Example 0.1. *A cat is looking at a laser pointer dot on a wall* $5m$ *away. The laser pointer dot is moving to the right and is currently* $2m$ *away from the point on the wall closest to the cat.*

The cat is turning her head at a rate of $0.005rad/s$. *How fast is the dot moving?*

This is our first example of a related rates problem that involves an angle. If we draw a picture, it is clear that the distance x from the dot to the closest point on the wall can be expressed as $5\tan\theta$ where θ is the angle.

We differentiate this equation with respect to t:

$$\frac{5}{(\cos\theta)^2}\frac{d\theta}{dt} = \frac{dx}{dt}$$

Finally, we plug in the values for $\frac{d\theta}{dt}$ and $\cos\theta$: At the time in the problem, we have that the adjacent side to the angle θ has length 5, and the hypotenuse has length $\sqrt{5^2+2^2} = \sqrt{29}$, so $(\cos\theta)^2$ comes out to $\frac{25}{29}$. We therefore get

$$\frac{29}{5}m * 0.005\frac{rad}{s} = \frac{dx}{dt}$$

this comes out to

$$\frac{29}{1000}\frac{m}{s}.$$

Example 0.2. *A ladder that is* $3m$ *long is slipping at a rate of* $0.1m/s$. *How fast is the angle between the ladder and the floor changing if the base of the ladder is* $1m$ *away from the wall?*

This time, we have $x = 3\cos\theta$. We therefore get

$$-3\sin\theta\frac{d\theta}{dt} = \frac{dx}{dt}$$

So

$$\frac{d\theta}{dt} = \frac{-dx/dt}{3\sin\theta}$$

Now, we're ready to plug everything in. $\sin\theta = \frac{\sqrt{8}}{3}$, and $\frac{dx}{dt} = 0.1m/s$, so we get

$$\frac{d\theta}{dt} = \frac{-0.1m/s}{3m * \sqrt{8}3}$$

which comes out to

$$\frac{d\theta}{dt} = \frac{-0.1}{\sqrt{8}}\frac{rad}{s}.$$

Finally, we give an example where things are moving along a shape that is more complicated than just a line.

Example 0.3. *For this problem, we will consider points* P *that lie along the parabola* $y = x^2$. *Let* O *be the origin* $(0,0)$. *Let* θ *be the angle between* OP *and the positive* x*-axis. Find the rate of change of* θ *with respect to* x *at the point* $P = (1,1)$.

We have that $\tan\theta = \frac{y}{x}$, so differentiating gives:

$$\frac{1}{(\cos\theta)^2}\frac{d\theta}{dx} = \frac{x\frac{dy}{dx} - y}{y^2}$$

The angle θ is equal to $\frac{\pi}{4}$ radians, so the cosine is $\frac{1}{\sqrt{2}}$. We have that $\frac{dy}{dx} = 2$ at $(1, 1)$, and x an dy are both 1, so we get:

$$2\cancel{\frac{1}{2}}\frac{d\theta}{dx} = \frac{2-1}{1}$$

so $\frac{d\theta}{dx} =$ 1/2

2 dtheta/dx = 2-1 =1 , therefore

We will start by giving a more complicated problem involving angles.

Example 0.1. *For this problem, we will consider points P that lie along the parabola $y = x^2$. Let O be the origin $(0, 0)$. Let θ be the angle between OP and the positive x-axis. Find the rate of change of θ with respect to x at the point $P = (1, 1)$.*

We have that $\tan\theta = \frac{y}{x}$, so differentiating gives:

$$\frac{1}{(\cos\theta)^2}\frac{d\theta}{dx} = \frac{x\frac{dy}{dx} - y}{y^2}$$

The angle θ is equal to $\frac{\pi}{4}$ radians, so the cosine is $\frac{1}{\sqrt{2}}$. We have that $\frac{dy}{dx} = 2$ at $(1, 1)$, and x an dy are both 1, so we get:

$$\frac{1}{2}\frac{d\theta}{dx} = \frac{2-1}{1}$$

so $\frac{d\theta}{dx} = 2$. Sometimes a problem that doesn't seem to be a related rates problem will turn out to be a related rates problem in disguise.

Example 0.2. *In an hourglass, the sand is flowing from the top segment to the bottom segment at a constant rate. The top of the hourglass is a cone with height $5cm$ and radius $3cm$. Assume the sand in the top of the hourglass always fills out a cone-shape. Currently, the sand in the top segment has height $2cm$, and the level of the sand is decreasing at a rate of $1mm/s$. How much more time remains until the top component of the hourglass is empty?*

This problem is somewhat harder than the related rates problems we have discussed in the past, in that it's indirect- it doesn't seem to be the case that we're looking for a rate of change.

Well, what we DO know is that the rate of change of volume of sand in the top component of the hourglass, $\frac{dV}{dt}$, is a constant. We can use this information in the following way: the total change in the volume over a time interval of t seconds will be $\frac{dV}{dt} \cdot t$. If we know $\frac{dV}{dt}$, we can calculate the time for the hourglass to empty out. The sand in the top of the hourglass is a cone of height h and radius r. Note that the cone of sand is similar to the top of the hourglass, so by a similar triangles argument, the ratio $\frac{r}{h}$ must be the value $\frac{3}{5}$. Therefore, the volume is

$$\begin{aligned}
V &= \frac{1}{3}\pi r^2 h \\
&= \frac{1}{3}\pi\left(\frac{3}{5}h\right)^2 h \\
&= \frac{1}{3}\pi \cdot \frac{9}{25}h^3 \\
&= \frac{3}{25}\pi h^3
\end{aligned}$$

So, when we differentiate, we get

$$\frac{dV}{dt} = \frac{9}{25}\pi h^2 \frac{dh}{dt}.$$

And we know that when $h = 2cm$, we have that $\frac{dh}{dt} = -0.1cm/s$. Thus,

$$\frac{dV}{dt} = \frac{9}{25}\pi(4cm^2)(-0.1cm/s)$$

which comes out to

$$\frac{dV}{dt} = -\frac{2}{5}\cdot\frac{9\pi}{25}cm^3/s$$

which simplifies to

$$\frac{dV}{dt} = \frac{-18\pi}{125}cm^3/s.$$

Now, we're ready to solve the problem. The current volume of sand is

$$\frac{3}{25}\pi(2cm)^3$$

which is

$$\frac{24}{25}\pi$$

and the rate of change is

$$\frac{-18\pi}{125}cm^3/s$$

so the total time is

$$\frac{5*24}{18} = \frac{120}{18}$$

seconds.

Let's do one more example from special relativity.

You watch two spaceships fly by in the same direction - In your reference frame, one is going at $0.1c$, and the one behind it is going at $0.05c$, where c is the speed of light. The ship going at $0.1c$ is accelerating at a rate of $0.01c/hour$, and the ship flying at $0.05c$ is moving at a constant speed. In the reference frame of the ship flying at $0.05c$, how fast is the ship moving at $0.1c$ accelerating? The faster ship's velocity in the slower ship's reference frame is given by

$$w = \frac{u - v}{1 - uv/c^2}$$

where u is the speed of the faster ship (in your reference frame) and v is the speed of the slower ship.

To solve this problem, we just need to differentiate with respect to time and use the fact that v is constant.

$$\frac{dw}{dt} = \frac{(1 - uv/c^2) + (u - v)(v/c^2)}{(1 - uv/c^2)}\frac{du}{dt}$$

The uv/c^2 terms cancel, so we get

$$\frac{1 - v^2/c^2}{(1 - uv/c^2)^2} \frac{du}{dt}$$

from here, we can just plug in u and v.