Amplitude-Dependent Wave Devices Based on Nonlinear Periodic Materials

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Contents

- Motivate study of nonlinear periodic structures
- Detail a perturbation approach for a set of infinite nonlinear difference equations
  - First order dispersion correction
- Present results for 1D and 2D lattices, to include potential devices based on nonlinear response
- Discuss wave-wave interactions and further device implications
- Present nonlinear string experiment
- Conclude with final thoughts on needed research

Nonlinear Periodic Structures

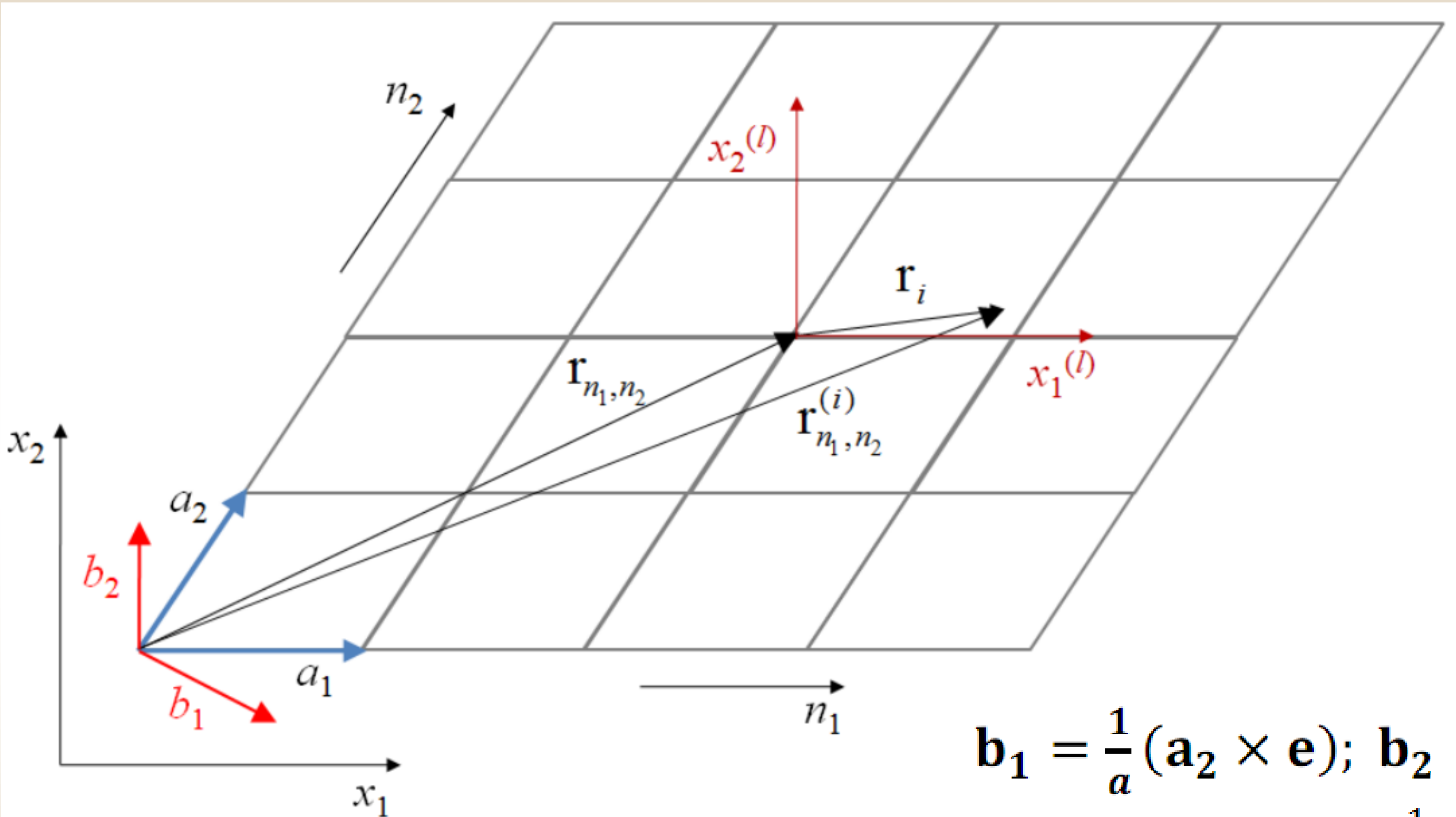
- Nonlinear periodic structures exhibit additional unique wave properties
  - Existence of highly stable localized solutions[1] even without defects
  - Solitary waves and solitons[2,3]
  - Variations in wave speeds and propagation direction related to wave amplitude and nonlinearity
- Our interest is in tunable phononic devices (frequency isolators, filters, logic ports, resonators, etc…)
- Most nonlinear analysis of discrete systems begins with a long wavelength approximation and then posing of an equivalent continuous system

[1] Vakakis A.F., King M.E., Pearlstein A.J., 1994, *Forced Localization in a periodic chain of nonlinear oscillators*, International Journal of Non-Linear Mechanics, *Vol.*29(3), pp. 429-447.
[2] Daraio C., Nesterenko V.F., Herbold E.B., Jin S., 2006,*Tunability of solitary wave properties in one-dimensional strongly nonlinear phononic crystals*, Physical Review, E 73, 026610.
[3] R.K. Bullough, P.J. Caudrey, *Solitons*, Springer, Berlin 1980.

Perturbation Approach

- Analytical treatment for weakly nonlinear media
- Treats the infinite, discrete system without reverting to the long wavelength limit
- Amounts to a Lindstedt Poincare' approach combined with Bloch Analysis
- A Multiple Scales perturbation approach is employed for wave-wave interactions

General Approach



$$\mathbf{r}_{n_1,n_2} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2$$

$$\mathbf{b}_1 = \frac{1}{a}(\mathbf{a}_2 \times \mathbf{e});\ \mathbf{b}_2 = \frac{1}{a}(\mathbf{e} \times \mathbf{a}_1)$$

where, $\mathbf{e} = \frac{1}{a}(\mathbf{a}_1 \times \mathbf{a}_2)$

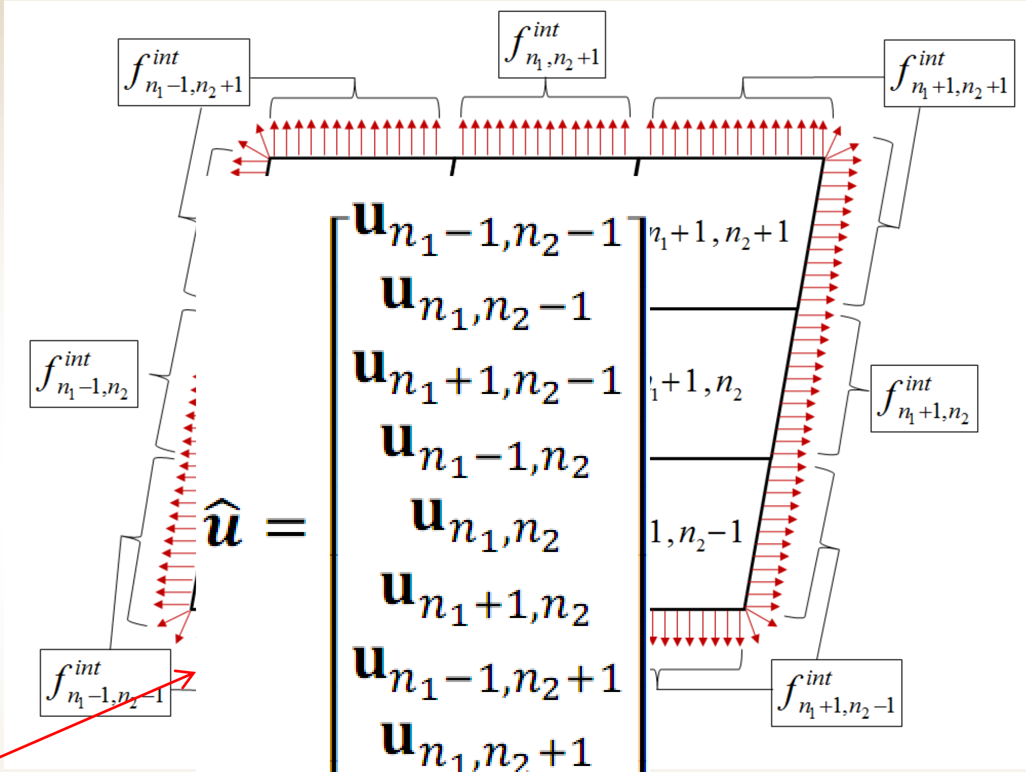
$$\mathbf{a}_i \cdot \mathbf{b}_j = \delta_{ij},\quad i,j = 1,2,$$

Unit Cell EOM

- Dynamic behavior is governed by,

$$\mathbf{u}_{n_1,n_2} = [u_1\ u_2\ u_3\ \ldots\ u_{N-1}\ u_N]^{\mathrm{T}}$$

N – denotes number of degrees of freedom for a unit cell



$$\hat{\boldsymbol{u}} = \begin{bmatrix} \mathbf{u}_{n_1-1,n_2-1} \\ \mathbf{u}_{n_1,n_2-1} \\ \mathbf{u}_{n_1+1,n_2-1} \\ \mathbf{u}_{n_1-1,n_2} \\ \mathbf{u}_{n_1,n_2} \\ \mathbf{u}_{n_1+1,n_2} \\ \mathbf{u}_{n_1-1,n_2+1} \\ \mathbf{u}_{n_1,n_2+1} \\ \mathbf{u}_{n_1+1,n_2+1} \end{bmatrix}$$

- For the 9 cell assembly,

$$\hat{\mathbf{M}}\frac{d^2\hat{\mathbf{u}}}{dt^2} + \hat{\mathbf{K}}\hat{\boldsymbol{u}} + \varepsilon \boldsymbol{f}_{NL}(\hat{\boldsymbol{u}}) = \hat{\boldsymbol{f}}^{int} + \hat{\boldsymbol{f}}^{ext}$$

Unit Cell EOM

- Equations of motion for the unit cell are extracted from the previous equation expressed for 9-cell assembly,

$$\omega^2 \sum_{\mathbf{p,q}=-1,0,1} \mathbf{M}^{(p,q)} \frac{d^2 \mathbf{u}_{n_1+p,n_2+q}}{d\tau^2} + \sum_{\mathbf{p,q}=-1,0,1} \mathbf{K}^{(p,q)} \mathbf{u}_{n_1+p,n_2+q} + \varepsilon \boldsymbol{f}_{NL}\left(\mathbf{u}_{n_1\pm p,n_2\pm q}\right) = \boldsymbol{f}^{ext}_{n_1,n_2}(\tau)$$

