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PRODUCT DIVERSIFICATION AND THE MULTIPRODUCT FIRM

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I. INTRODUCTION

It could reasonably be argued that the multiproduct firm is the rule rather than the exception in the modern capitalist economy. Its empirical relevance and the peculiar cost and demand situations usually faced by such a kind of firm should invite theoretical examination of its decision-making process and development of tools to that effect. Surprisingly little has been written [1; 2; 3; 4; 5; 6], however, with the result that concepts and methods of analysis of the conventional single-product firm have not been applied to problems of the multiproduct firm, and consequently no suitable modification of them has become available.

This paper is a step towards this direction. Its purpose is to construct diagrams and develop simple mathematical analysis of the decision-making process of the multiproduct firm. Specifically, we describe the decision of the firm regarding the number of products it should produce. In the multiproduct firm the decision on profit maximization involves a further dimension in comparison with the single-product firm. One strand of analysis is the choice of the set of prices which will maximize profits given the number of products. The other strand is the choice of the total number of products to produce, given the set of prices. Bailey [1] has examined the first in considerable detail, but the second has scarcely been attempted. This paper concentrates on the second question. Results are contrasted with the work done previously on the multiproduct firm and with those of the analysis of a single-product firm contemplating whether to begin operations.

Another distinguishing characteristic of this paper within the theory of the multiproduct firm is the assumption that a block of indivisible[1] inputs not previously used is associated with the introduction of an additional product. This entails a lump-sum expenditure for such inputs, and the resulting cost characteristic is seen to be a factor in the decision for diversification.

Our paper does not assume costs of transfer of fixed inputs from one product to another, and in that sense it does not produce results related to such a dependence of production sets [5].

[1] Indivisibility may be technical necessity or simply a matter of technology conforming to economic conditions.

The simple distinction between units of product and units of output of each product seems central for the understanding of the peculiar cost and demand conditions facing the multiproduct firm. Work in this area requires concepts heavily dependent on this distinction. For example, the concept of marginal cost can be applied to a new product and also to units of output within this product. The same holds for marginal revenue. Our costs are defined with this distinction in mind, as the conventional concept of costs with respect to units of output not only fails to yield familiar maximization rules for the multiproduct firm but also runs into fundamental difficulties when the demands for the various products are assumed interdependent.

II. THE MULTIPRODUCT FIRM: ASSUMPTIONS CONCERNING COST AND DEMAND

For the construction of a simple, readable diagram illustrating the profit-maximizing decision of the multiproduct firm when the number of products is the decision variable, we make the following assumptions.

A multiproduct firm can produce x number of differentiated products. The units of output q_i sold from each product i are beyond its control and depend on demand conditions. The costs of production comprise, first, an overall amount of overhead costs, F, which are not affected by either the introduction of a new product or the variation of output within each product. Second, the continued production of a new product involves a fixed product cost, f_i, not affected either by the output of this or other products or by the introduction of other products. Third, there are the costs associated with output, q_i, within each product i, and varying with that output. These will be called output costs, $C_i(q_i)$.

On the demand side, it is assumed that all units of output of all products command the same price p, constant and insensitive to the firm's variations in the number of products. This implies that the firm's products are, in this case, closely related although not identical. In general the products are assumed to be gross substitutes except when they are independent. Gross substitutability means that the cross elasticity of demand is positive, which would imply that the introduction of a new product may have an adverse effect on the sales of the previous products.

For the purpose of mathematical presentation, in the case where demands are interdependent it is convenient to use the symbol $q_{i/x}$, which represents the output of good i when x products are being produced.

The total revenue when x products are produced is then

$$TR_x = p \sum_{i=1}^{x} q_{i/x}. \tag{1}$$

If demands for the products are independent (1) can be written more simply as

$$TR_x = p \sum_{i=1}^{x} q_i. \tag{2}$$

It is convenient to number the products in the order they would be brought into production if first the single most profitable good, then the two most profitable goods, then the three most profitable goods had to be chosen, and so on.[2] By most profitable we mean jointly most profitable.[3]

[2] Ordering in terms of profitability within the chosen most profitable products is immaterial. Suppose A is the single most profitable good. The production of B may so reduce sales from A that the profit from B is greater than the profit from A with both A and B being produced. This is no problem for the diagram or the analysis. Further, we do not require that if the two most profitable goods A and B are chosen the profit from B, $\pi_{B/A,B}$, be greater than that of any other good, say C, $\pi_{C/A,C}$. This is because $\pi_{C/A,C} > \pi_{B/A,B}$ does not necessarily imply that A and C are jointly the most profitable pair. $\pi_{A/A,C}$ may be less than $\pi_{A/A,B}$. $\pi_{C/A,C}$ is the profit from C given that both A and C are produced. The other symbols can be interpreted similarly. For further discussion regarding the ordering see footnote 3.

[3] As stated above in the text, the products are arranged in the order in which they would be brought into production if first the single most profitable good, then the two most profitable goods had to be chosen, and so on. Here most profitable means jointly most profitable. For the purposes of the mathematical presentation it is convenient to number the products in

A basic assumption of this paper on the cost side is the existence of the fixed product cost f_i defined above. For the convenience of the diagram and because there are empirical counterparts to this assumption (see below), it is assumed that this cost is the same for all products which the firm contemplates, although it is very easy to specify that f_i depends on the number of products x. It is also assumed, for the sake of this diagram, that output cost functions are the same for all products; i.e., $C_i(q_i)$ can be written $C(q_i)$. Again there is empirical relevance to this assumption; although it is not necessary for the analysis, the more general case is set out mathematically in equation (16) below.

