Nov 29

* linear approximation: $y = f(a) + (x-a)f'(a)$

* overestimate or underestimate

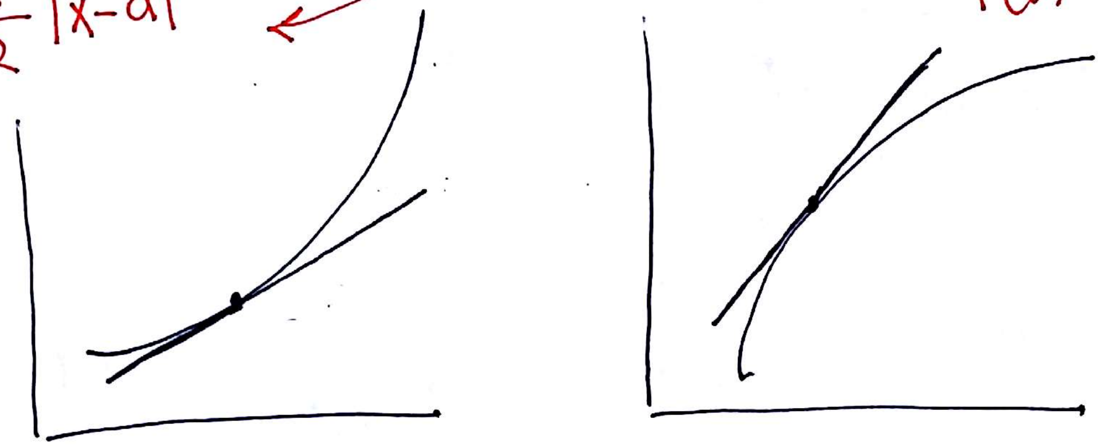
$f''(a) > 0 \rightarrow$ underestimate

max of $|f''|$ ~~over~~ between x and a

$\uparrow$

$\frac{M}{2}|x-a|^2$ $\leftarrow$ * Maximum Error.

$f''(a) < 0 \rightarrow$ overestimate



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1) estimate $\sqrt[4]{90}$, 2) Say if your estimation is ~~a~~ underestimate or overestimate.

3) give an Error for your approximation.

$f(x) = \sqrt[4]{x}$, linearize at $a = 81$

$\mathcal{L}(x) = f(a) + f'(a)(x-a)$

$f(x) = \sqrt[4]{x} = x^{1/4} \longrightarrow f(81) = \sqrt[4]{81} = 3$

$f'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4(\sqrt[4]{x})^3} \rightarrow f'(81) = \frac{1}{4(\sqrt[4]{81})^3} = \frac{1}{4(27)}$

$= \frac{1}{108}$

$f(x) \simeq \mathcal{L}(x) = \boxed{3 + \frac{1}{108}(x-81)}$

$$f(90) \sim L(90) = 3 + \frac{1}{108}(90-81) =$$

$$3 + \frac{1}{108} 9 = 3 + \frac{1}{12} = \underline{\underline{\frac{37}{12}}}$$

2)

$$f''(x) = \frac{-3}{16} x^{-\frac{7}{4}} = \frac{-3}{16x^{7/4}} = \frac{-3}{16(x^{1/4})^7} = \frac{-3}{16(\sqrt[4]{x})^7}$$

$$f''(81) = \frac{-3}{16(\sqrt[4]{81})^7} = \frac{-3}{16(3^7)} = \frac{-1}{16(3^6)}$$

$f''(a) < 0 \longrightarrow$ over estimate.

3)

$$\frac{M}{2}(x-a)^2$$

max of $|f''|$ between a, and x.

* find absolute maximum of $\frac{+3}{16} x^{-\frac{7}{4}}$ in interval $[81, 90]$

$$\frac{3}{16x^{7/4}} \rightarrow M = \frac{3}{16(81)^{7/4}} = \frac{1}{16(3^6)}$$

$$\text{Error} = \frac{1}{2} \cdot \frac{1}{16(3^6)} (90-81)^2 = \frac{9^2}{32(3^6)} = \boxed{\frac{1}{32(9)}} =$$

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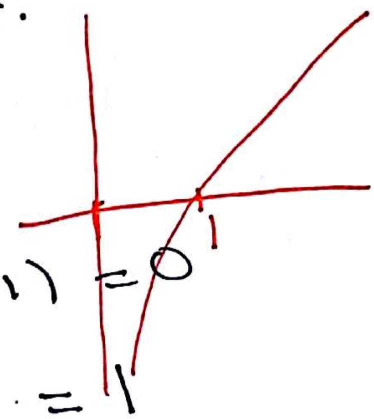
$$\frac{37}{12} - \frac{1}{32(9)} < \sqrt[4]{90} < \frac{37}{12} + \cancel{\frac{1}{32(9)}}$$

Estimate $\ln(0.9)$, Say if it is overestimate underestimate, give a Worst case Error.



