Dynamics of pentamode structures using beam theory

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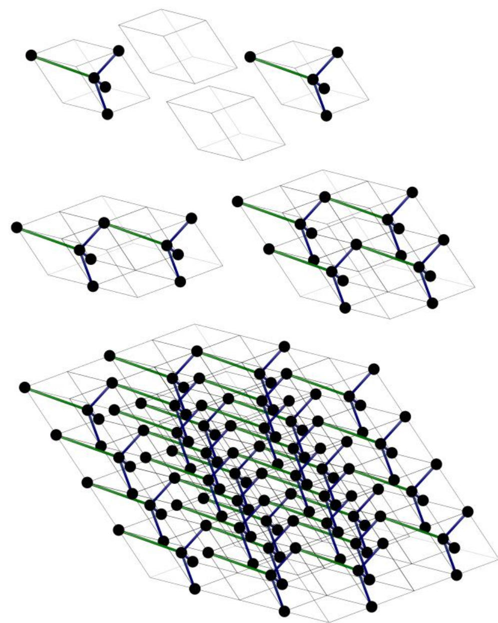

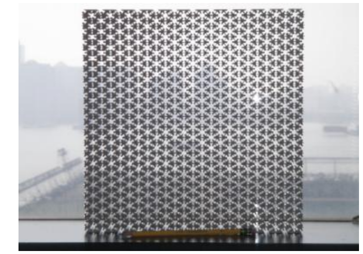

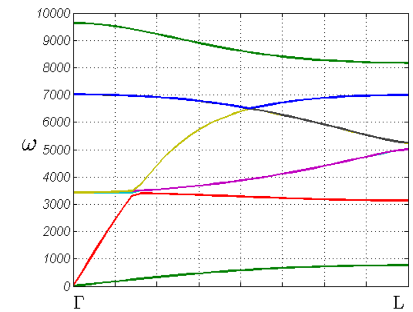

AmeriMech 2014 (2 – 4 April 2014)

Dynamics of periodic materials and structures

Georgia Tech Hotel and Conference Center, Atlanta, Georgia, USA

Sponsors

pentamode materials

pentamode lattice: statics

pentamode lattice: dynamics

Kelvin (1856)

$$C_{ijkl} = \sum_{\alpha=1}^{6} K_\alpha \, P^\alpha_{ij} P^\alpha_{kl} \quad \rightarrow \quad \mathbf{C} = \sum_{\alpha=1}^{6} K_\alpha \, \mathbf{P}^\alpha \otimes \mathbf{P}^\alpha$$

Positive definite strain energy : $K_\alpha > 0$

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Positive definite strain energy : $K_\alpha > 0$

Maxwell (1864)

Necessary and sufficient conditions for truss to be rigid

Necessary and sufficient condition for rigidity of 2D and 3D frameworks is Z > 6 and Z > 12, respectively
Deshpande et al. *JMPS* (2001).

Kelvin (1856)

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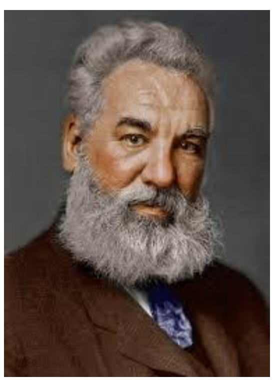
Maxwell(1864)

Necessary and sufficient conditions for truss to be rigid

Necessary and sufficient condition for rigidity of 2D and 3D frameworks is Z > 5 and Z > 11, respectively

Deshpande et al. *JMPS* (2001).

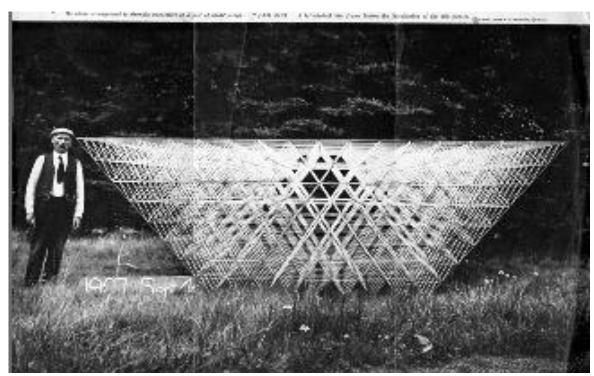


Bell (1907)

A. Graham Bell
Has New Idea
In Architecture

Opening of the Tetrahedral Tower, Seventy Feet High on Beinn Bhreagh.

It May Become an Important Factor in Building of the Future.

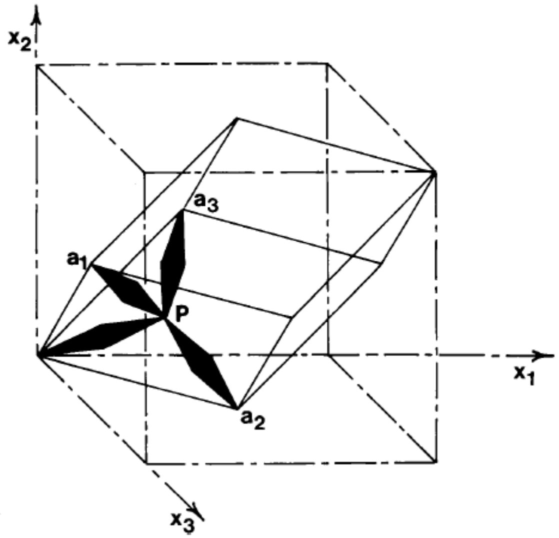


Kelvin

$$C_{ijkl} = \sum_{\alpha=1}^{6} K_\alpha P_{ij}^\alpha P_{kl}^\alpha \quad \rightarrow \quad \mathbf{C} = \sum_{\alpha=1}^{6} K_\alpha \mathbf{P}^\alpha \otimes \mathbf{P}^\alpha$$

$$K_\alpha > 0, \ \alpha = 1, 2, \ldots 6$$

Milton and Cherkaev (1995)



proposed diamond structure

$$K_1 = 0, \ K_\alpha > 0, \ \alpha = 2, \ldots 6$$ Unimode

$$K_1 = K_2 = 0,$$ Bimode

……. ……

$$K_1 = \ldots = K_5 = 0, \ K_6 > 0$$ **Pentamode (PM)**

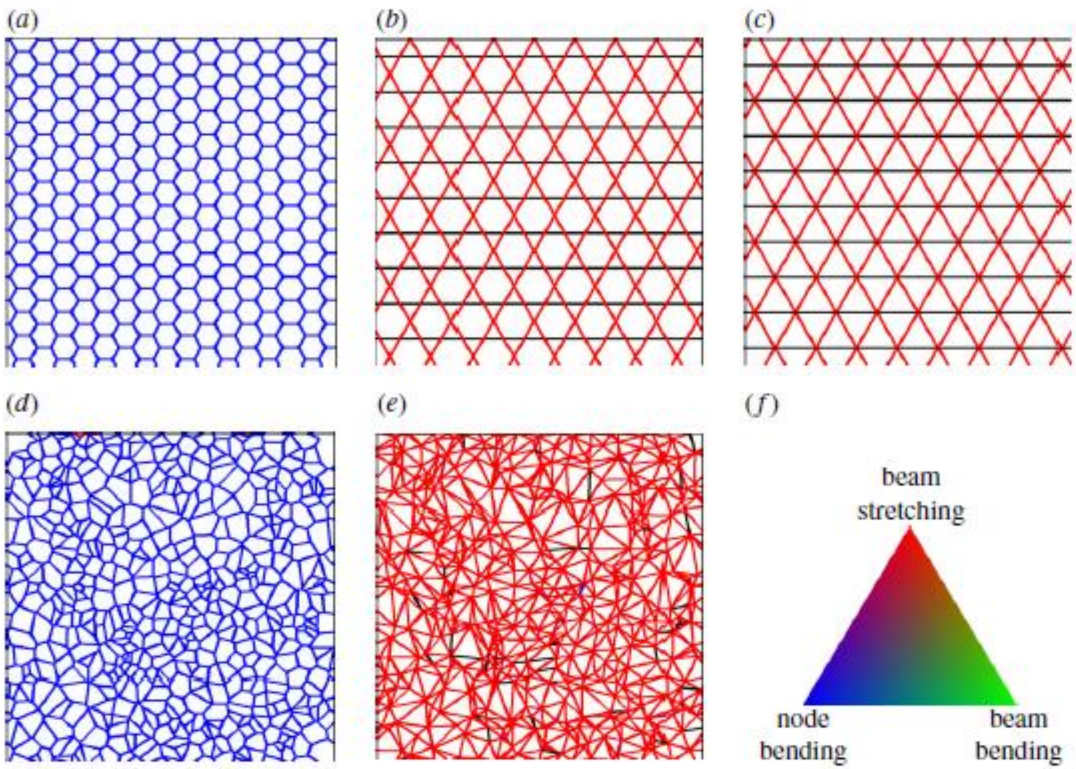
PM: five of the eigen-stiffnesses are zero

five (*penta)* easy/soft modes

Warren & Kraynik (1988, ..) Ashby, Deshpande, Hutchinson & Fleck, Christensen (1995, 2000)

