Today, we will be discussing some more examples involving implicit differentiation.

Problem 0.1. *A cylinder is getting taller and thinner, but the surface area is held constant at* $50cm^2$*. Find* $\frac{dr}{dh}$ *when the radius is equal to* $2cm$*.*

The surface area of a cylinder is $2\pi r^2 + 2\pi rh$. So we take the derivative with respect to h to get

$$4\pi r\frac{dr}{dh} + 2\pi h\frac{dr}{dh} + 2\pi r = 0.$$

Therefore, we get

$$\frac{dr}{dh} = \frac{-r}{2r+h}.$$

But when $r = 2$, we have

$$8\pi + 4\pi h = 50$$

so

$$h = \frac{50 - 8\pi}{4\pi}.$$

We now plug in r and h to get

$$\frac{dr}{dh} = \frac{-2}{4 + \frac{50-8\pi}{4\pi}}.$$

Again, it's important to take the derivative *before* trying to plug in any numbers. Otherwise, you will get the wrong answer.

Problem 0.2. *Suppose we are on the curve* $y^2 = x^3 + 5x - 6$*.*

- *Are there any places where the curve has a vertical tangent?*
- *Find the second derivative of* y *with respect to* x *at an arbitrary place on the curve.*

$$2y\frac{dy}{dx} = 3x^2 + 5$$
$$\frac{dy}{dx} = \frac{3x^2+5}{2y}$$

Vertical tangent lines can only occur if $y = 0$. Clearly $(1, 0)$ is a point on the curve for which $y = 0$. Let's check if there are any others: factoring gives that $x^3 + 6x - 6 = (x - 1)(x^2 + x + 6)$, and $x^2 + x + 6$ is irreducible because the discriminant $b^2 - 4ac$ is negative. Therefore, $x = 1, y = 0$ is the only possibility.

To verify that there is actually a vertical tangent line here, we need to compute $\frac{dx}{dy}$, which comes out to be $\frac{2y}{3x^2+5}$, which is zero at $(1, 0)$. So there is indeed a vertical tangent line at $(1, 0)$.

Now, let's work on finding the second derivative. We have

$$2y\frac{dy}{dx} = 3x^2 + 5.$$

Let's take another derivative:

$$2\left[\frac{dy}{dx}\right]^2 + 2y\frac{d^2y}{dx^2} = 6x$$

So we solve for $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = \frac{6x - 2\left(\frac{dy}{dx}\right)^2}{2y}$$

and plug in our expression for $\frac{dy}{dx}$ to get this whole thing in terms of just x and y:

$$\frac{d^2y}{dx^2} = \frac{6x - 2\left(\frac{3x^2+5}{2y}\right)^2}{2y}.$$

Problem 0.3. *We will do another geometric example now: An ellipse can be described as the set of points P in the plane such that, for two special points called the foci, the sum of the distance from P to the foci is a constant. Consider the ellipse with foci $(-3, 0)$ and $(3, 0)$ with distance-sum equal to 10. Find $\frac{dy}{dx}$ at the points where $x = 3$.*

We have:

$$\sqrt{(x-3)^2 + y^2} + \sqrt{(x+3)^2 + y^2} = 10.$$

Upon differentiating, we get:

$$0 = \frac{1}{2\sqrt{(x-3)^2 + y^2}}\left(2(x-3) + 2y\frac{dy}{dx}\right) + \frac{1}{\sqrt{(x+3)^2 + y^2}}\left(2(x+3) + 2y\frac{dy}{dx}\right).$$

At $x = 3$, we have $\pm y + \sqrt{36 + y^2} = 10$. A little algebra gives that $y = \pm\frac{64}{20}$. We can plug this into the expression for $\frac{dy}{dx}$.

Let's do a geometric example that works in 2 dimensions:

Example 0.1. *You're running at* $5km/hr$. *You run on a road that is* 1 *meter away from a tree. You run past the tree. How quickly is your distance from the tree changing when you're* 50 *meters away from the tree?*

We have

$$D = x^2 + y^2$$

by Pythagoras. So by differentiating:

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt}$$

so

$$\frac{dD}{dt} = \left(\frac{x}{D}\right)\frac{dx}{dt}$$

Now, use $D = 50$ and $x = \sqrt{2499}$ to get $\frac{\sqrt{2499}}{10}km/hr$.

The point is that these kinds of distance problems really lend themselves to related rates. Another classic example is one where *both* of the objects you're dealing with the distance of are moving.

Example 0.2. *At noon, boat A is* $100km$ *north of boat B. Boat A is sailing East at a constant speed of* $30\frac{km}{hr}$ *and boat B is sailing West at a constant speed of* $35\frac{km}{hr}$. *How quickly is the distance between the boats changing at 4:00 PM?*

To do this, it really helps to draw a picture of what's going on. Let's draw a picture illustrating the boats at both times.

Now, we know that boat A is sailing East and boat B is sailing west. Let's let a be the distance that boat A has travelled since noon and b be the distance that boat B has travelled since noon. We know the speed of both boats:

$$\frac{da}{dt} = 30$$
$$\frac{db}{dt} = 35$$

We want to compute the rate at which the distance between the two boats is changing. Looking at the picture and applying the Pythagorean theorem, we conclude that the distance D between the two boats is related to the quantities a and b by the equation:

$$D^2 = (a + b)^2 + 100^2.$$

We differentiate both sides to get:

$$2D\frac{dD}{dt} = 2(a + b)\left(\frac{da}{dt} + \frac{db}{dt}\right)$$

We can find a and b at $4PM$: a is equal to $30 \cdot 4 = 120$; b is equal to $35 \cdot 4 = 140$. By Pythagoras, we get that $D = \sqrt{120^2 + 140^2} = \sqrt{14400 + 19600} = \sqrt{34000}$. So we plug and chug to get:

$$\frac{dD}{dt} = \frac{(260)(65)}{\sqrt{34000}}.$$

The point of this problem is that the Pythagorean theorem is useful when you are doing related rates problems involving distances.

Another important example involving similar triangles is the "shadow" problem:

A woman with height $2m$ is walking toward a lamppost at a rate of $2\frac{m}{s}$. The lamppost is 3 meters tall. How quickly is the length of her shadow decreasing when she is $1m$ away from the lamppost.

Here, we know *both* rates, but we're interested in a different quantity.

Let s be the length of the woman's shadow and x be the distance from the woman to the lamppost. Drawing the picture, we can conclude using similar triangles that

$$\frac{3}{2} = \frac{x+s}{s}.$$

Solving for s we get that $s = 2x$. Therefore, $\frac{ds}{dt} = 2\frac{dx}{dt}$. We can plug in $\frac{dx}{dt} = -2\frac{m}{s}$ to get a rate of $-4\frac{m}{s}$. (Note that the distance to the lamppost is not relevant for computing this rate of change).