Determination of Acoustic Scattering from a 2D Finite Phononic Crystal

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PHONONIC CRYSTALS

Phononic crystals:

A **phononic crystal** (PC) is a metamaterial (engineered material) with periodic variations in mass or stiffness . Acoustic phononic crystals possess interesting properties such as

1. Dispersive behavior due to multiple scattering effects
2. Frequency and angular band gaps
3. Ability to focus waves.

MOTIVATION

The emergence of fluid phononic crystals requires development of new analysis tools. Many tools employ the Bloch theorem (for infinite periodic media) to describe the acoustic field and give information of the wave field using a single unit cell. Interaction of the infinite PC with an acoustic plane wave yields information applicable to a finite PC.

The effect of the backscattered waves from a **large finite PC** will be explored with:

1. Finite size effects
2. Weak internal disorder

Knowledge of the semi-infinite solution aids in the finite PC solution.

BLOCH WAVE EXPANSION

Consider a semi-infinite PC half-space subject to an incident plane wave. The semi-infinite PC contains a complete set of Bloch wave functions to employ as an expansion. Bloch wave expansion allows one to find the reflected and transmitted (into PC) wave fields.

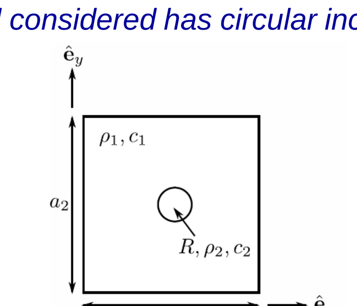
The Bloch theorem: pressure field = periodic function (called a mode) x a plane wave

$$p(\mathbf{x}) = \tilde{p}(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

Benefits of the BWE

1. Valid for **any** frequency and incident angle
2. Arbitrary geometric/material inclusion properties considered via finite element method (FEM)
3. Is significantly faster than direct FEM solution

Unit cell considered has circular inclusion



Incident plane wave

$$p_i(\mathbf{x}) = e^{i\mathbf{k}_i\cdot\mathbf{x}}$$

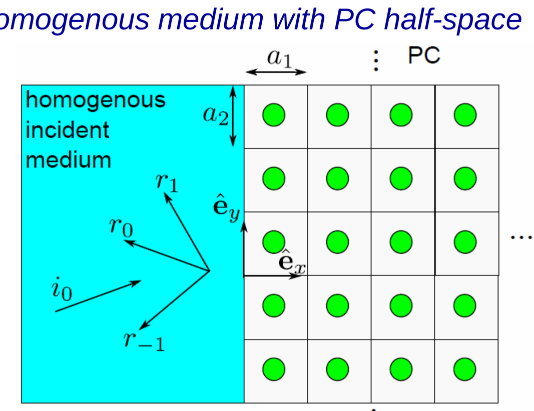
A plane wave expansion for the reflected wave field

$$p_r(\mathbf{x}) = \sum_{m=-N/2}^{N/2} r_m e^{i\mathbf{k}_m\cdot\mathbf{x}}$$

A Bloch wave expansion for the transmitted wave field

$$p_t(\mathbf{x}) = \sum_{n=1}^{N} t_n \tilde{p}_n(\mathbf{x}) e^{i\mathbf{k}_n\cdot\mathbf{x}}$$

Homogeneous medium with PC half-space



BLOCH WAVE EXPANSION

The BWE enables calculation of the reflected/transmitted pressure fields

1. Discretize acoustic wave equation by finite element method (FEM)
2. Invoke the Bloch Theorem
3. Solve eigenvalue problem for dispersion relation and Bloch waves

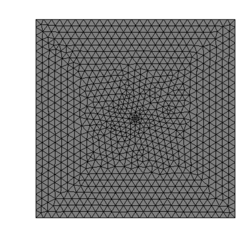
Bloch waves used in expansion satisfy

1. Wavenumber conservation (in *y* direction)
2. Real part of k (within first Brillouin zone): $-\frac{\pi}{a_1} \le \operatorname{Re} k \le \frac{\pi}{a_1}$
3. Imag. part of k (decaying wave): $\operatorname{Im} k \ge 0$
4. Group velocity vector pointing into PC (power transmitted into PC)

Complete solution arrived at by enforcement of boundary conditions: pressure and normal particle velocity