Stretch dominated, bending dominated, collapse mechanisms, ….
………..soft modes, easy modes, isostatic

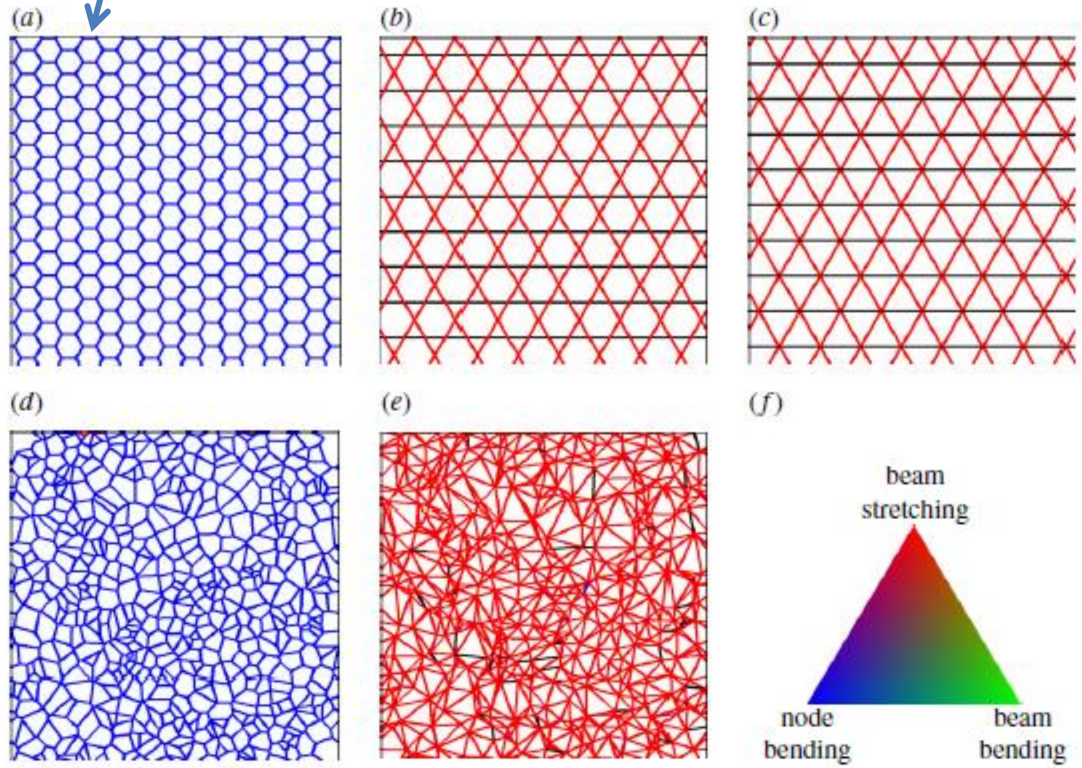
Stretch dominated, bending dominated, collapse mechanisms, ….
………..soft modes, easy modes, isostatic



Gurtner & Durand, “Stiffest elastic networks”, *PRSA* 2014 doi: 10.1098/rspa.2013.0611 soft modes are bending dominated

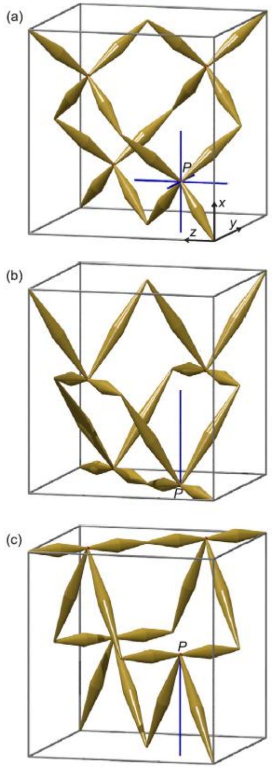
$K_\alpha > 0$ i.e. effective static moduli, are stretch dominated

soft/easy modes

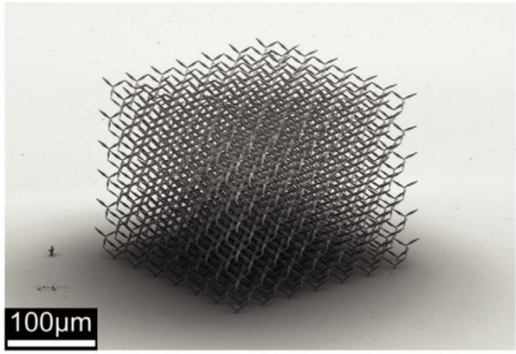


Gurtner & Durand, "Stiffest elastic networks", PRSA 2014 doi: 10.1098/rspa.2013.0611

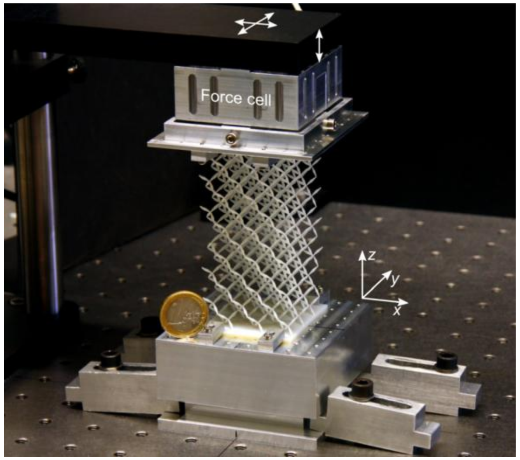
Pentamode lattice structures



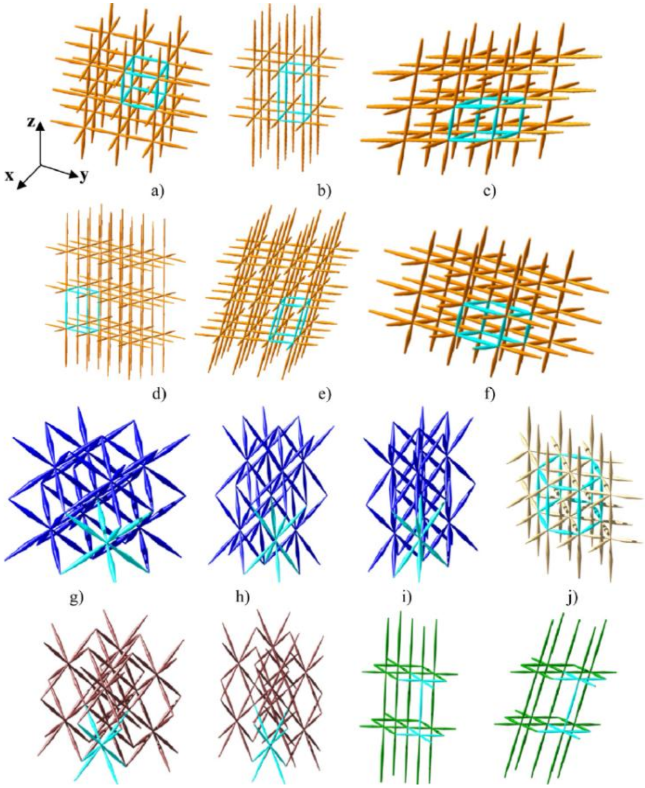
Kadic et al., *NJP* 2013



Kadic et al., *APL* 2012



Schnitty et al., *APL* 2013



Mejica and Lantada *Smart Mat. Struct*. 2013

water as an elastic "solid"

elastic equation of motion $\text{div}\boldsymbol{\sigma} = \boldsymbol{\rho}\ddot{\mathbf{u}}$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad \rightarrow \quad \boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$$