- Weakly nonlinear model is governed by,

$$\omega^2 \mathbf{M} \frac{d^2 \mathbf{u}_{n_1,n_2}}{d\tau^2} + \left[\sum_{p,q=-1,0,1} \mathbf{K}^{(p,q)} \mathbf{u}_{n_1+p,n_2+q}\right] + \varepsilon \mathbf{f}_{NL}\left(\mathbf{u}_{n_1\pm p,n_2\pm q}\right) = \boldsymbol{f}^{ext}_{n_1,n_2}(\tau).$$

small parameter

Lumped mass

=0

Nonlinear force interactions

- Free wave propagation is analyzed by setting the external forcing to zero

Asymptotic Expansions

- Asymptotic expansion of frequency and displacement,

$$\mathbf{u}_{n_1,n_2} = \mathbf{u}^{(0)}_{n_1,n_2} + \varepsilon \mathbf{u}^{(1)}_{n_1,n_2} + O(\varepsilon^2)$$

$$\omega = \omega_0 + \varepsilon\omega_1 + O(\varepsilon^2)$$

- Substituting the above expansions leads to ordered equations,

$$\varepsilon^0: \quad \omega_0^2 \mathbf{M} \frac{d^2\mathbf{u}^{(0)}_{n_1,n_2}}{d\tau^2} + \sum_{p,q=-1}^{+1} \mathbf{K}^{(p,q)} \mathbf{u}^{(0)}_{n_1+p,n_2+q} = \mathbf{0}$$

$$\varepsilon^1: \quad \omega_0^2 \mathbf{M} \frac{d^2\mathbf{u}^{(1)}_{n_1,n_2}}{d\tau^2} + \sum_{p,q=-1}^{+1} \mathbf{K}^{(p,q)} \mathbf{u}^{(1)}_{n_1+p,n_2+q} = -2\omega_0\omega_1 \mathbf{M} \frac{d^2\mathbf{u}^{(0)}_{n_1,n_2}}{d\tau^2} - \boldsymbol{f}_{NL}\left(\mathbf{u}^{(0)}_{n_1,n_2}, \mathbf{u}^{(0)}_{n_1\pm p,n_2\pm q}\right),$$

- 0^{th} order equation can be solved for Bloch waves

Solution Approach

- Zeroth order solution is obtained using Bloch wave assumption

$$\varepsilon^0: \quad \omega_0^2 \mathbf{M} \frac{d^2 \mathbf{u}_{n_1,n_2}^{(0)}}{d\tau^2} + \sum_{p,q=-1}^{+1} \mathbf{K}^{(p,q)} \mathbf{u}_{n_1+p,n_2+q}^{(0)} = \mathbf{0}$$

- Bloch wave theorem is imposed by assuming the following displacement expression,

$$\mathbf{u}_{n_1,n_2}(\tau) = \mathbf{u}_0 e^{i\,\mathbf{k}\cdot\mathbf{r}_{n_1,n_2}} e^{i\tau}$$

$$\mathbf{u}_{n_1 \pm p,\, n_2 \pm q}(\tau) = \mathbf{u}_{n_1,n_2}(\tau) e^{i\,(\pm p\mu_1 \pm q\mu_2)} \text{- } \mu_2 n_2$$

- Substituting above into the zeroth order equation,

$$\omega_0^2 \mathbf{M} \frac{d^2 \mathbf{u}_{n_1,n_2}^{(0)}(\tau)}{d\tau^2} + \left[\sum_{p,q=-1}^{+1} \mathbf{K}^{(p,q)} e^{i\,(\pm p\mu_1 \pm q\mu_2)}\right] \mathbf{u}_{n_1,n_2}^{(0)}(\tau) = \mathbf{0}$$

leads to, $\left[-\omega_0^2 \mathbf{M} + \widetilde{\mathbf{K}}(\mathbf{k})\right] \mathbf{u}_0(\mathbf{k}) = \mathbf{0}$ ← Eigenvalue problem

1st Order Correction

$$\varepsilon^1: \quad \omega_0^2 \mathbf{M} \frac{d^2 \mathbf{u}_{n_1,n_2}^{(1)}}{d\tau^2} + \sum_{p,q=-1}^{+1} \mathbf{K}^{(p,q)} \mathbf{u}_{n_1+p,n_2+q}^{(1)} = -2\omega_0\omega_1 \mathbf{M} \frac{d^2 \mathbf{u}_{n_1,n_2}^{(0)}}{d\tau^2} - \boldsymbol{f}_{NL}\left(\mathbf{u}_{n_1,n_2}^{(0)}, \mathbf{u}_{n_1\pm p,n_2\pm q}^{(0)}\right)$$

Reference unit cell, $(n_1, n_2) = (0,0)$

$$\mathbf{u}^{(0)}(\tau) = \frac{A_0}{2} \mathbf{u}_{0,j}(\mathbf{k}) e^{i\tau} + c.c.$$

normalized wave mode

Can be easily seen that nonlinear force is periodic in τ

$$\boldsymbol{f}_{NL}\left(\mathbf{u}^{(0)}(\tau), \mathbf{u}_{p,q}^{(0)}(\tau)\right) = \boldsymbol{f}_{NL}\left(\mathbf{u}^{(0)}(\tau+2\pi), \mathbf{u}_{p,q}^{(0)}(\tau+2\pi)\right)$$

Therefore, the RHS of ε^1 order equation with $e^{i\tau}$ dependence is

$$\omega_{0,j}^2 \mathbf{M} \frac{d^2 \mathbf{u}^{(1)}}{d\tau^2} + \sum_{p,q=-1}^{+1} \mathbf{K}^{(p,q)} \mathbf{u}_{p,q}^{(1)} = \left[\omega_{0,j}\omega_1 A_0 \mathbf{M} \mathbf{u}_{0,j}(\mathbf{k}) - \boldsymbol{c}_1(A_0)\right] e^{i\tau}$$

For j^{th} mode, $\omega_{0,j}^2 \mathbf{M} \frac{d^2 \mathbf{u}^{(1)}}{d\tau^2} + \sum_{p,q=-1}^{+1} \mathbf{K}^{(p,q)} \mathbf{u}_{p,q}^{(1)} = \boldsymbol{f}_j e^{i\tau}$

1st Order Correction

Solvability condition for the j^{th} mode

$$\mathbf{u}_{0,j}^{\mathrm{H}}\boldsymbol{f}_j = 0$$

Finally, the first order correction to frequency for any j^{th} mode :

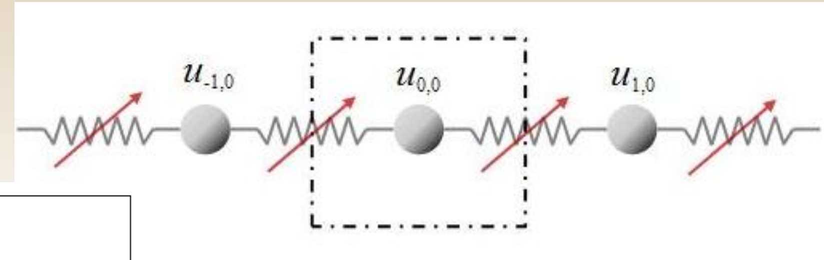
$$\omega_{1,j}(A_0, \mathbf{k}) = \frac{\mathbf{u}_{0,j}^{\mathrm{H}}(\mathrm{k})\mathbf{c}_1(A_0)}{\omega_{0,j}A_0\mathbf{u}_{0,j}^{\mathrm{H}}(\mathbf{k})\mathbf{M}\mathbf{u}_{0,j}(\mathbf{k})}$$

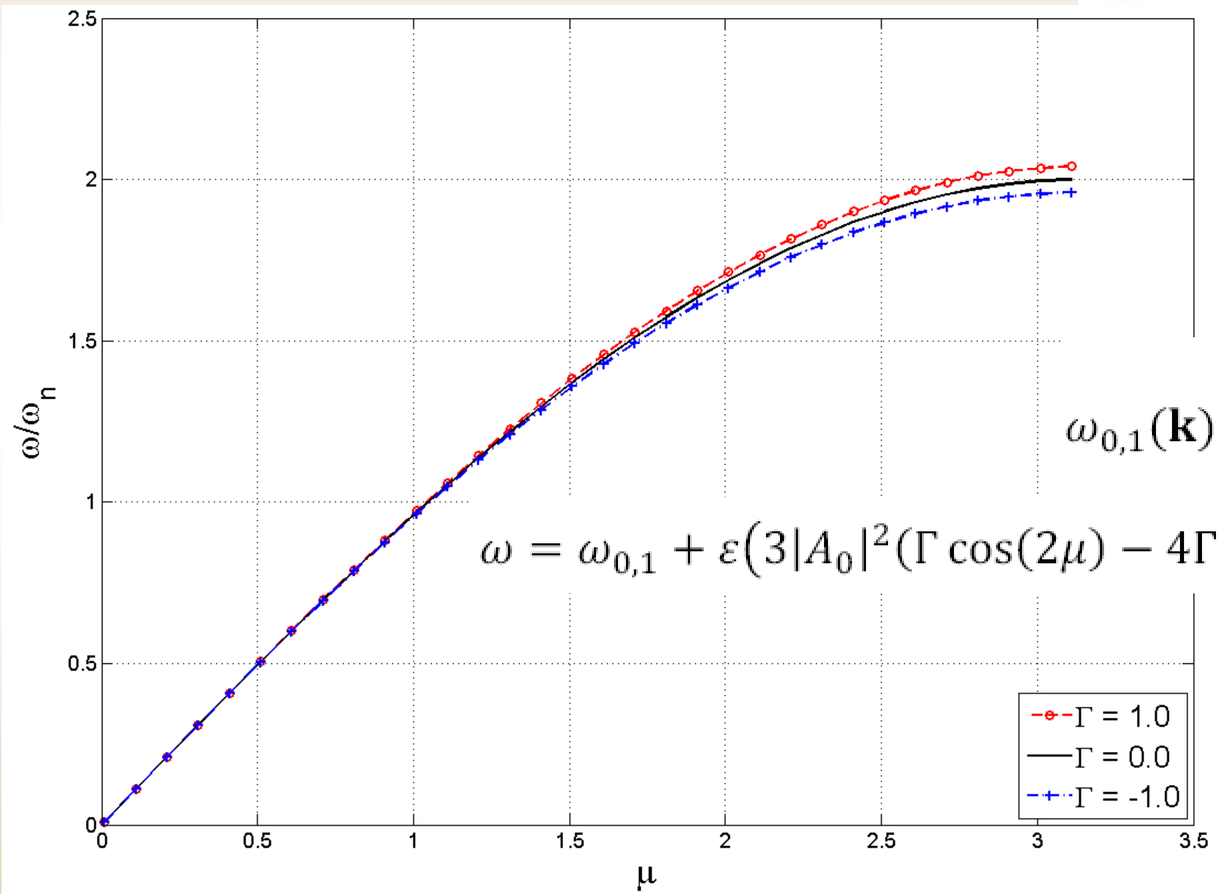
$$\omega_j = \omega_{0,j} + \varepsilon\omega_{1,j}(A_0, \mathbf{k}) + O(\varepsilon^2)$$