$f(x) = \ln x \quad \longrightarrow \quad a = 1, \quad f(a) = \ln(1) = 0$

$f'(x) = \frac{1}{x} \quad \longrightarrow \quad f'(a) = \frac{1}{1} = 1$

$$L(x) = f(a) + (x-a) f'(a)$$

$$f(x) \sim L(x) = (x-1)\cdot 1 = x - 1$$

$$\ln(0.9) = f(0.9) \sim L(0.9) = 0.9 - 1 = -0.1$$

$f''(x) = \frac{-1}{x^2} \qquad f''(1) = -1 < 0 \rightarrow$ overestimate

$y' = \frac{-2x}{x^4} = \frac{2}{x^3} \qquad \boxed{\cancel{x=0}}$ critical Point

$M =$ max of $y = \frac{1}{x^2}$ in $\underbrace{[0.9, 1]}$

Max happens at $\boxed{x = 0.9}$ $\qquad M = \boxed{\frac{1}{0.81}}$

$$\text{Error} = \frac{1}{2}\,\frac{1}{0.81}(0.9-1)^2 = \frac{1}{2}\cdot\frac{100}{81}\cdot 0.01 = \frac{1}{162}$$

π, e, $\sqrt{2}$,

We approximate b/c some important numbers can only be approximated.

We use Polynomials to approximate.

$$P_n(x) = A_0 + A_1 x^1 + A_2 x^2 + \dots + A_n x^n$$

Why? Polynomial

Polynomials only involve $+$, $*$
easy to calculate.

How to Approximate $f(x)$ with $P_n(x)$

at $x=a$.