Verification performed via specialized COMSOL finite element model of large semi-infinite PC

Discretized FEM mesh of unit cell



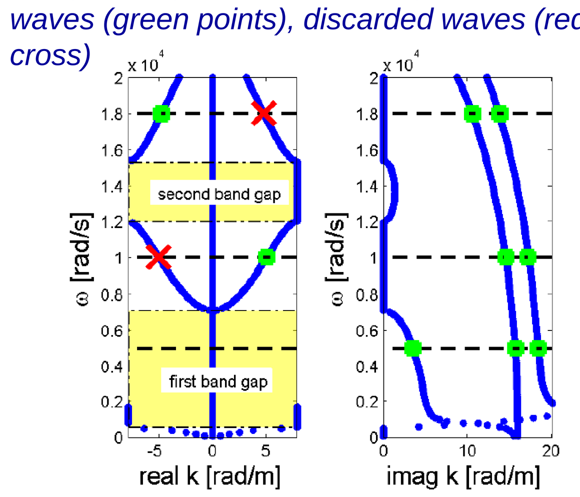
Bloch theorem

$$p(\mathbf{x}) = \tilde{p}(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

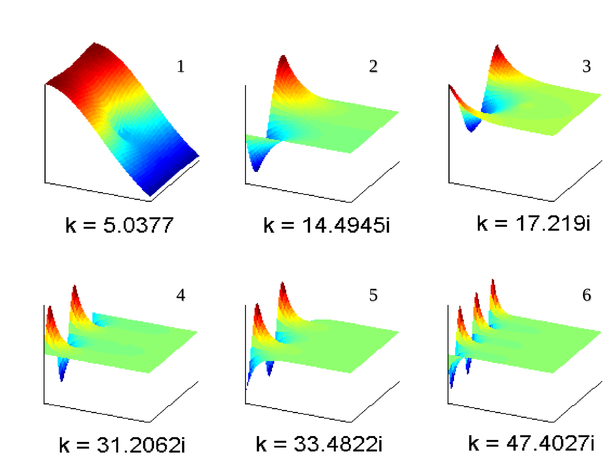
Eigenvalue problem

$$\left[k^2\mathbf{B} + k\mathbf{A}'(\omega) + \mathbf{D}'(\omega)\right]\tilde{\mathbf{p}} = \mathbf{0}$$

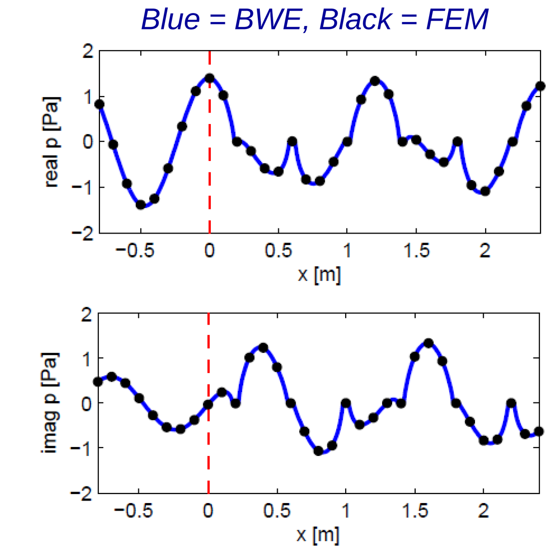
Complex dispersion relationship, excited Bloch waves (green points), discarded waves (red cross)



Six excited Bloch waves



Verification vs. FEM
Blue = BWE, Black = FEM



SCATTERING FROM A FINITE CRYSTAL

Scattered pressure from a finite crystal can be predicted through **Helmholz-Kirchhoff integral** (HKI)

$$p_s(\mathbf{x}_f) = \int_S [G\nabla p - p\nabla G]\cdot\hat{\mathbf{n}}\,dS$$

$$G(\mathbf{x}_s,\mathbf{x}_f) = \frac{i}{4}H_0^{(1)}\left(k_i\left|\mathbf{x}_f - \mathbf{x}_s\right|\right)$$