Acoustics $C_{ijkl} = K\,\delta_{ij}\delta_{kl} \quad \rightarrow \quad \mathbf{C} = K\,\mathbf{I}\otimes\mathbf{I}$

Water is a **pentamodal elastic material**

transformation acoustics: **isotropic PM** ⟹ **anisotropic PM**

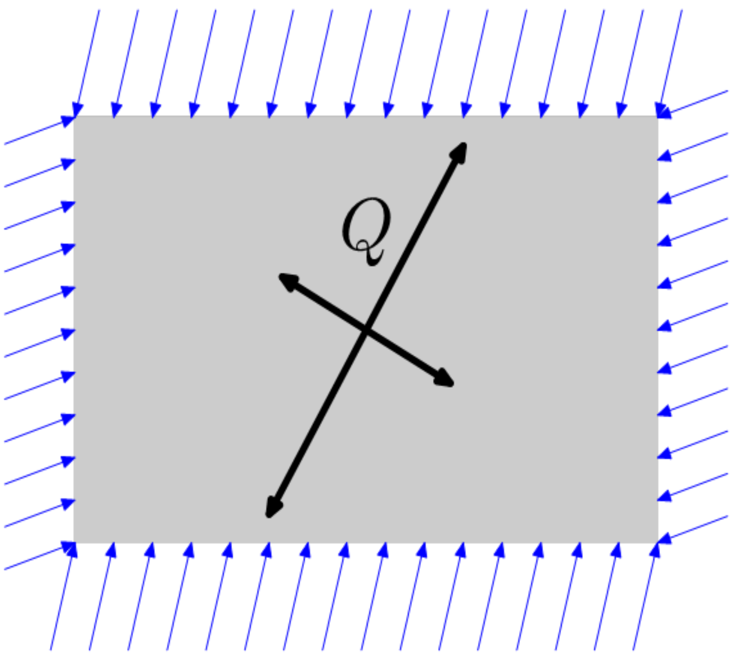
Norris (2008, 2009)

pentamode form of stiffness: $\boldsymbol{C} = K\boldsymbol{S}\otimes\boldsymbol{S}$

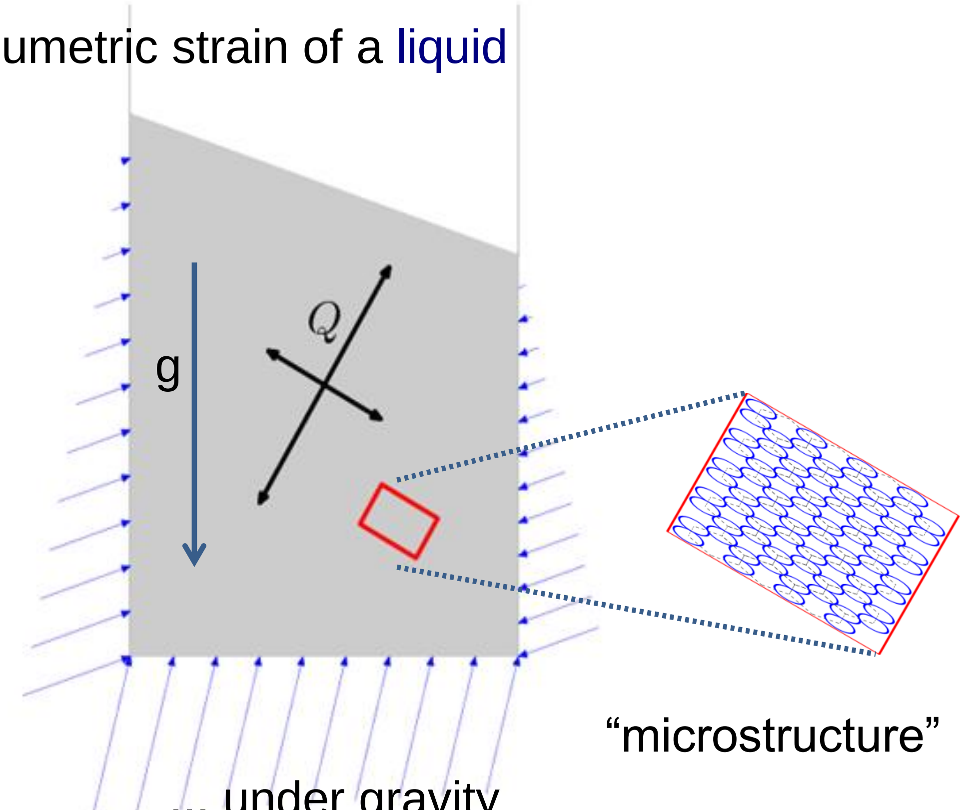
mechanical behavior of **pentamode materials** (**PM**) $C_{ijkl} = K\, Q_{ij} Q_{kl}$

a single type of stress (and strain)

- like hydrostatic stress and volumetric strain of a liquid



static equilibrium



“microstructure”

... under gravity

PM = limiting case of anisotropic solids with zero “shear“ rigidity

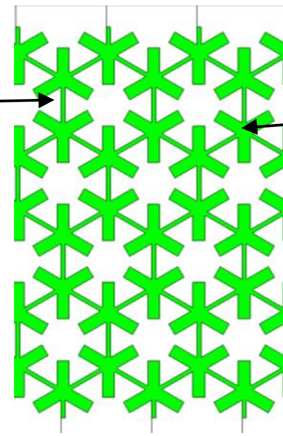
Metal Water

generic structure for **transformation acoustics** in water

Norris, Nagy (2011)