linear approximation

$f(a) = P(a)$ → Same value at $x=a$

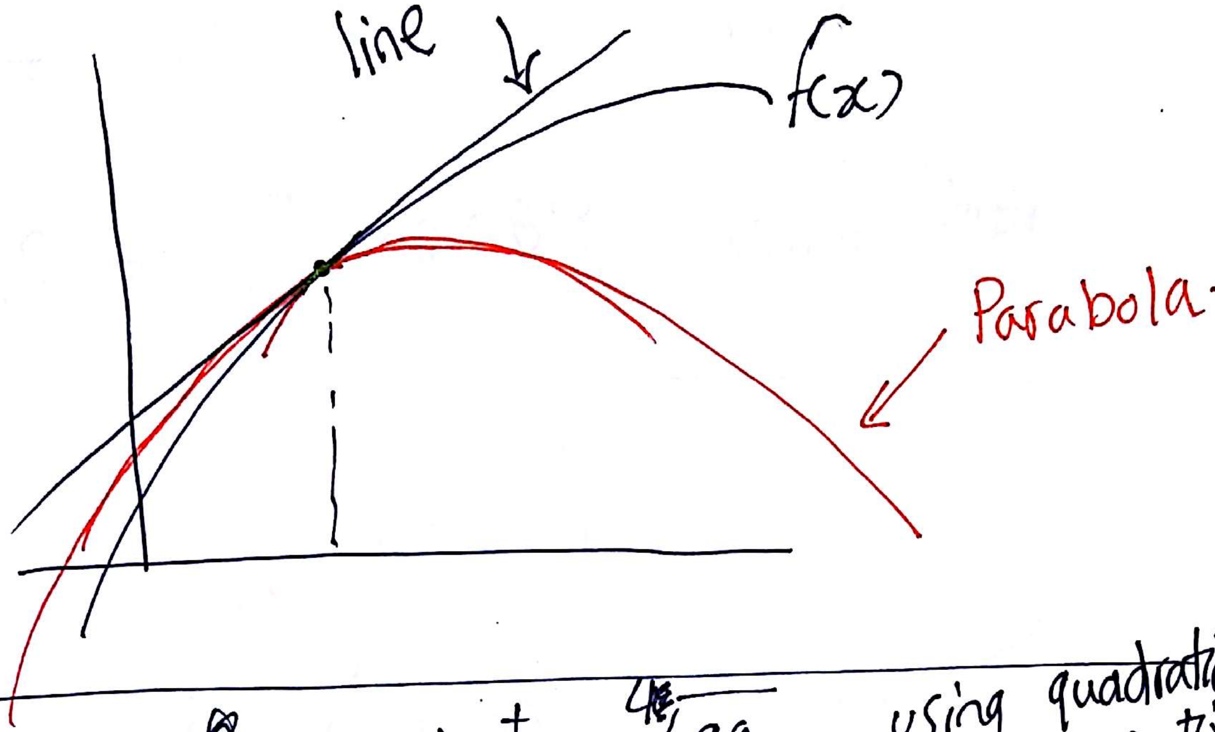
$f'(a) = P'(a)$ Same slope at $x=a$

$f''(a) = P''(a)$ Same Concavity at $x=a$

→ quadratic approximation

$\vdots$

$f^{(n)}(a) = P^{(n)}(a)$



approximate $\sqrt[4]{90}$ using quadratic approximation

$$f(x) = \sqrt[4]{x} \qquad f(81) = 3$$

$$P = Ax^2 + Bx + C$$

$$f'(x) = \frac{1}{4(\sqrt[4]{x})^3} \qquad f'(81) = \frac{1}{108}$$

$$P'(x) = 2Ax + B$$

$$f''(x) = \frac{-3}{16(\sqrt[4]{x})^7} \qquad f''(81) = \frac{-1}{16(3^6)}$$

$$P''(x) = 2A$$

$$P(81) = f(81) = 3 \rightarrow A(81)^2 + B(81) + C = 3 \rightarrow C = \frac{63}{32}$$

$$p'(81) = f'(81) = \frac{1}{108} \rightarrow 2A(81) + B = \frac{1}{108} \rightarrow B = \frac{7}{432}$$

$$p''(81) = f''(81) = \frac{-1}{16(3^6)} \rightarrow 2A = \frac{-1}{16(3^6)}$$

$$\Downarrow$$

$$A = \frac{-1}{32(3^6)} = \frac{-1}{23328}$$

$$\boxed{f(x) \sim P(x) = \frac{-1}{23328}x^2 + \frac{7}{432}x + \frac{63}{32}}$$

$\sqrt[4]{90} \approx P(90) = 3.07986 \leftarrow$ using quadratic

$\sqrt[4]{90} \approx L(90) = \dfrac{37}{12} \simeq 3.0813$ using linear.

Taylor's Idea to shift ~~P(x)~~ by "a" and then find Coefficients

$$P = A'(x-a)^2 + B'(x-a) + C'$$

$f(a) = P(a) = C' \longrightarrow C' = f(a) = 3$

$f'(a) = P'(a) = B' \longrightarrow B' = f'(a) = \dfrac{1}{108}$

$f''(a) = P''(a) = 2A' \longrightarrow A' = \dfrac{f''(a)}{2} = \dfrac{-1}{23328}$

$$P = 3 + \frac{1}{108}(x-81) - \frac{1}{23328}(x-81)^2$$

linear apporoximation quadratic

in general

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

n^{th} order Taylor Polynomial at $x=a$

$n! = n(n-1)(n-2)\cdots(1)$ $\boxed{0! = 1}$ by definition

Find third order Taylor Polynomial of $f(x) = e^x$

at $a = 1$

$f(x) = e^x \rightarrow f(1) = e$

$f'(x) = e^x \quad f'(1) = e$

$f''(1) = e$

$\vdots$

$f'''(x) = e^x \quad f'''(1) = e$

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3$$

$$= e + e(x-1) + e\frac{(x-1)^2}{2} + \frac{e}{6}(x-1)^3$$

Taylor Polynomials at $a=0$ are called Mac laurin Polynomials.

Maclaurin Polynomial of $f(x) = e^x$ is:

$f(x) = e^x \rightarrow f(0) = 1$

$\vdots \qquad \vdots$

$f^{(n)} = e^x \quad f^{(n)}(0) = 1$

$$T_n(x) = 1 + 1(x-0) + \frac{1}{2}(x-0)^2 + \frac{1}{6}(x-0)^3 + \frac{1}{(24)}(x-0)^4 + \cdots \frac{1}{n!}(x-0)^n$$

M.P of e^x

$$= \boxed{1 + x + x^2/2 + x^3/6 + \frac{x^4}{24} + \cdots + \frac{x^n}{n!}}$$

Find second order Polynomial $f(x)=x^{2/3}$ at $x=1$

$$f(x)=x^{2/3} \longrightarrow f(1)=1$$

$$f'(x)=\frac{2}{3}x^{-1/3} \longrightarrow f'(1)=2/3$$

$$f''(x)=\frac{-2}{9}x^{-4/3} \longrightarrow f''(1)=-2/9$$

$$T_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2$$

$$T_2(x)=1+\frac{2}{3}(x-1)-\frac{2}{9}/_2\,(x-1)^2$$

$$\underbrace{\underbrace{1+\frac{2}{3}(x-1)}_{\text{linear}}-\frac{1}{9}(x-1)^2}_{\text{quadratic.}}$$

$$\lim_{x\to 0}\frac{x-\sin x}{1-\cos x}=\lim_{x\to 0}\frac{x-\left(x-\frac{x^3}{6}\right)}{1-\left(1-\frac{x^2}{2}\right)}=\lim_{x\to 0}\frac{\frac{x^3}{6}}{\frac{x^2}{2}}=$$

$$\lim_{x\to 0}\frac{x}{3}=0$$