Monatomic Chain

Nonlinear force interaction can be described by:

$$f = k\delta + \Gamma\delta^3$$

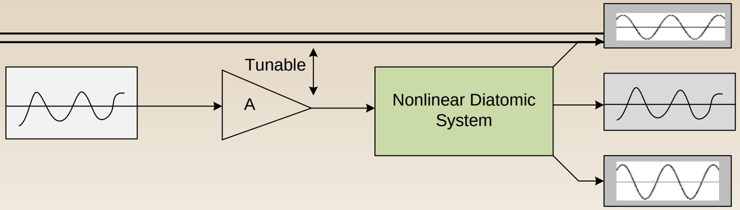


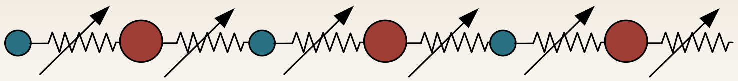


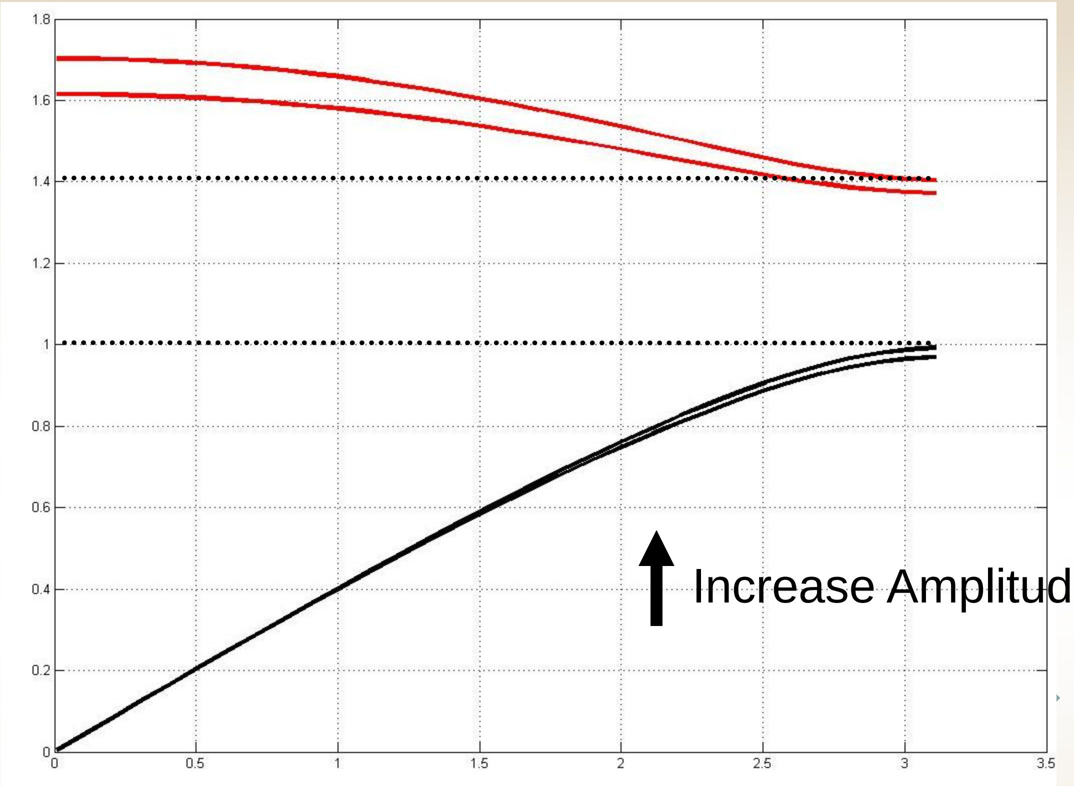
$$\omega_{0,1}(\mathbf{k}) = \sqrt{2k(1-\cos(\mu))/m}$$

$$\omega = \omega_{0,1} + \varepsilon\left(3|A_0|^2(\Gamma\cos(2\mu) - 4\Gamma\cos(\mu) + 3\,\Gamma)/4m\omega_{0,1}\right) + O(\varepsilon^2).$$

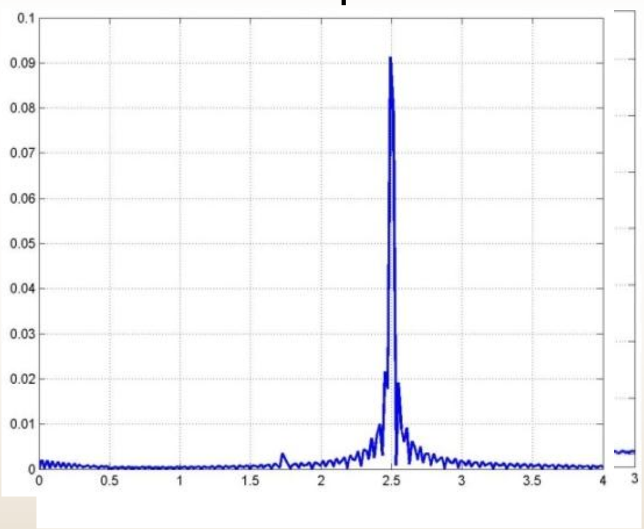
Nonlinear Device



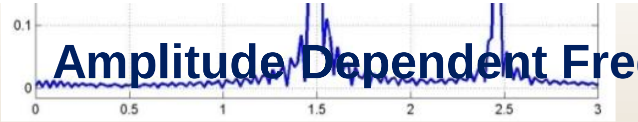




Output



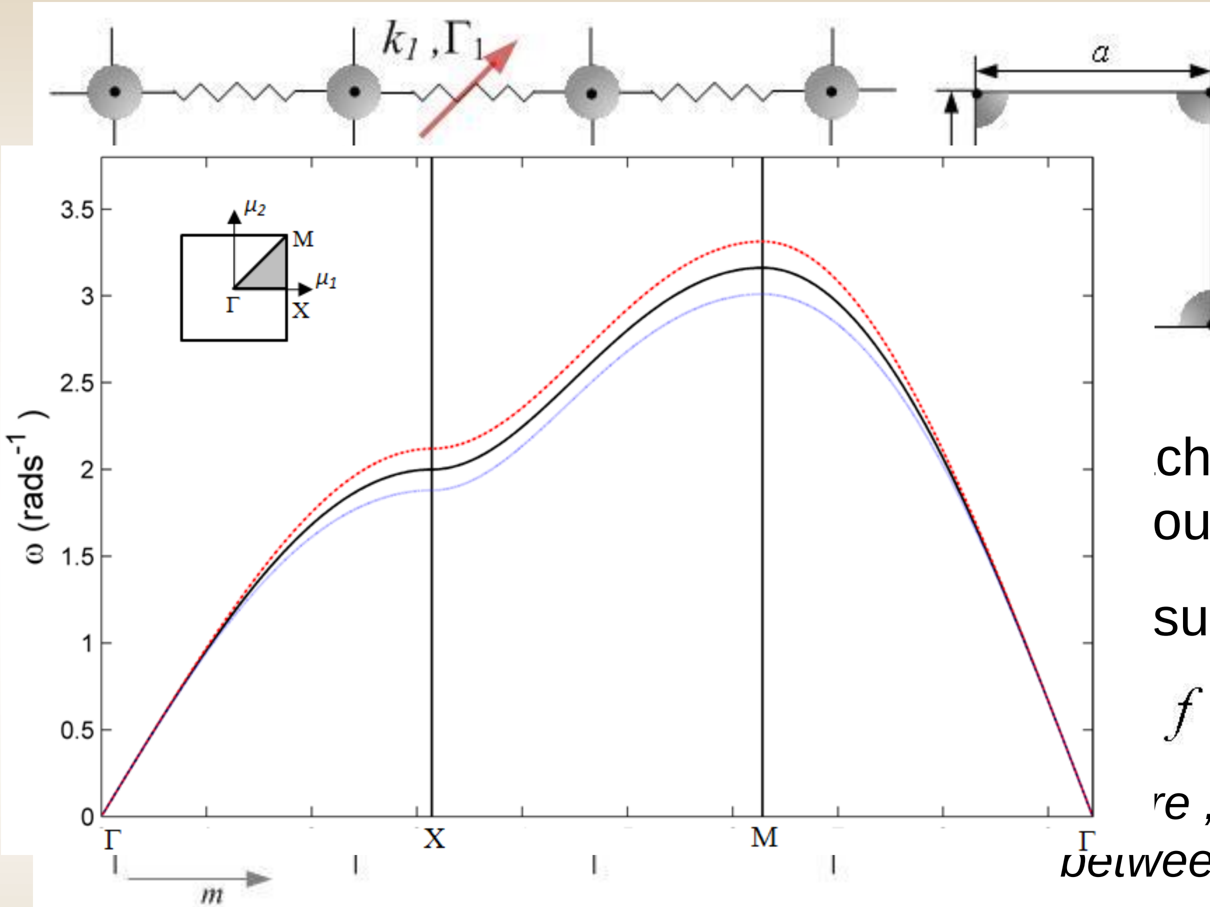
Low Gain



Amplitude Dependent Frequency Isolator

- Dispersion in one-dimensional nonlinear periodic chains

Square Monatomic Lattice



ch mass is connected to 4 ounding masses

sumed force interaction,

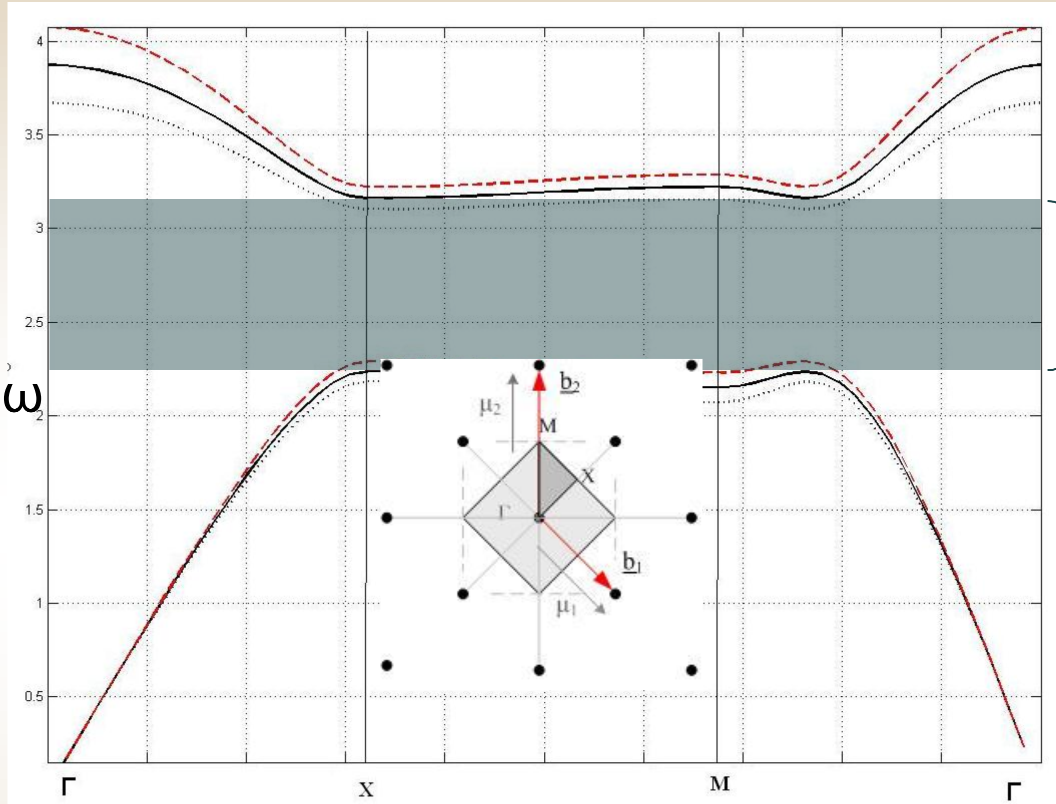
$$f = k\delta + \Gamma\delta^3$$

re , δ is relative displacement between two masses

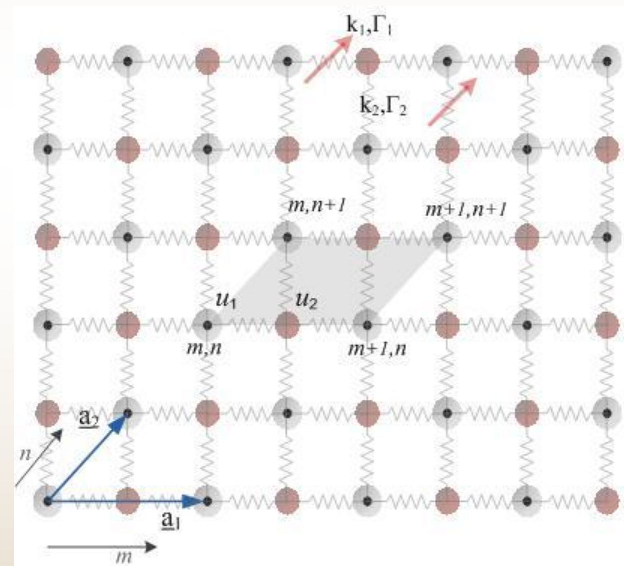
Stiffness parameters k_1 =1.0 N/m, k_2 =1.5 N/m, A_0 = 2.0.