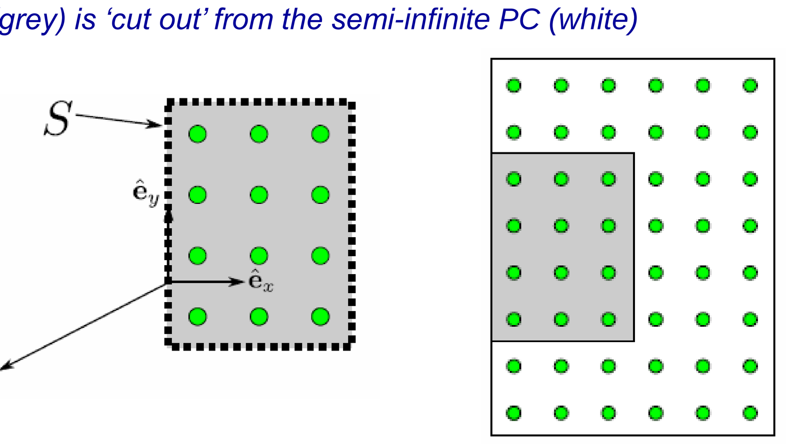
G = free space Green's function

In the small wavelength regime (large PC), assume surface pressure is described by Bloch wave expansion

$$p \to p_{BWE}$$
$$\nabla p \to \nabla p_{BWE}$$

The BWE informs the HKI for integral computations

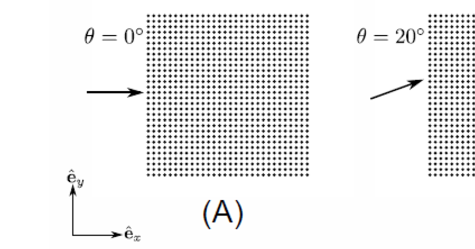
Finite PC (grey) is 'cut out' from the semi-infinite PC (white)



1.	**Finite PC of interest**
2.	**Compute BWE for semi-infinite PC half-space**
3.	**Compute pressure on surface S**
4.	**Compute scattered field using HKI**