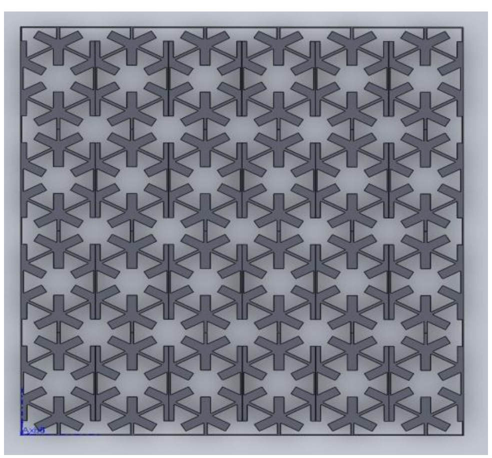
struts

- for stiffness mainly



islands

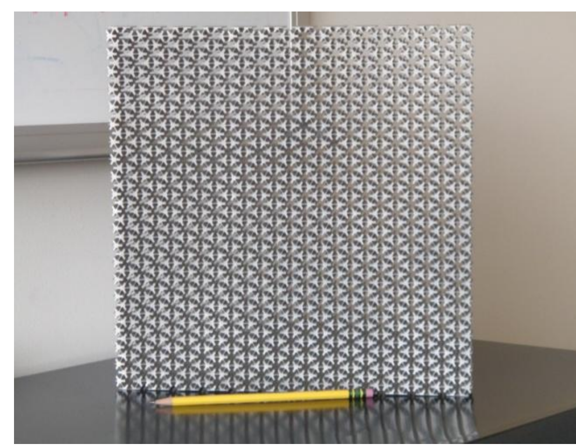
- inertial role mainly



bulk modulus = 2.25 Gpa

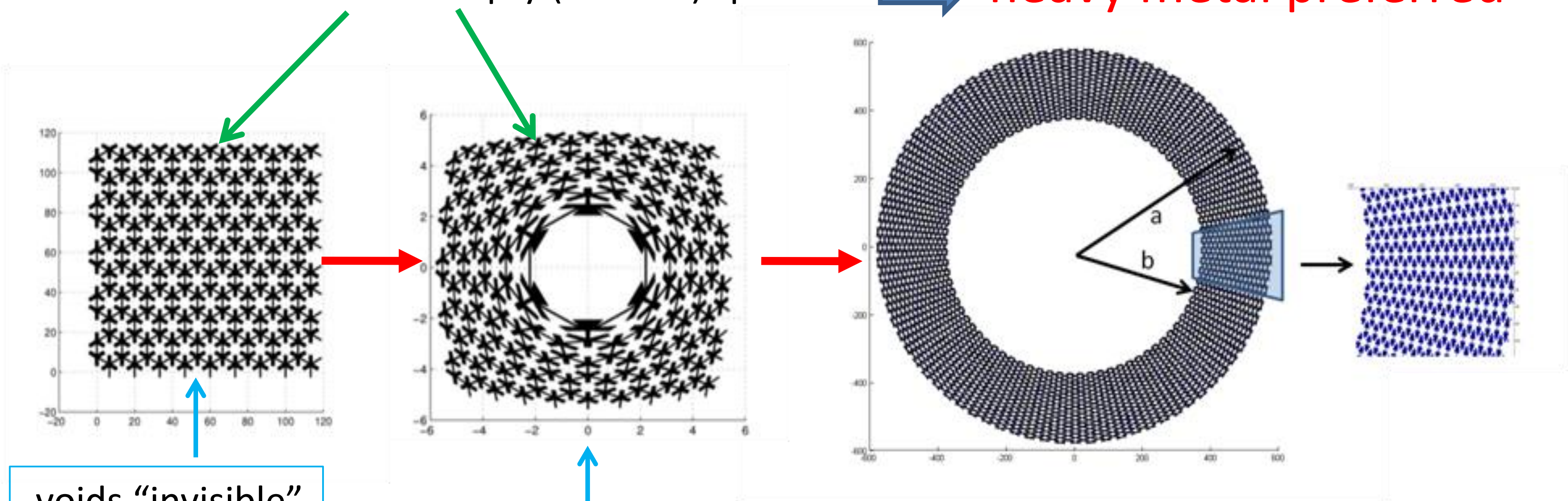
density = 1000 kg/m^3

shear modulus = 0.065 Gpa
(i.e. small)



Pentamode material and transformation acoustics

same amount of total empty (cloaked) space → heavy metal preferred



voids "invisible"

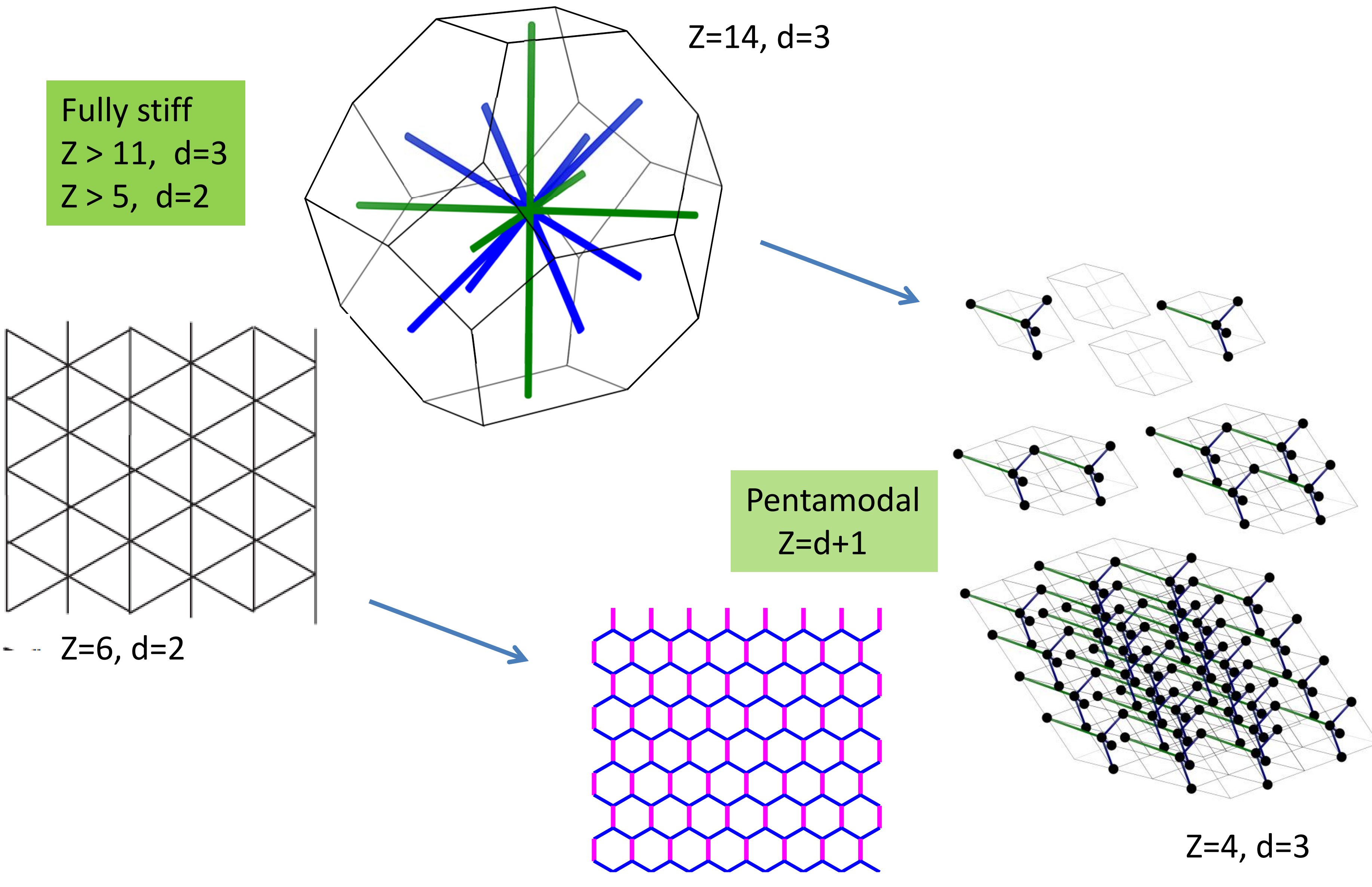
volume of empty space remains constant

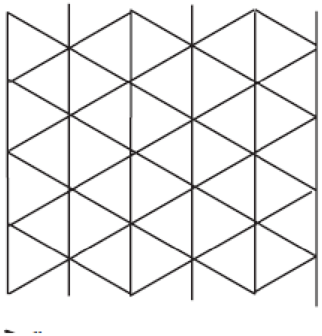
conservation of empty/cloaked space = conservation of mass pentamode materials

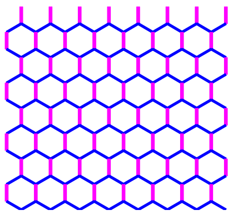
pentamode lattice: statics

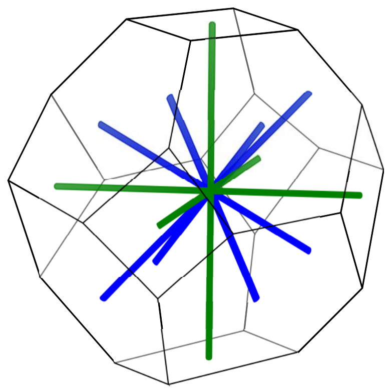
pentamode lattice: dynamics

stretch dominated effective elastic moduli









Ingredients:

Z = coordination #

$R_i, \ \mathbf{e}_i$ length, direction

$M_i = \int_0^{R_i} \frac{dx}{E_i A_i}$ axial compliance

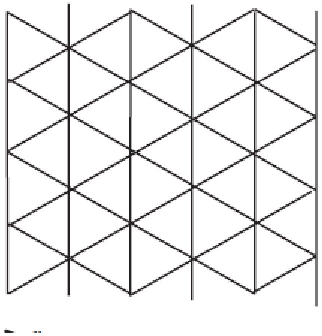
V = unit cell volume

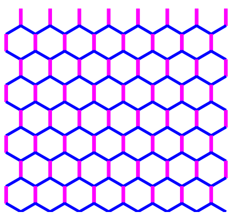
Effective elastic moduli

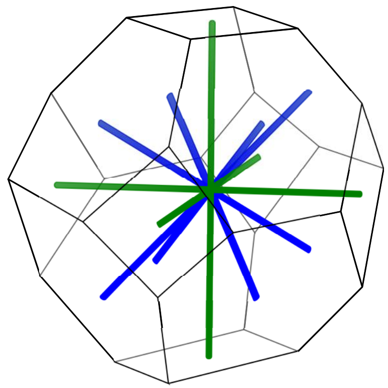
$$\mathbf{C} = \frac{1}{V} \sum_{i,j=1}^{Z} R_i R_j \sqrt{M_i M_j}\, P_{ij} (\mathbf{v}_i \otimes \mathbf{v}_i) \otimes (\mathbf{v}_j \otimes \mathbf{v}_j) \quad \text{where}$$

$$P_{ij} = \delta_{ij} - \mathbf{v}_i \cdot \Big(\sum_{k=1}^{Z} \mathbf{v}_k \otimes \mathbf{v}_k \Big)^{-1} \cdot \mathbf{v}_j, \qquad \mathbf{v}_i \equiv M_i^{-1/2} \mathbf{e}_i$$

Effective elastic moduli of stretch dominated lattices







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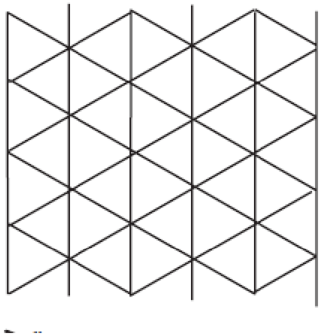
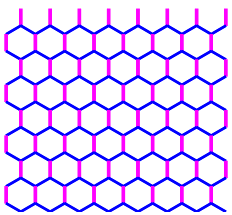
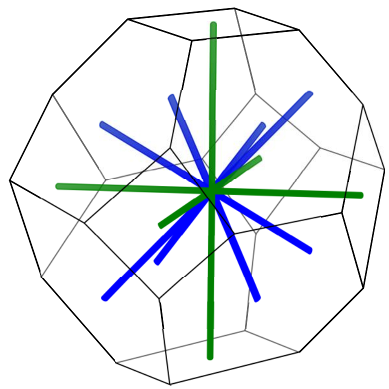
P is a projector $\quad \mathbf{P}^2 = \mathbf{P}, \quad \mathrm{tr}\mathbf{P} = Z - d,$

$\Rightarrow \quad \mathrm{rank}\, \mathbf{C} \le Z - d$

Effective elastic moduli of stretch dominated lattices







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$M_i = \int_0^{R_i} \frac{dx}{E_i A_i}$ axial compliance

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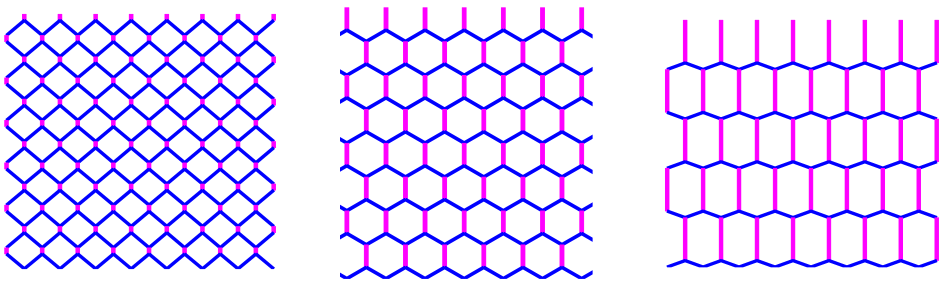
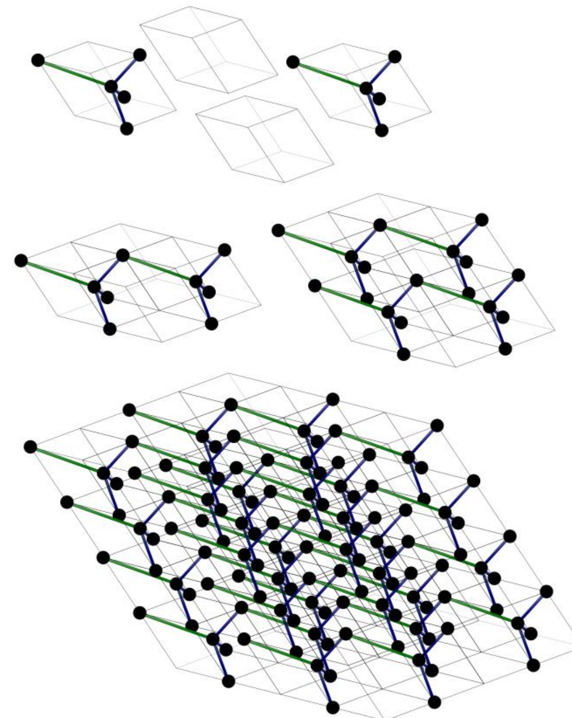
Pentamode: Z=d+1

$\mathrm{rank}\, \mathbf{C} = 1$

Explicit static PM moduli for lattices with Z=d+1



Z=3, d=2 2D PM lattices (all isotropic)



Z=4, d=3

Pentamode: $\mathbf{C} = K\mathbf{S} \otimes \mathbf{S}$

$$\mathbf{C} = \Big(V\sum_{k=1}^{d+1}\gamma_k\Big)^{-1}\Big(\sum_{i=1}^{d+1}\gamma_i\,\mathbf{e}_i\otimes\mathbf{e}_i\Big)\otimes\Big(\sum_{j=1}^{d+1}\gamma_j\,\mathbf{e}_j\otimes\mathbf{e}_j\Big)$$

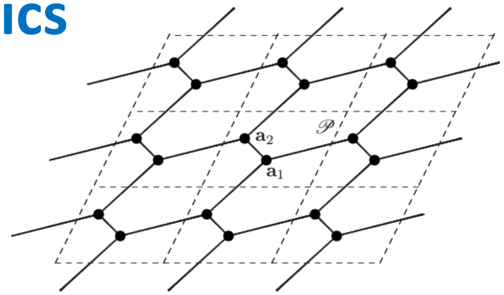
$$\text{where}\quad \gamma_i = \frac{R_i^2}{M_i} - \frac{\mathbf{R}_i}{M_i}\cdot\Big(\sum_{k=1}^{d+1}\frac{1}{M_k}\mathbf{e}_k\otimes\mathbf{e}_k\Big)^{-1}\cdot\sum_{j=1}^{d+1}\frac{\mathbf{R}_j}{M_j}$$

$R_i,\ \mathbf{e}_i$ length, direction, $M_i = \int_0^{R_i}\frac{dx}{E_iA_i}$ axial compliance, V = cell volume pentamode materials

pentamode lattice: statics

pentamode lattice: dynamics

Semi-analytical methods for lattice dynamics



Colquitt et al. *Proc. R. Soc. A,* 2011 and 2013.