- - - - $\Gamma_1 = \Gamma_2$ = +1.0, —— Linear ($\Gamma_1 = \Gamma_2 = 0$), ······· $\Gamma_1 = \Gamma_2$ = -1.0

Diatomic Lattice



Amplitude Dependent Band Gap



$\beta = m_1 / m_2 = 2$, Stiffness parameters: $k_1 = 1.0\ Nm^{-1}$, $k_2 = 1.5\ Nm^{-1}$, $A_0 = 2.0$

- - - - $\Gamma_1 = \Gamma_2 = +1.0$ (hard), ——— Linear ($\Gamma_1 = \Gamma_2 = 0$),

·········· $\Gamma_1 = \Gamma_2 = -1.0$ (soft)

Group Velocity Plots

• Group velocity defines energy flow as wave propagates

$$\boldsymbol{c}_g = \nabla\omega(\mathbf{k})$$

Gradient of dispersion relation

From nonlinear dispersion, we know that

$$\omega_j = \omega_{0,j} + \varepsilon\omega_{1,j}\left(\left|A_{0,j}\right|, \mathbf{k}\right) + O(\varepsilon^2)$$

Hence,

$$\mathbf{c}_{g,j}\left(\mathbf{k}, \left|A_{0,j}\right|\right) = \nabla\omega_{0,j}(\mathbf{k}) + \varepsilon\,\nabla\omega_{1,j}(\mathbf{k}, \left|A_{0,j}\right|) + O(\varepsilon^2)$$

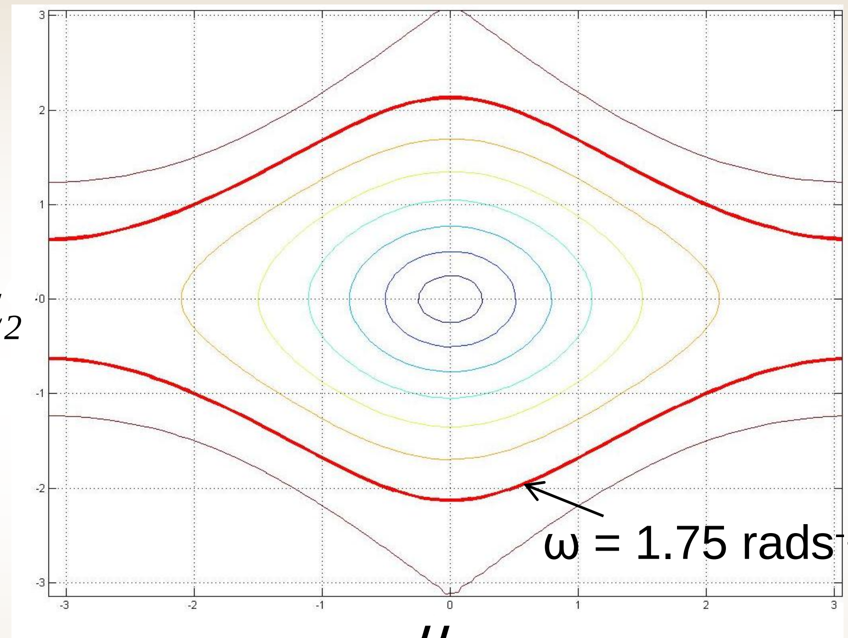
• Group velocity contours are also amplitude dependent

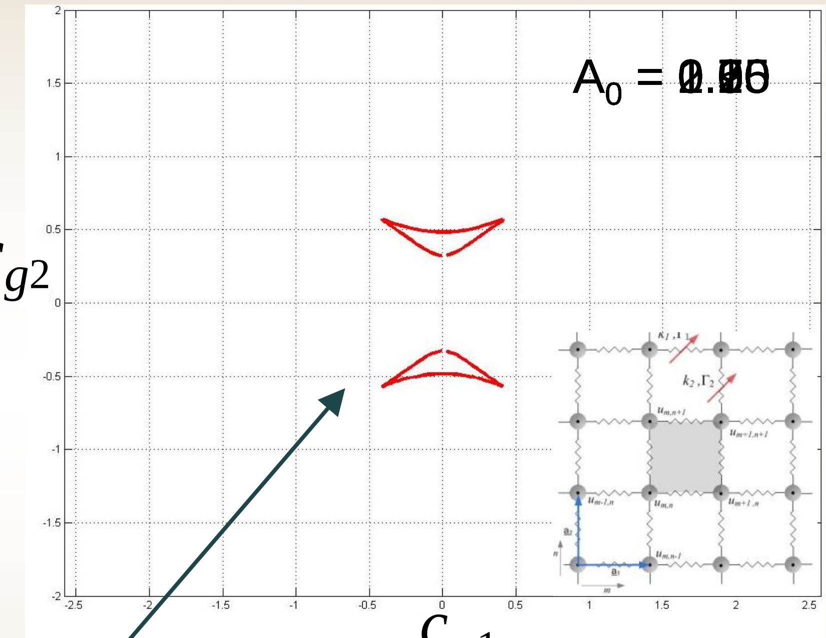
• Useful for predicting energy flow in nonlinear structures

Amplitude-Dependent c_g

$$\omega = f(A, \mathbf{k})$$

$$\boldsymbol{c}_g = \nabla\omega(\mathbf{k})$$





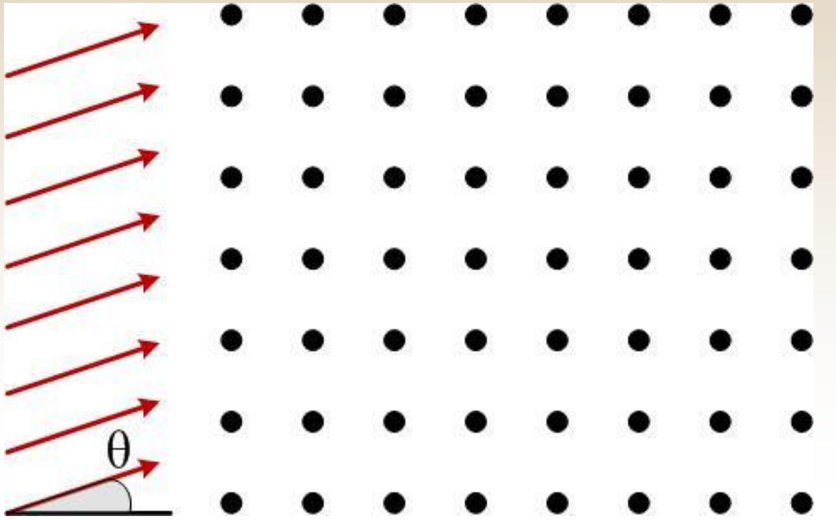
$$\mathbf{k} = \mu_1\mathbf{b_1} + \mu_2\mathbf{b_2}$$

Wave is impeded along $\underline{a}_1$ axis

Numerical Validation

- Imposing the displacement on the left boundary at frequency ω_0 and phase shift
- The phase shift determines the angle at which wave is injected θ and also the wavenumber along a_2 axis

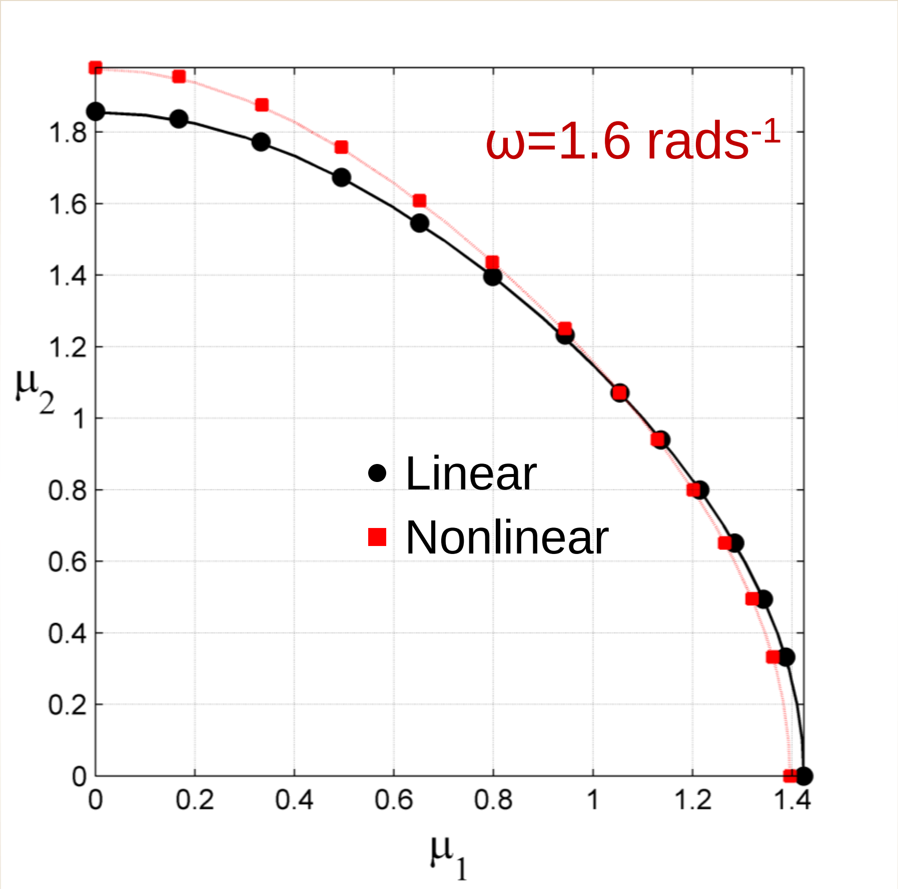
Single frequency denotes particular iso-frequency contour on dispersion surface



A plane wave is injected into a finite spring-mass lattice at incident angle θ

- Numerical integration of equations of motion
- From the response, the propagation constants are computed using FFTs in space
- θ is varied from 0 to $\pi/2$ to determine iso-frequency contour in one quadrant