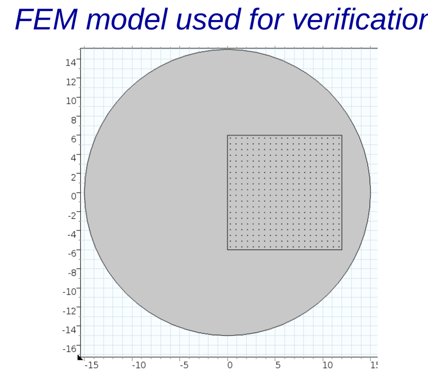
RESULTS

Verification of BWE and HKI done vs. direct FEM



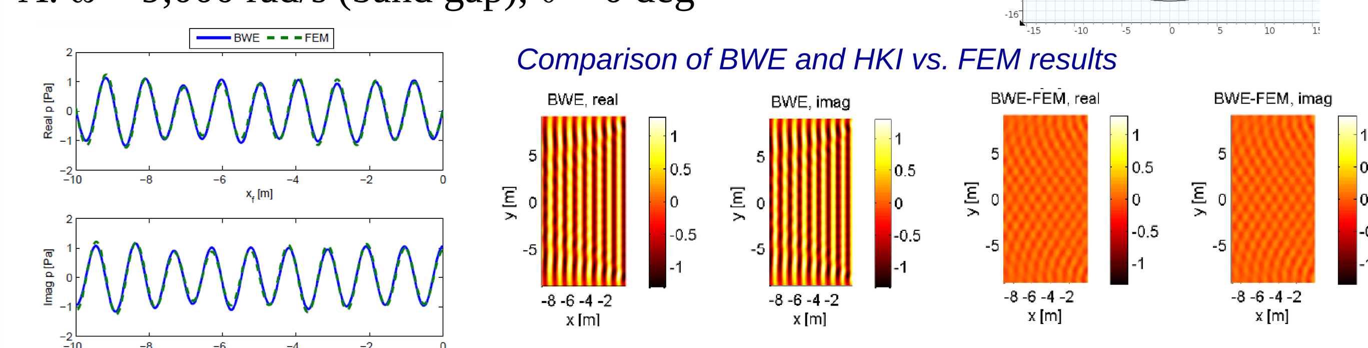
Two finite PCs considered, 900 cells

FEM model used for verification



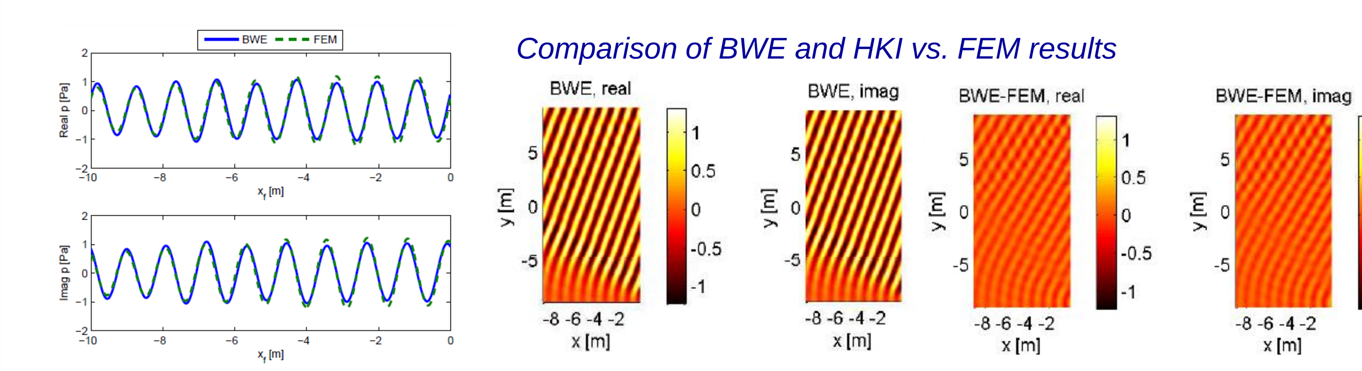
A. ω = 9,000 rad/s (band gap), θ = 0 deg

Comparison of BWE and HKI vs. FEM results



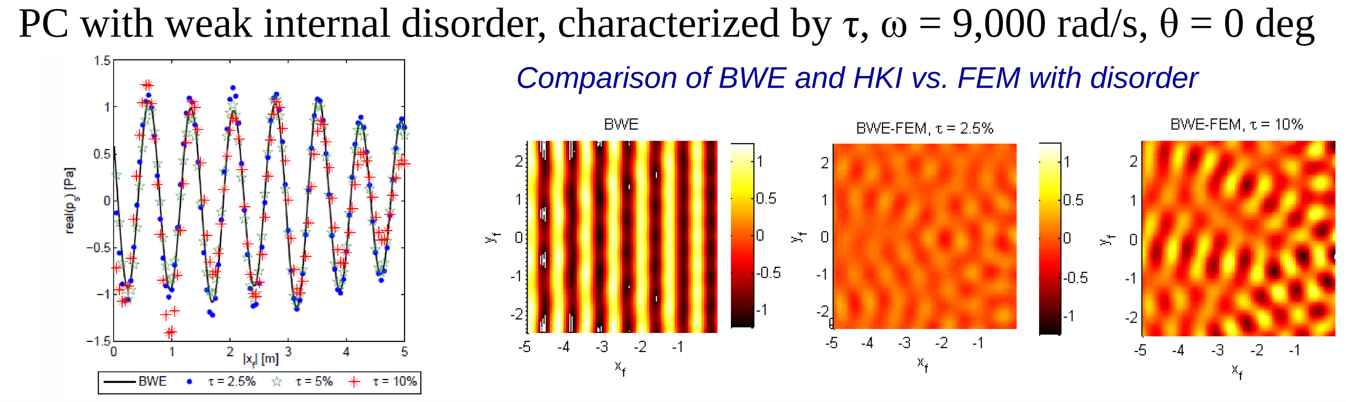
B. ω = 9,000 rad/s (band gap), θ = 20 deg

Comparison of BWE and HKI vs. FEM results



PC with weak internal disorder, characterized by τ, ω = 9,000 rad/s, θ = 0 deg

Comparison of BWE and HKI vs. FEM with disorder



CONCLUSIONS

Analytical and numerical tools have been introduced to further study the interaction of acoustic waves and a finite phononic crystal

Specifically:

1. Bloch wave expansion introduced for study of semi-infinite media reflection/transmission
2. HKI theorem, using BWE, applied to study scattering from large finite PCs
3. Results indicate good agreement for scattering from real, disordered PCs

Next steps include:

1. Effects of PC geometry, including voids, on scattering
2. At frequencies not in a band gap, Bloch waves reflect from far end of the PC. Do these waves PC play a significant role in scattering?

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