'(x) = 0. \to \begin{cases} 3x^2 - 6x = 0 \to 3x(x-2) = 0, x = 0,2 & 1 < x < 3 \\ \frac{1}{x} = 0\ , no\ solution & 0 < x < 1 \end{cases}$

Note that $x = 0$ is not in the domain of that branch of the function. So the only acceptable solution is $x = 2$.

We also need to see if $f'(x) not\ defined$ has any solution. Obviously the only point that we need to investigate is $x = 1$

At $x = 1$ The slope of tangent line on the left branch is $\frac{1}{1} = 1$ and slope of tangent line on the right branch $3 - 6 = -3$. That means derivative at $x = 1$ does not exist. So $x = 1$ is a critical point. Our critical points are $x = 1$ and $x = 2$.

- Evaluate critical points

$$f(1) = -2$$
$$f(2) = 2^3 - 3(2^2) = -4$$

- Therefore absolute maximum = max(0,0,-2,-4) = 0 and absolute minimum = min(0,0,-2,-4) = -4
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