Monatomic Lattice Results



m = 1, k_1 = 1.5 Nm^{-1}, k_2 = 1.0 Nm^{-1},

Γ_1 = +1.0 Nm^{-3}, Γ_2 = -1.0 Nm^{-3},

k – Linear Stiffness
Γ – Nonlinear Stiffness

——— A_0 = 0.1 (Perturbation Analysis), ● A_0 = 0.1 (Numerical Estimation),

.... A_0 = 2.0 (Perturbation Analysis), ■ A_0 = 2.0 (Numerical Estimation)

Monatomic Lattice Results

$m = 1$, $k_1 = 1.5\ \mathrm{Nm^{-1}}$, $k_2 = 1.0\ \mathrm{Nm^{-1}}$,

$\Gamma_1 = +1.0\ \mathrm{Nm^{-3}}$, $\Gamma_2 = -1.0\ \mathrm{Nm^{-3}}$

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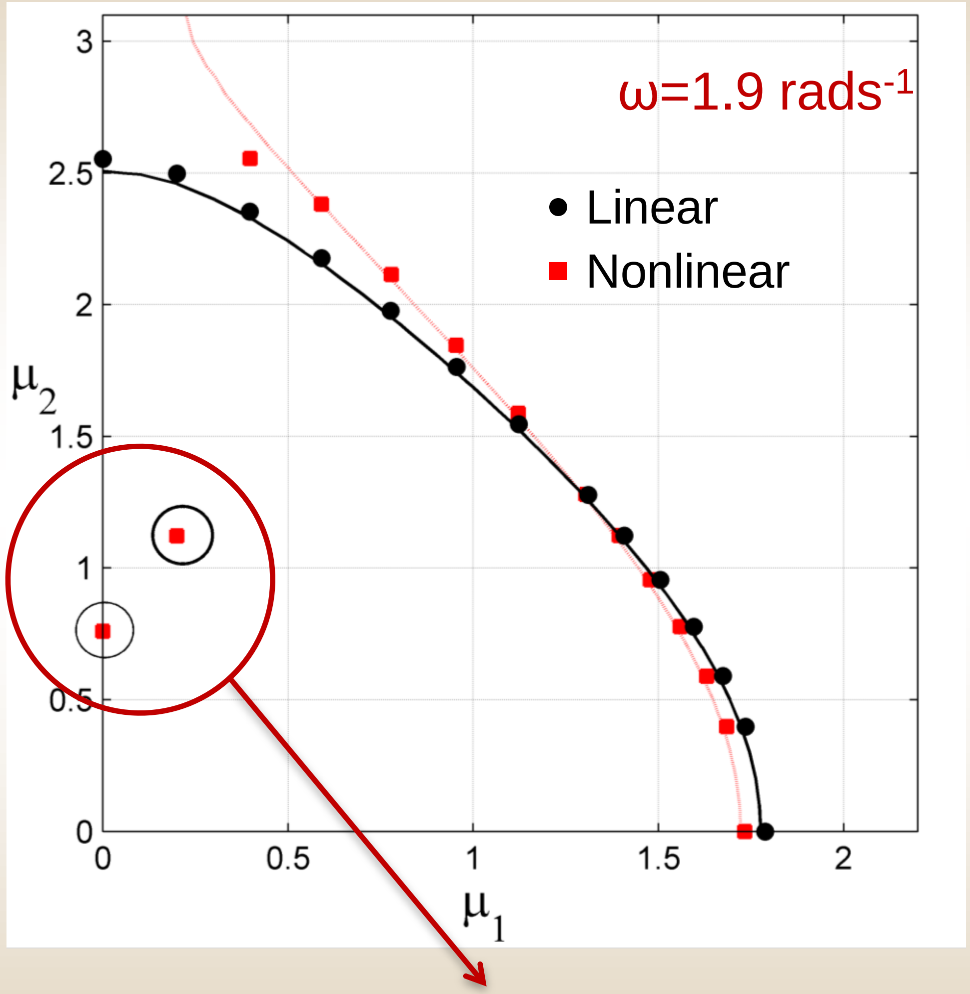
■ $A_0 = 2.0$ (Numerical Estimation)



Outliers indicate evanescent waves

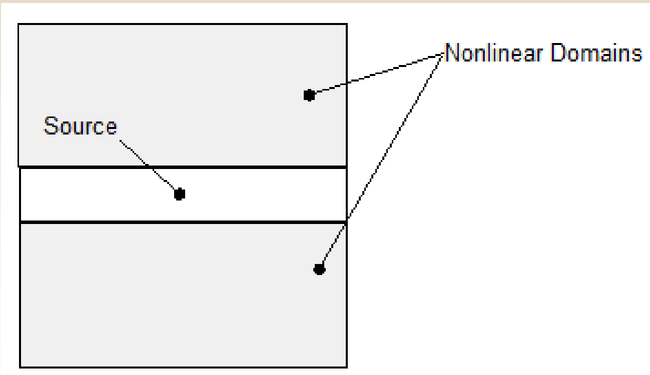
Monatomic Lattice Results

Low Amplitude

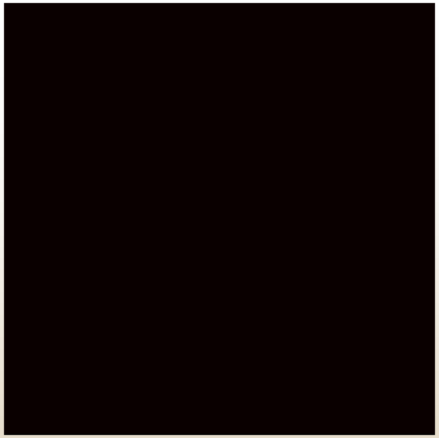
High Amplitude

- Point harmonic forcing in mono-atomic lattice generates spherical wave front
- Quasi-symmetric linear stiffness but asymmetric in nonlinear stiffness
- Asymmetric nonlinear stiffness generates “dead zone” along $\underline{a}_1$ axis with amplitude increase

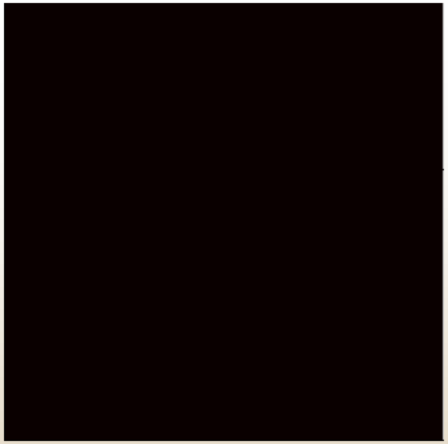
Nonlinearly Activated Waveguide



- “Low” Amplitude vs. “High” Amplitude

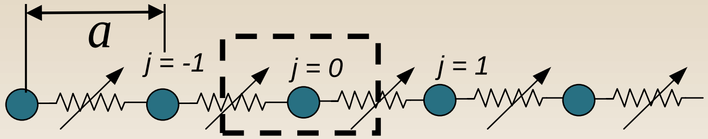


Low-Amplitude Excitation



High-Amplitude Excitation

Wave-Wave Interactions



- Two waves (A and B) introduced
- Results in additional term due to wave-wave interaction (Method of Mult. Scales)
  - Similar relation holds for ω_B with indices A and B switched

$$\omega_A = \sqrt{2-2\cos(\kappa_A a)} + \varepsilon \cdot \frac{3}{8} A^2 \left(2-2\cos(\kappa_A a)\right)^{3/2} + \varepsilon \cdot \frac{3}{4} B^2 \left(2-2\cos(\kappa_A a)\right)^{1/2} \left(2-2\cos(\kappa_B a)\right)$$

Previous correction term — *Wave-interaction term*

- Additional waves result in the same wave-wave interaction term (with appropriate indices)

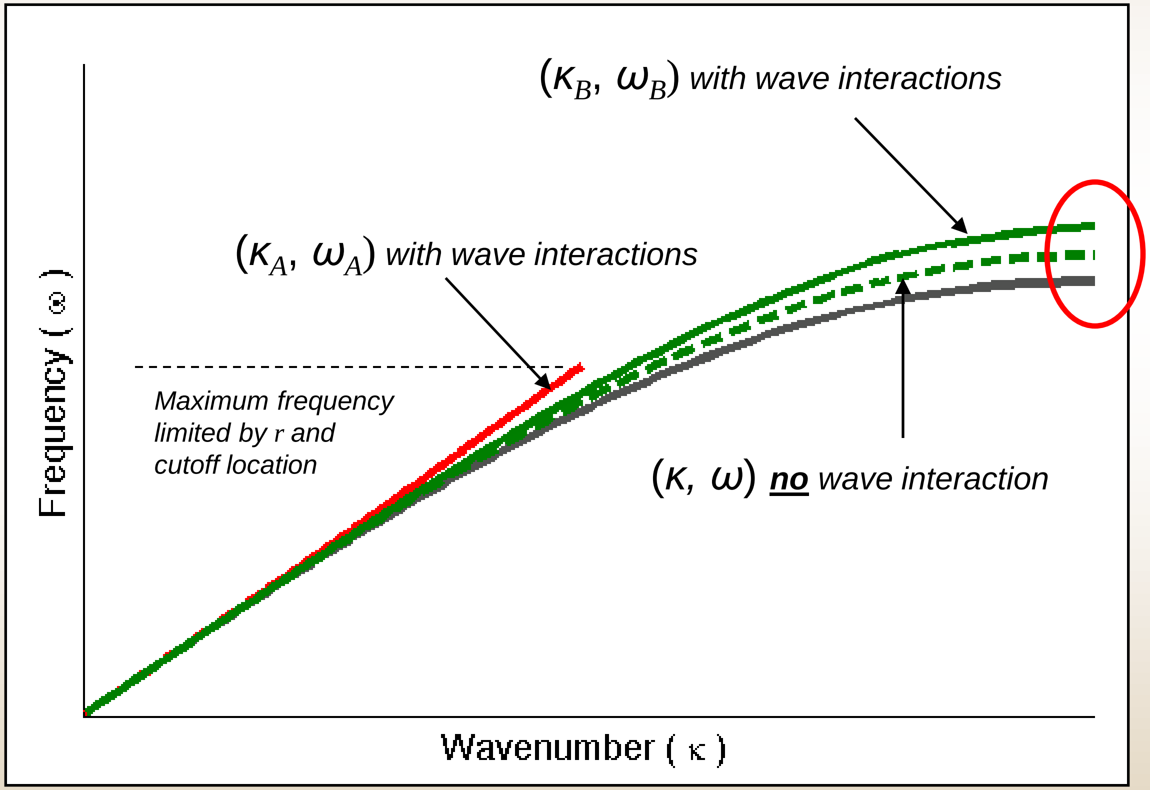
$$\omega_A = \sqrt{2-2\cos(\kappa_A a)} + \varepsilon \cdot \frac{3}{8} A^2 \left(2-2\cos(\kappa_A a)\right)^{3/2} + \varepsilon \sum_i \left(\frac{3}{4} B_i^2 \left(2-2\cos(\kappa_A a)\right)^{1/2} \left(2-2\cos(\kappa_i a)\right) \right)$$

Multiple wave-interaction terms

Manktelow et. al., 2010, *Nonlinear Dynamics*

Wave Interaction Significance

- *Tunable* dispersion relation by introducing a second wave
- Display both dispersion relations on the same plot:
  - *Let* $\omega_B > \omega_A$ and $\omega_B = r^*\omega_A$
- Wave interactions provide additional latitude in device design