- 2D
- Longitudinal and flexural waves
- Effective mass underestimated if flexural waves left out

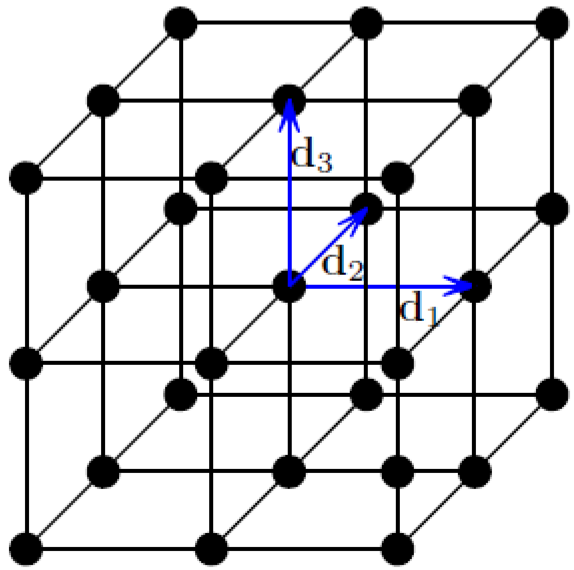
Leamy, *J. Sound. Vib.*, 2012

- Wave based approach, 2D, using reflection & transmission

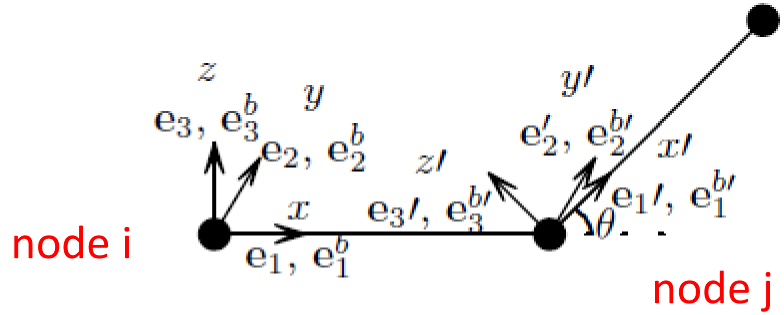
Here – 2D and 3D, L and flex waves

- Consistent method 2D, 3D
- Low frequency asymptotics (2D)
- Correct effective mass

Example: cubic lattice



Lattice dynamics: for each rod



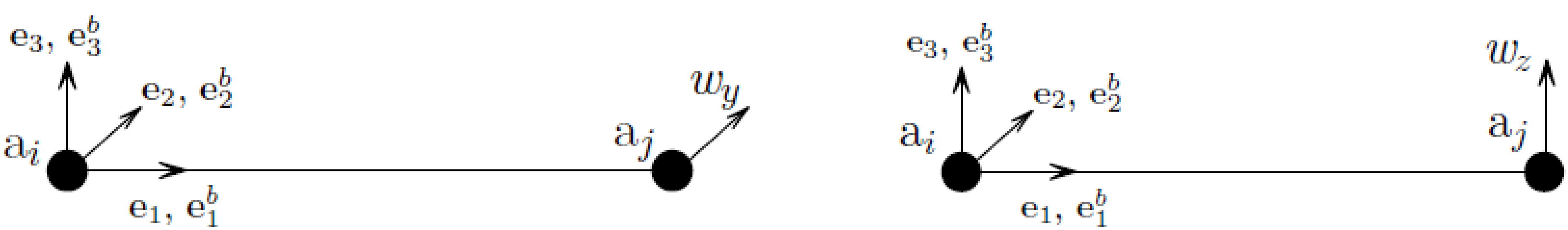
1) **Longitudinal wave equation**

$$\mu_{ij}\frac{\partial^2}{\partial x^2}u_{ij} = -\omega^2 d_{ij}u_{ij}, \quad u_{ij}(0) = \mathbf{e}_1\cdot\mathbf{u}_i, \quad u_{ij}(l_{ij}) = \mathbf{e}_1\cdot\mathbf{u}_j$$

$$u_{ij}(x) = \frac{\mathbf{e}_1\cdot\mathbf{u}_i\sin(s_{ij}\omega(l_{ij}-x)) + \mathbf{e}_1\cdot\mathbf{u}_j\sin(s_{ij}\omega x)}{\sin(s_{ij}\omega l_{ij})}, \quad s_{ij} = \sqrt{\frac{\rho_{ij}}{\mu_{ij}}}$$

2) **Flexural wave equation**

$$\frac{\partial^4 w}{\partial x^4} - \gamma^4 w = 0, \quad x\in[0,l]$$



Bending in orthogonal directions

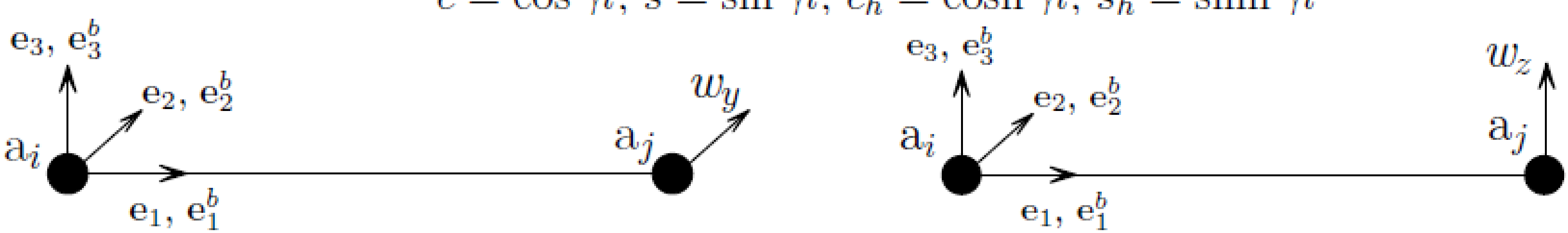
2) **Flexural wave equation**

$$\frac{\partial^4 w}{\partial x^4} - \gamma^4 w = 0, \quad x \in [0, l]$$

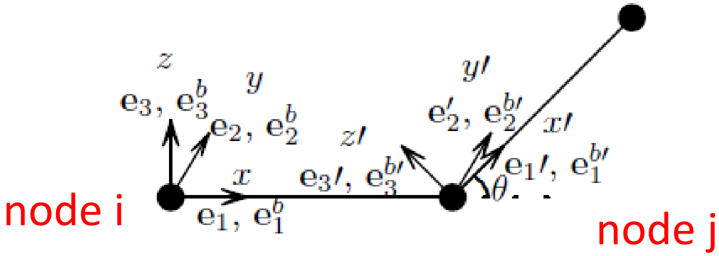
$$\begin{pmatrix} w'''(0) \\ -w''(0) \\ -w'''(l) \\ w''(l) \end{pmatrix} = \mathbf{K}(\omega) \begin{pmatrix} w(0) \\ w'(0) \\ w(l) \\ w'(l) \end{pmatrix}$$

$$\mathbf{K}(\omega) = \frac{1}{1 - cc_h} \begin{pmatrix} \gamma^3(cs_h + sc_h) & \gamma^2 ss_h & -\gamma^3(s + s_h) & \gamma^2(c_h - c) \\ \gamma^2 ss_h & \gamma(sc_h - cs_h) & \gamma^2(c - c_h) & \gamma(s_h - s) \\ -\gamma^3(s + s_h) & \gamma^2(c - c_h) & \gamma^3(cs_h + sc_h) & -\gamma^2 ss_h \\ \gamma^2(c_h - c) & \gamma(s_h - s) & -\gamma^3 ss_h & \gamma(sc_h - cs_h) \end{pmatrix} \quad \mathbf{K} = \begin{pmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_2^T & \mathbf{K}_3 \end{pmatrix}$$

$$c = \cos\gamma l,\ s = \sin\gamma l,\ c_h = \cosh\gamma l,\ s_h = \sinh\gamma l$$



Bending in orthogonal directions



longitudinal   bending

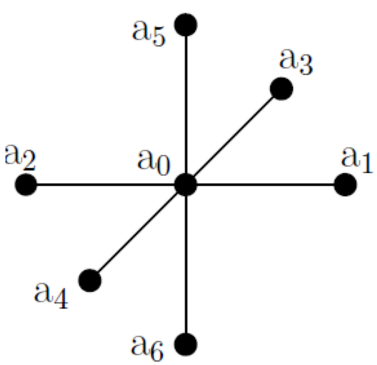
Total force at point i from rod ij: $$\mathbf{f}_{ij} = \mathbf{f}^{(1)}_{ij}(0) + \mathbf{f}^{(2)}_{ij}(0) + \mathbf{f}^{(3)}_{ij}(0)$$