Potentially significant shift in the band gap that could be utilized in metamaterial design

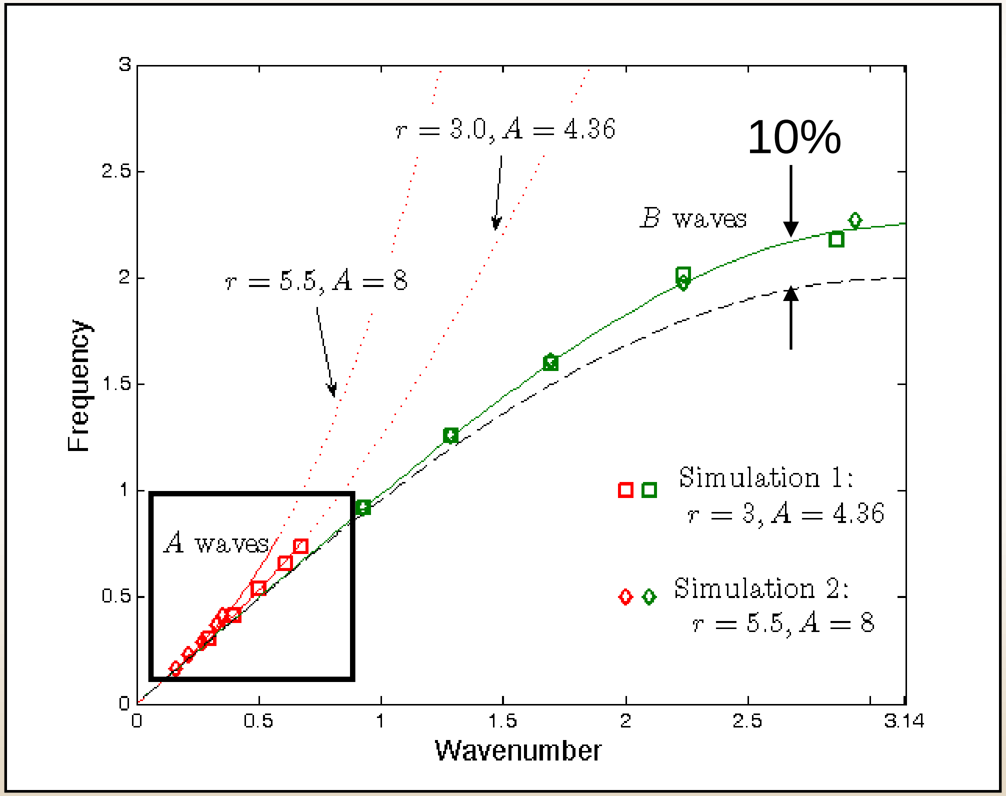
Parameters: $r = \sqrt{2}$

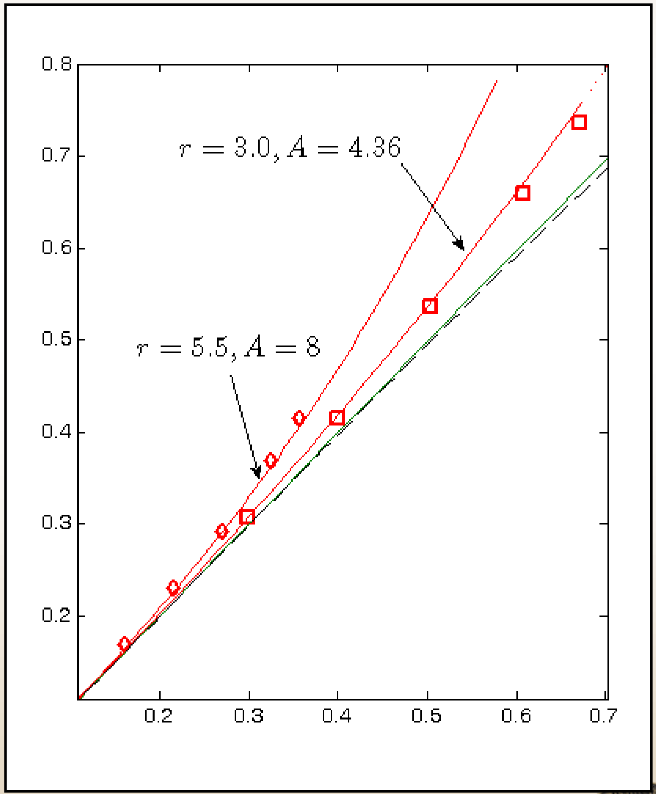
$A = 2$

$B = 2$

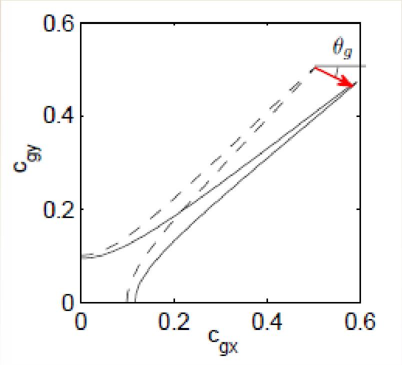
Numerical Simulation

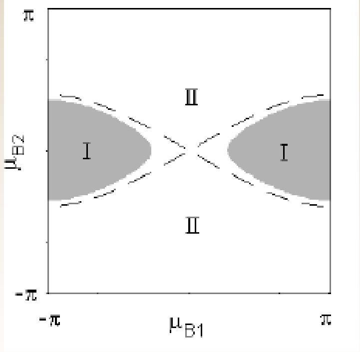
- **Example**: B wave $2*\cos(\kappa_B aj - \omega_B t)$ may be shifted by *10%* using nonlinear wave interactions:
  - Simulation 1: $r=3, A=4.36$
  - Simulation 2: $r=5, A=8.00$





Application: Beaming Control

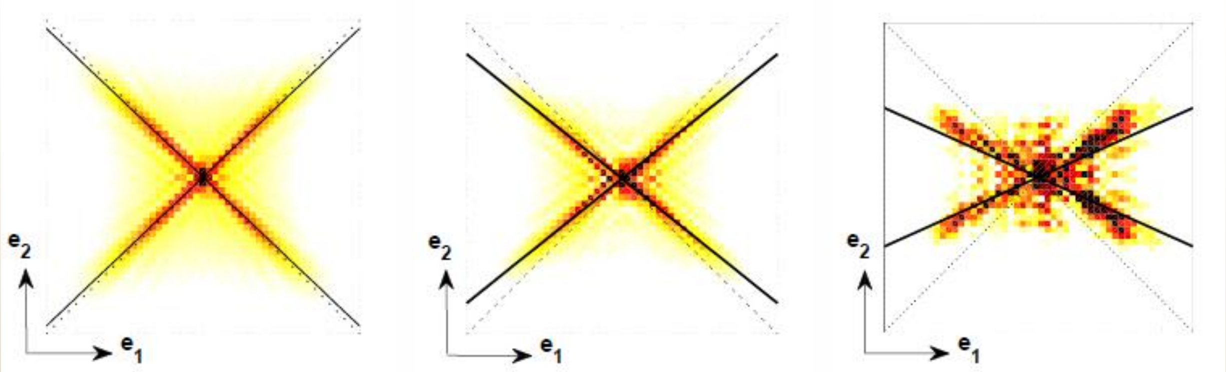




Region I: Negative group velocity corrections

Region II: Positive group velocity corrections

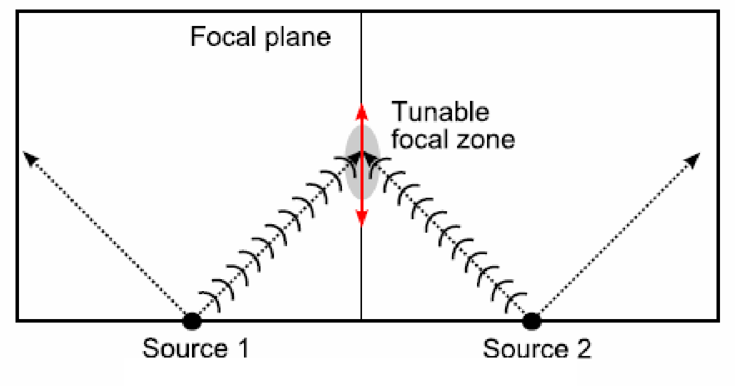
- Numerical simulations validate the expected direction shift
  - Control wave field in horizontal direction
  - (Image filtering to remove control wave from view)

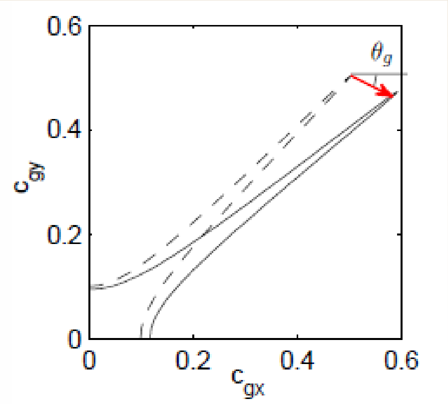


(a) $\alpha_B = 1.0$ (b) $\alpha_B = 2.0$ (c) $\alpha_B = 4.0$

Application: Tunable Focusing

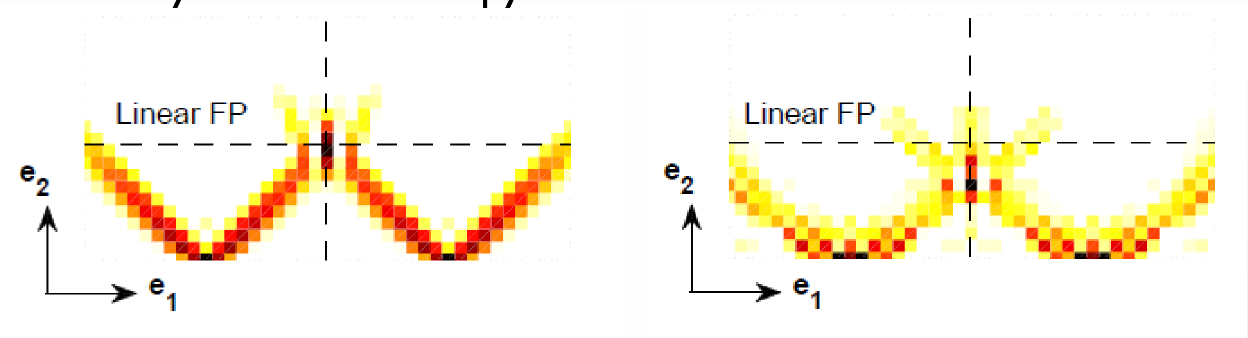
- Device schematic: two sources at a wave-beaming frequency produces a high-intensity region





- Numerical simulation of monoatomic lattice
  - Control wave field introduces dynamic anisotropy
  - Increased stiffness

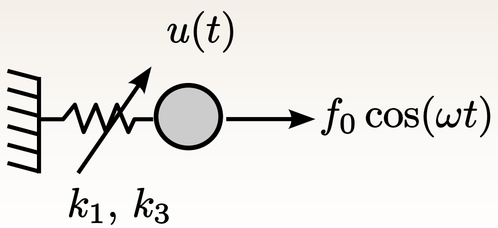
  from control wave alters the beam direction

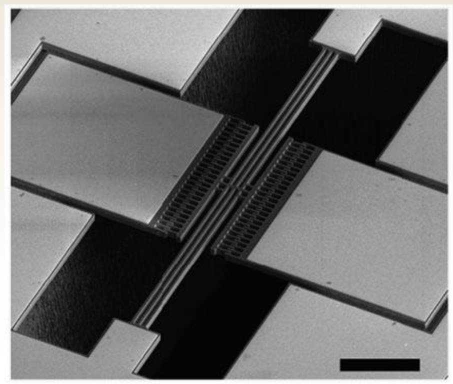


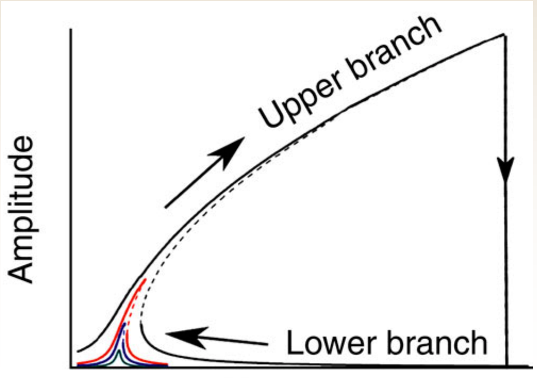
(a) $\alpha_B = 0.1$ (b) $\alpha_B = 2.5$

Experimental Verification

- The classical Duffing oscillator exhibits a well-known frequency shift and models many physical resonators







Ref [9]

$$m\ddot{u} + k_1 u + \varepsilon k_3 u^3 = \varepsilon f(t)$$

$$\omega = \omega_n + \varepsilon\sigma$$

$$\sigma = \frac{3}{8}\frac{k_3 A^2}{m\omega_n} \pm \sqrt{\frac{f_0^2}{4\omega_n^2 A^2}}$$

- What about a chain of oscillators?

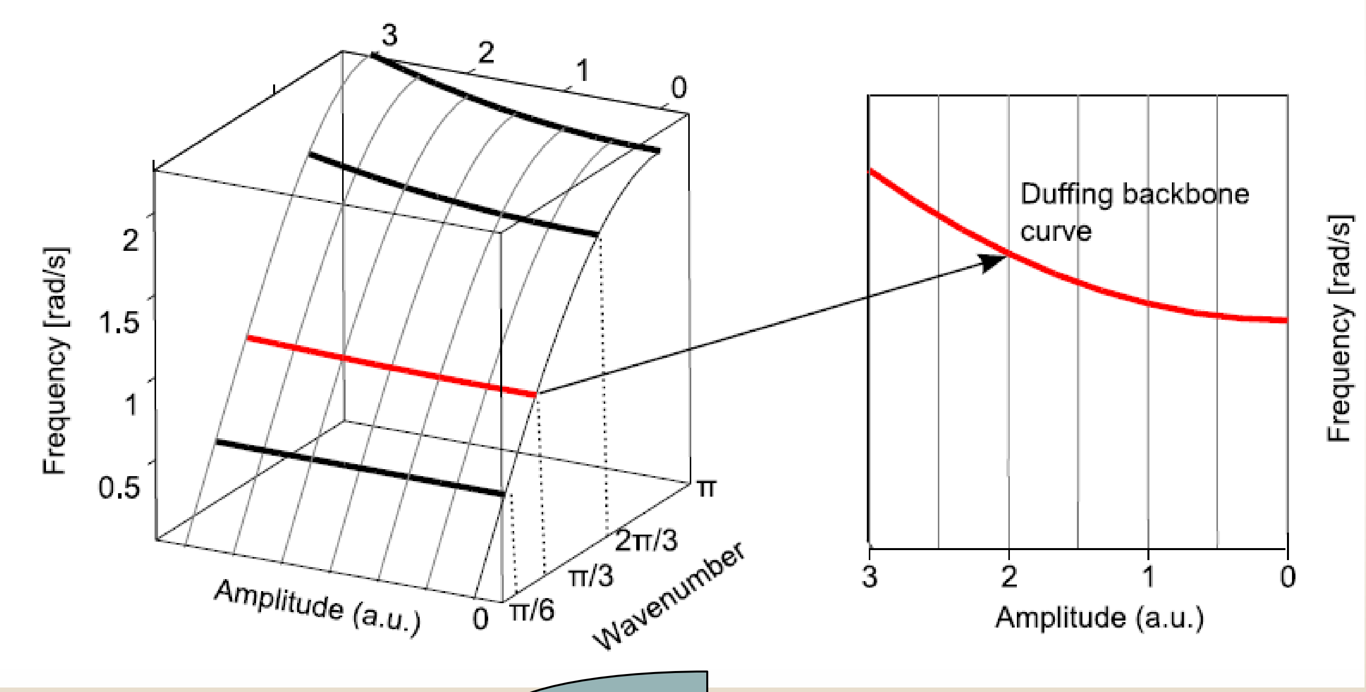
$$m\ddot{u}_p + k_1(2u_p - u_{p+1} - u_{p-1}) + \varepsilon k_3(u_p - u_{p+1})^3 + \varepsilon k_3(u_p - u_{p-1})^3 = 0$$

$$\omega(\mu) = \omega_n\sqrt{2 - 2\cos(\mu)} + \varepsilon\frac{3}{8}\frac{k_3 A^2}{m\omega_n}(2 - 2\cos(\mu))^{3/2} + O(\varepsilon^2)$$

Backbone curve looks remarkably like the dispersion frequency shift in the monoatomic chain

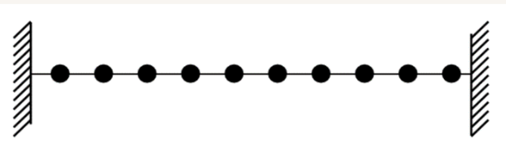
Experimental Verification

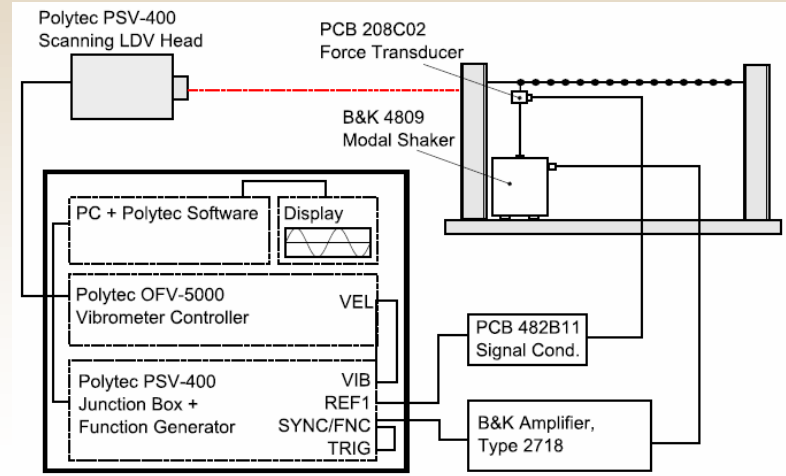
- Observe that for $\mu = \pi/3$ the dispersion shift is identical to the Duffing *backbone curve*
  - Dispersion shifts associated with free-wave propagation are analogous to backbone curves in finite systems.
  - Provides a means for experimentally measuring dispersion shifts



Experimental Verification

- Wire/mass system approximates a monoatomic chain
- Measure resonances at large amplitudes to determine dispersion shifts



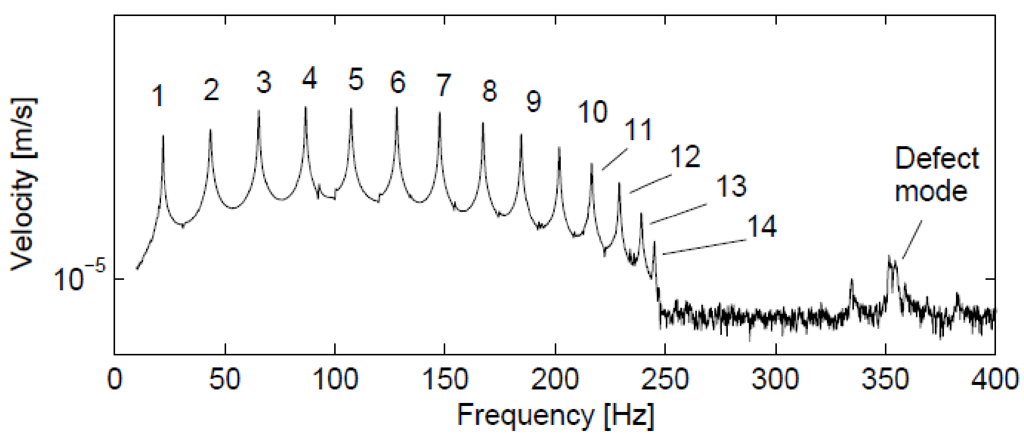


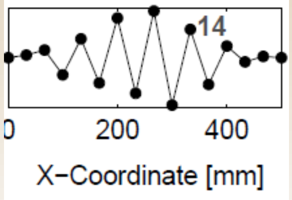
$$m\ddot{v}_p + \frac{T_0}{a}(v_p - v_{p+1}) + \frac{T_0}{a}(v_p - v_{p-1}) + \frac{EA_c}{2a^3}(v_p - v_{p+1})^3 + \frac{EA_c}{2a^3}(v_p - v_{p-1})^3 = 0$$

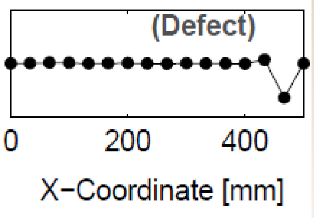
Parameter	D [mm]	a [mm]	E [GPa]	m [g]	ρ_V [kg/m^3]	T_0 [N]
Value	0.254	32.5	205	1.57	7850	21.8

(a) Mass element (b) Apparatus (c) Scanning head

Experimental Verification

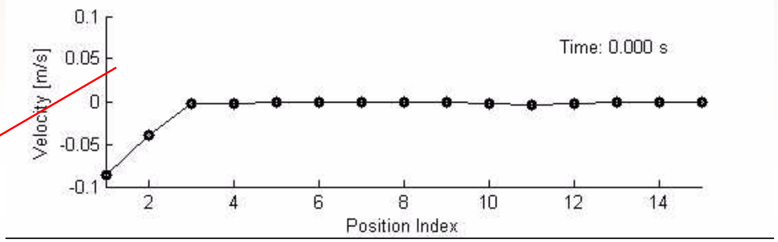


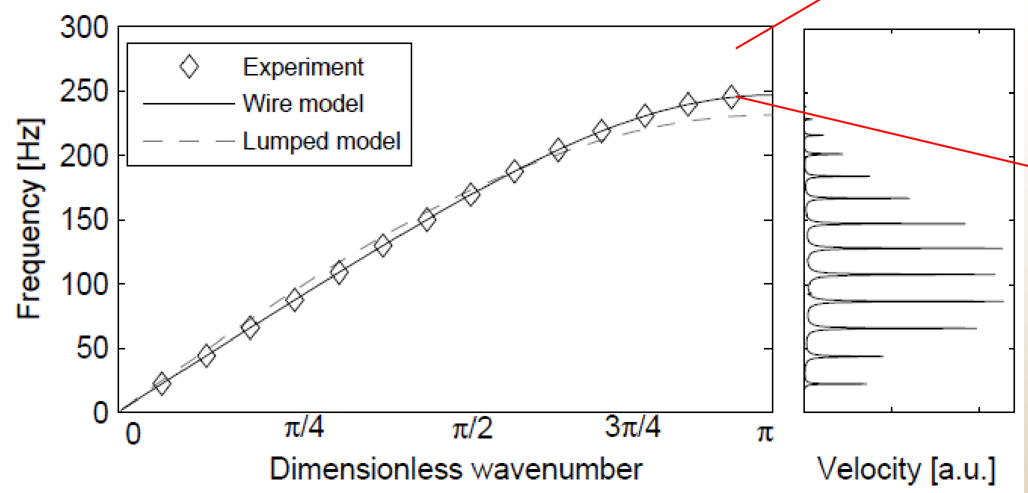




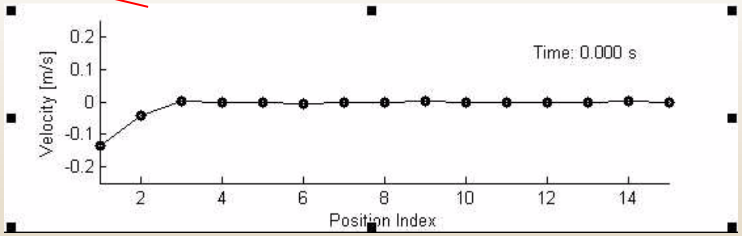
Resonant peaks in a finite periodic system fall on dispersion branches [10]