Equilibrium: $$\sum_{j\in\mathcal{N}_i} \mathbf{f}_{ij} = -\omega^2 \mathbf{M}_i \mathbf{u}_i, \quad \mathbf{M}_i = \mathrm{diag}(m_i, m_i, m_i, I_i, I_i, I_i)$$

Floquet conditions

$$\sum_{j\in\mathcal{N}_1} \left(\mathbf{P}^{(2)}_{1j} \exp(i\mathbf{k}\cdot\mathbf{g}_j)\,\mathbf{u}_2 - \mathbf{P}^{(1)}_{1j}\mathbf{u}_1\right) = -\omega^2 \mathbf{M}_1 \mathbf{u}_1$$

$$\sum_{j\in\mathcal{N}_2} \left(\mathbf{P}^{(2)}_{2j} \exp(i\mathbf{k}\cdot\mathbf{g}_j)\,\mathbf{u}_1 - \mathbf{P}^{(1)}_{2j}\mathbf{u}_2\right) = -\omega^2 \mathbf{M}_2 \mathbf{u}_2$$



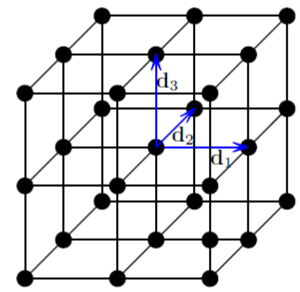
$\mathcal{N}_i$ = set of nodes connected to node i

$$\mathbf{g}_j = \mathbf{a}_j - \mathbf{a}_1, \quad j \in \mathcal{N}_2$$
$$\mathbf{g}_j = \mathbf{a}_j - \mathbf{a}_2, \quad j \in \mathcal{N}_1$$

$$\mathbf{P}^{(1)}_{ij} = \tilde{\mu}_{ij}\tilde{s}_{ij}\cot\tilde{s}_{ij}\mathbf{e}_1\mathbf{e}_1^T + \lambda_{ij}(\mathbf{e}_2,\, \mathbf{e}^b_3)\mathbf{K}_1(\mathbf{e}_2,\, \mathbf{e}^b_3)^T + \lambda_{ij}(\mathbf{e}_3,\, -\mathbf{e}^b_2)\mathbf{K}_1(\mathbf{e}_3,\, -\mathbf{e}^b_2)^T$$
$$\mathbf{P}^{(2)}_{ij} = \tilde{\mu}_{ij}\tilde{s}_{ij}\csc\tilde{s}_{ij}\mathbf{e}_1\mathbf{e}_1^T - \lambda_{ij}(\mathbf{e}_2,\, \mathbf{e}^b_3)\mathbf{K}_2(\mathbf{e}_2,\, \mathbf{e}^b_3)^T - \lambda_{ij}(\mathbf{e}_3,\, -\mathbf{e}^b_2)\mathbf{K}_2(\mathbf{e}_3,\, -\mathbf{e}^b_2)^T$$
$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_2^T & \mathbf{K}_3 \end{pmatrix}$$

equation/dispersion relation

$$\mathbf{H}\mathbf{u} = \omega^2 \mathbf{M}\mathbf{u}$$



$$\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}, \quad \mathbf{M} = \mathrm{diag}(\mathbf{M}_1, \mathbf{M}_2), \quad \mathbf{H} \equiv \mathbf{H}(\omega, \mathbf{k}) = \begin{pmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_2^+ & \mathbf{H}_3 \end{pmatrix}$$

$$\mathbf{H}_1 = \sum_{j \in \mathcal{N}_1} \mathbf{P}_{1j}^{(1)}, \quad \mathbf{H}_2 = -\sum_{j \in \mathcal{N}_1} \mathbf{P}_{1j}^{(2)} \exp(i\mathbf{k} \cdot \mathbf{g}_j), \quad \mathbf{H}_3 = \sum_{j \in \mathcal{N}_1} \mathbf{P}_{1j}^{(3)}$$

equation/dispersion relation

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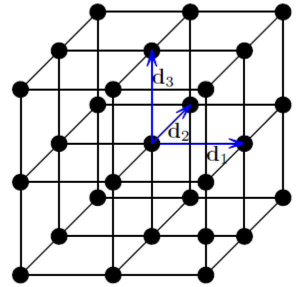

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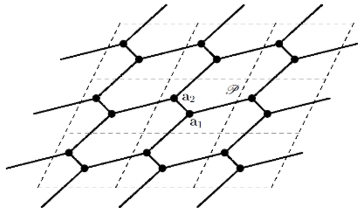
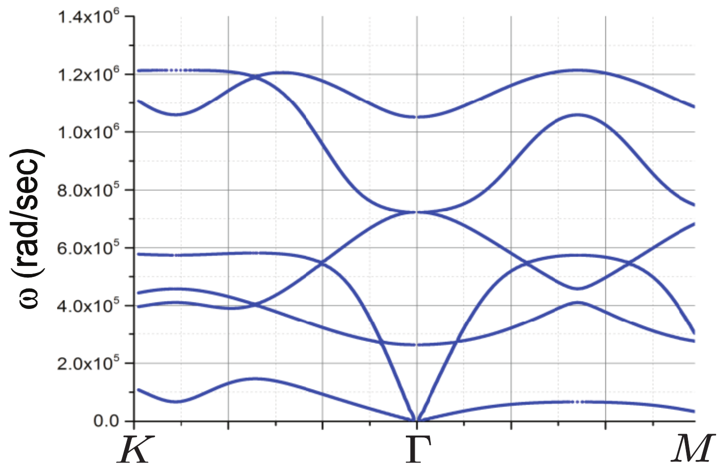
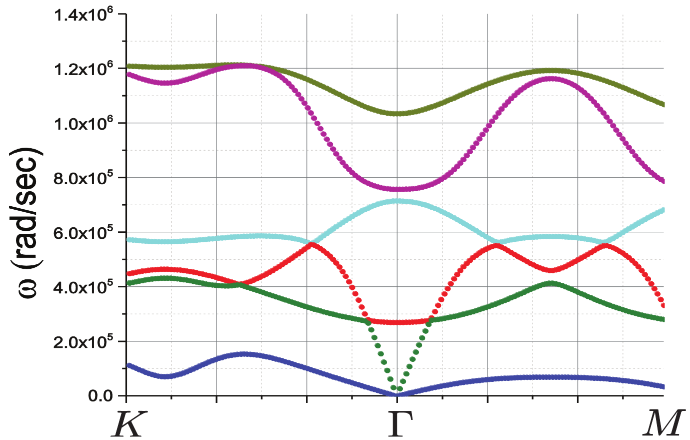
Honeycomb: 2D pentamode





Beam theory



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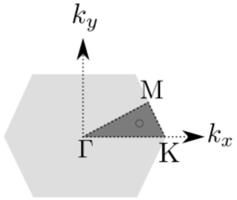
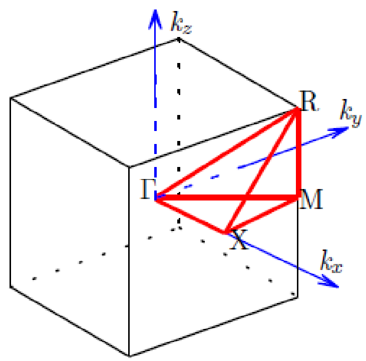
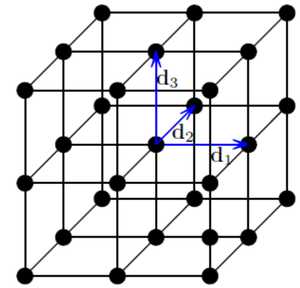
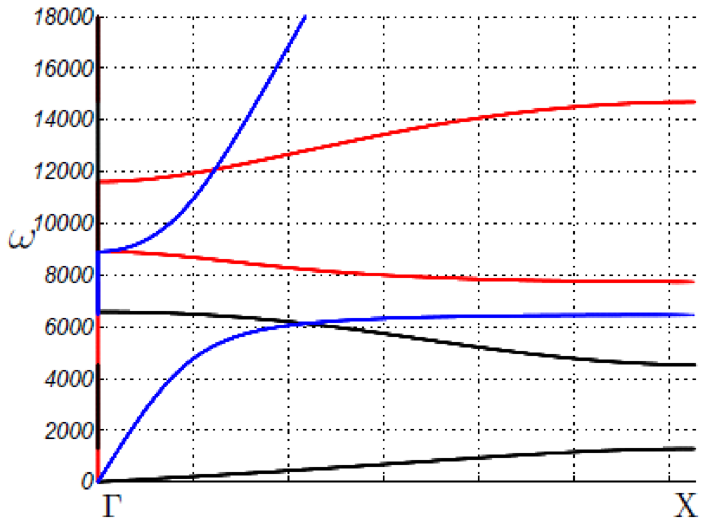