Evanescent wave (300 Hz, stop band)



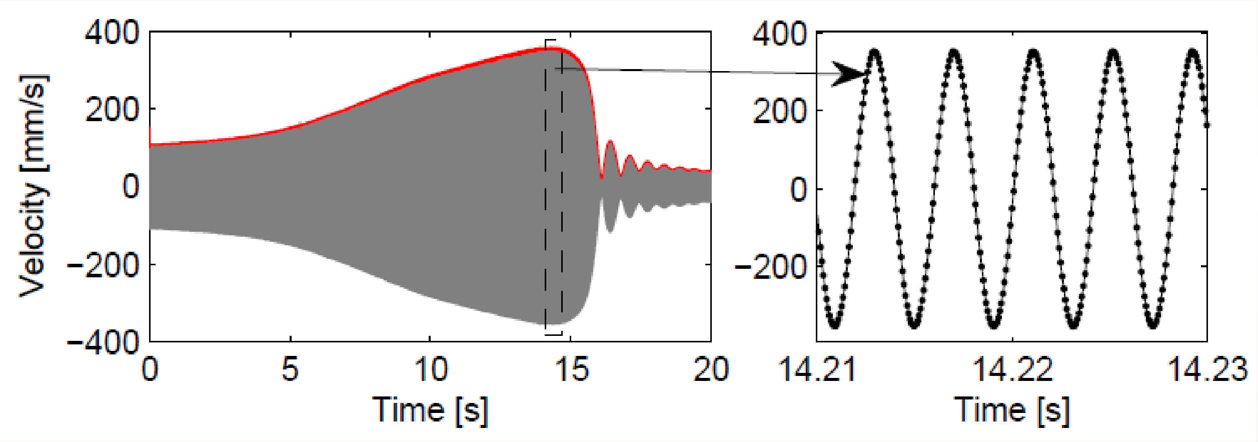


Propagating wave (240 Hz, pass band)



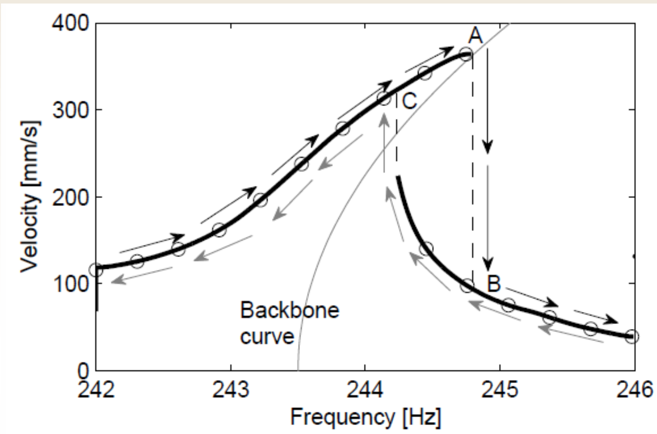
Experimental Verification

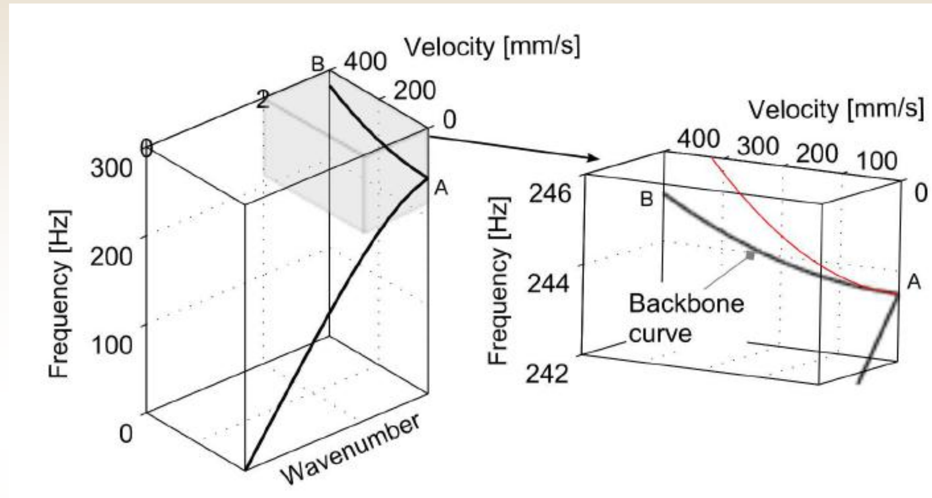
- Slow time-domain frequency sweeps over natural frequencies illustrates Duffing nonlinearity
  - Despite large amplitudes near resonance, signal is essentially monochromatic
  - Hilbert transform converts time-domain signal into an analytic signal



Experimental Verification

- Slow up/down frequency sweeps (~0.2 Hz/s) yield backbone curve, which relates to nonlinear dispersion shifts





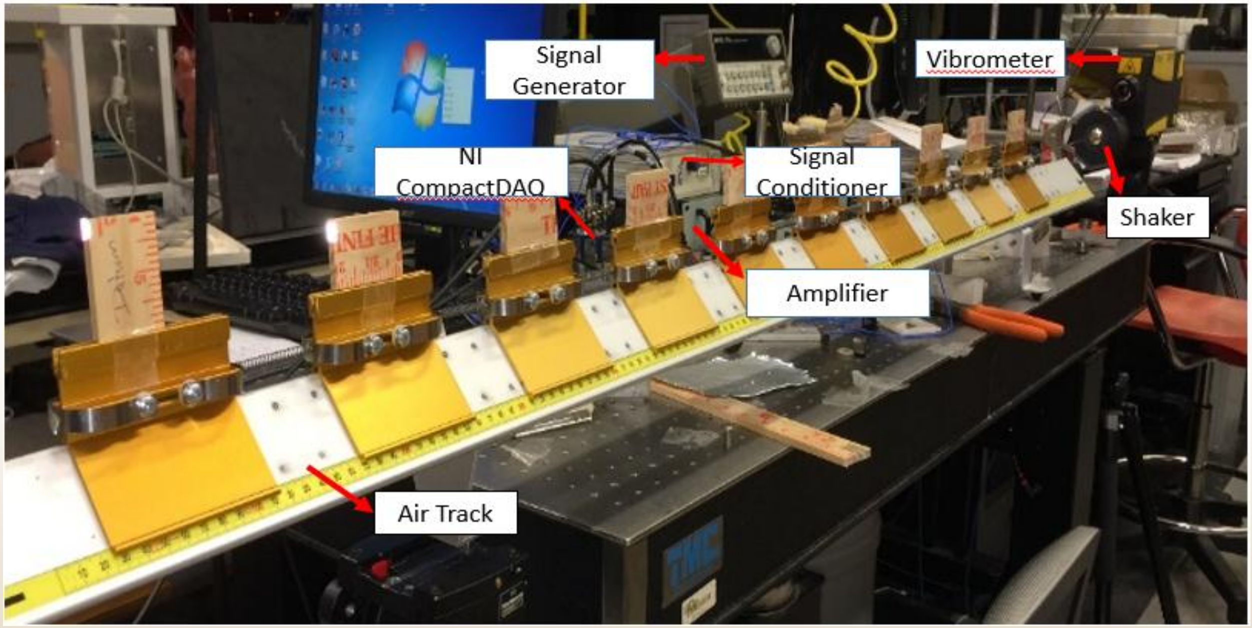
$$\omega_1 = \frac{3}{16}\frac{EA_cA^2}{m\omega_n a^3}(2 - 2\cos(\mu))^{3/2}$$

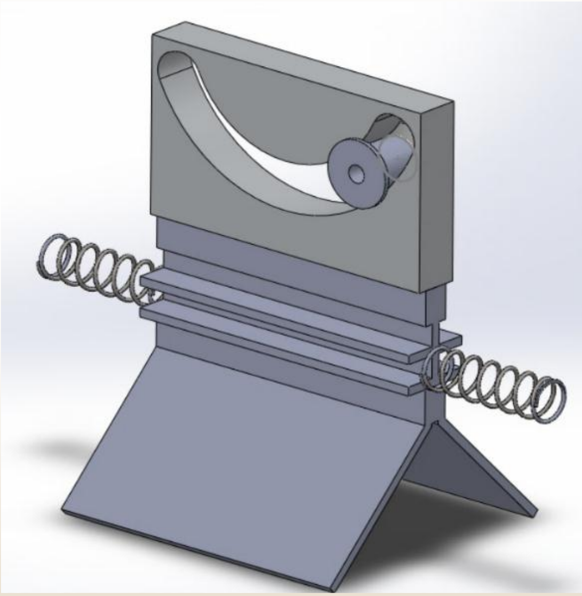
Conclusions

- Resonance backbone curves are related to free-wave propagation
- Resonances in *finite* periodic systems can be analyzed via the dispersion relation of a unit cell

Follow-On Research

- Experimental verification
  - 1D string is very limited
  - 2D offers opportunity to study wave-wave interactions (shifting focus, etc.) and amplitude-dependent group velocity





Follow-On Research

- Device construction
  - Perhaps RF devices?

- Strongly nonlinear periodic materials/structures
  - Stability of plane waves
  - Reconfigurability
  - Solitons

Publications

Manktelow, K., Leamy, M.J., Ruzzene, M., 2014, “Analysis and Experimental Estimation of Nonlinear Dispersion in a Periodic String,” *Journal of Vibration and Acoustics*, in press.

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Manktelow, K., Leamy, M.J., Ruzzene, M., 2013, “Topology Design and Optimization of Nonlinear Periodic Materials,” *Journal of the Mechanics and Physics of Solids*, **61**: 2433-2453.

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Manktelow, K., Narisetti, R.K., Leamy, M.J., Ruzzene, M., 2012, “Finite-Element Based Perturbation Analysis of Wave Propagation in Nonlinear Periodic Structures,” *Mechanical Systems and Signal Processing*, **39** (1-2): 32-46.

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