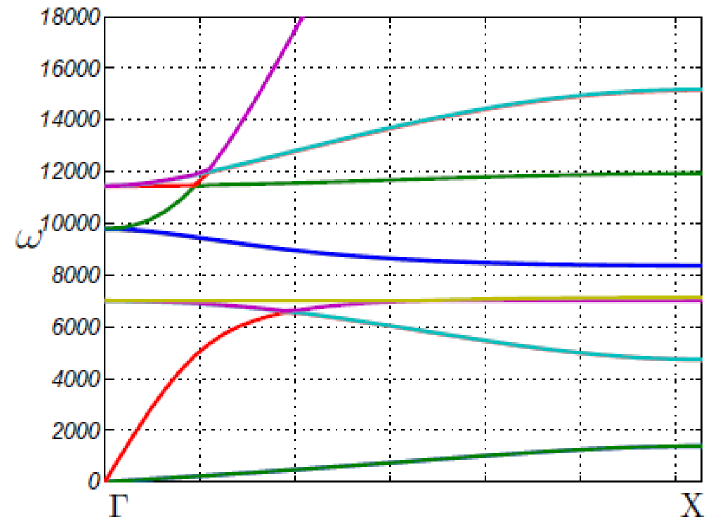
Aluminum beams   length : thickness = 12.5 : 1







Beam theory



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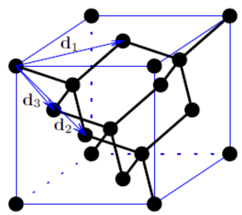
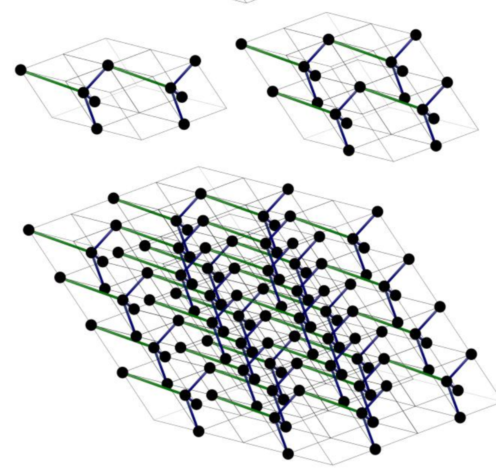
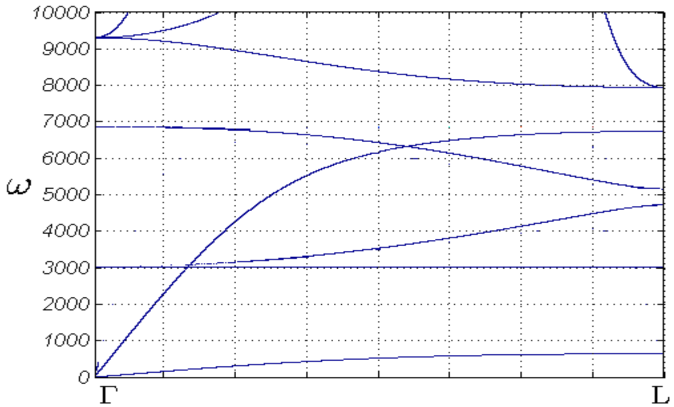
Aluminum beams length : thickness = 20 : 1

Diamond lattice : 3D pentamode

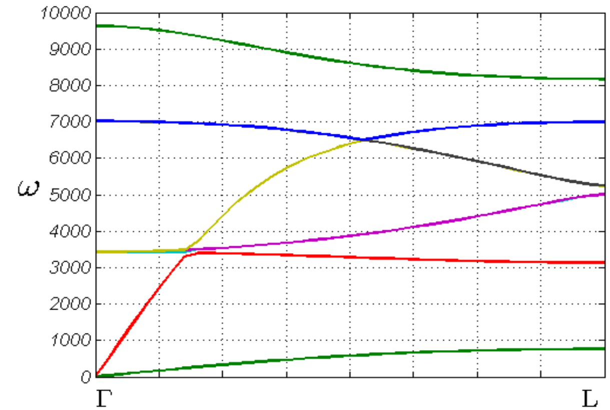




steel rods length : thickness = 20 : 1



Beam theory



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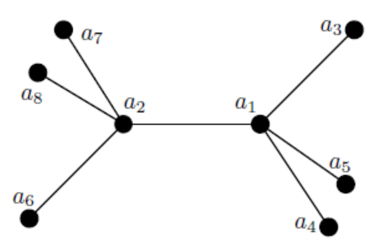
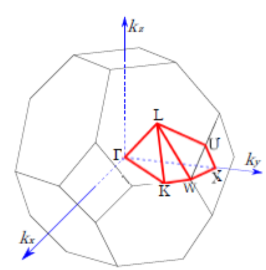
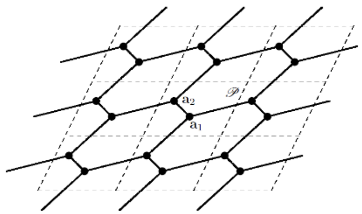


(A) Diamond lattice



(B) IBZ

Low frequency asymptotics : 2D



Gives correct effective mass: $$m = m_1 + m_2 + \sum_{j \in \mathcal{N}_1} \rho_{1j} l_{1j}$$

Gives correct quasistatic wave speeds $$c_T^2 = \frac{3l}{2m}\left(\frac{1}{\mu} + \frac{l^2}{12\lambda}\right)^{-1}, \quad c_L^2 = c_T^2 + \frac{3l}{4m}\mu$$

Challenges

dynamics:

$$\mathbf{H}\mathbf{u} = \omega^2 \mathbf{M}\mathbf{u} \qquad \mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}, \quad \mathbf{M} = \mathrm{diag}(\mathbf{M}_1, \mathbf{M}_2), \quad \mathbf{H} \equiv \mathbf{H}(\omega, \mathbf{k}) = \begin{pmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_2^+ & \mathbf{H}_3 \end{pmatrix}$$

- low frequency asymptotics for 3D
- use as semi-analytical tool, e.g. relate to "dynamics homogenization", Willis equations (Norris et al., *PRSA* 2012)

statics

$$\mathbf{C} = \frac{1}{V} \sum_{i,j=1}^{Z} R_i R_j \sqrt{M_i M_j}\, P_{ij} (\mathbf{v}_i \otimes \mathbf{v}_i) \otimes (\mathbf{v}_j \otimes \mathbf{v}_j) \quad \text{where}$$

$$P_{ij} = \delta_{ij} - \mathbf{v}_i \cdot \Big(\sum_{k=1}^{Z} \mathbf{v}_k \otimes \mathbf{v}_k \Big)^{-1} \cdot \mathbf{v}_j, \quad \mathbf{v}_i \equiv M_i^{-1/2} \mathbf{e}_i$$

- stretch + **bending**
- relate to asymptotics of dynamic model

thanks:

G. Cipolla, N. Gokhale Weidlinger
A. Shuvalov U. Bordeaux 1

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and to you

for listening!