The total variable product cost associated with x number of products and q_i units of output within each product is then

$$TVPC_x = fx + \sum_{i=1}^{x} C(q_{i/x}). \qquad (3)$$

The total cost, including the overhead costs F, is

$$TC_x = F + fx + \sum_{i=1}^{x} C(q_{i/x}). \qquad (4)$$

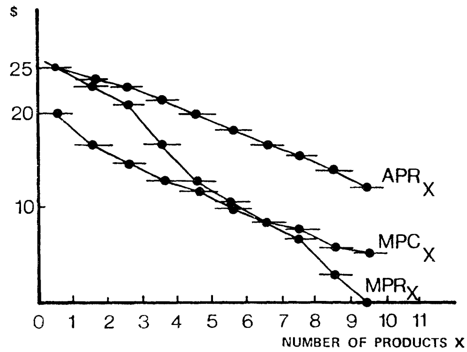

FIGURE 1a.

III. NEW CONCEPTS IN FAMILIAR-LOOKING DIAGRAMS

Figure 1b illustrates the choice by the multiproduct firm of the number of products to produce to maximize profits. The number of products (and not the output of each) is measured on the horizontal axis, while costs and revenues are measured on the vertical. The products are in order of increasing profitability, as explained in footnote 3.

Strictly, each line in Figure 1b should be drawn as a step function with each segment of length one unit rather than as a smooth curve. For instance, even in the simplest case of independent demands where the extra revenue from an extra product (MRP_x) is simply the value of sales from that product (see equation (7)), an extra product will cause a discrete change in MRP_x. APR_x, MRP_x and MPC_x (defined and explained below) are illustrated as step functions in Figure 1a. The smooth curves in Figure 1b are obtained by joining up the midpoints of each step. This is done purely for ease of reading.[4]

Rather interestingly, Figure 1b looks perfectly familiar and identical to the standard diagram of the single-product firm, but in actual fact the economic meaning of the

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this order provided no product once chosen drops out of the most profitable group with an increase in the number of products. That is, the single most profitable product is called number 1, the extra product chosen when two products are produced is called number 2, and so on. In symbols, suppose we have a group of goods, A, B, $\cdots$ Z. If A is at least as profitable as any other good, then

$$\pi_A \geq \pi_i \qquad i = B, C, \cdots Z,$$

where π_i is the profit from the ith good alone; good A can be given number 1. If A and B together are at least as profitable as any other pair of goods, then

$$\pi_{B/A,B} + \pi_{A/A,B} \geq \pi_{i/A,i} + \pi_{A/A,i}$$

where $\pi_{B/A,B}$ is the profit from B given that A and B are produced simultaneously. $\pi_{A/A,B}$ and $\pi_{i/A,i}$ can be similarly interpreted. Good B can be given the number 2.

If a product is included when x products are chosen but not when $x + 1$ products are chosen there is a problem in the numbering of the products but not in the arrangement of the groups of products along the axis of the graph. In this situation marginal product cost is the difference between producing $x + 1$ best products in combination and the x best products in combination.

[4] A similar simplification is often made in the analysis of the single product firm in the case where each unit of output is indivisible. Marginal revenue can then only decline in discrete steps with each extra unit sold. Note that in Figure 1b the revenue curves are drawn as straight lines (as is common in the case of the single product firm) also purely for convenience.

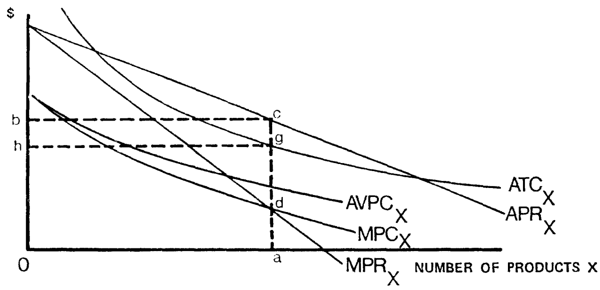

FIGURE 1b.

curves of the multiproduct firm is quite different. Consider, for example, the Average Product Revenue curve (labelled APR_x): it shows, as its name suggests, the average revenue per product when x number of products are chosen.

$$APR_x = p \sum_{i=1}^{x} q_{i/x}/x \tag{5}$$

The assumption made above that the products are gross substitutes means that, as the number of products increases, the average number of units of output sold per product (and hence the average revenue per product) decreases. Thus, the APR_x declines not because of the price (which is constant per unit of output sold from each product) but because of the assumed demand conditions.[5] This condition, it will be noted, can obtain even with independent demands (where the addition of a new product does not affect the sale of the previous ones) provided that the intensity of demand for additional products declines and, hence, at the constant price, fewer units of output are sold from each additional product.

The MPR_x curve shows, in the case of interdependent demands, the total revenue from product x minus the revenue lost from the $x - 1$ products when the xth product is introduced.

$$\begin{aligned} MPR_x &= TR_x - TR_{x-1} \\ &= p(q_{x/x} + \sum_{i=1}^{x-1} \{q_{i/x} - q_{i/x-1}\}) \qquad (6) \\ &= pn(x), \end{aligned}$$

where $n(x)$ are the net new sales, that is, the sales of the xth product $q_{x/x}$ less the loss of sales from the other $x - 1$ products.

On the assumption that the demands are independent, the marginal revenue from the xth product is simply equal to the value of sales from that product; that is,

$$MRP_x = pq_x. \tag{7}$$

With a uniform price, MPR_x will usually be positive. It may reach zero when the introduction of a new product simply detracts sales from the previous products but does not bring any net new sales. It may also become negative if the introduction of a new product x cuts down the sales of the previous products by a number of units greater than those sold of the xth product.[6]

When demands are independent, the MPR_x curve will decline (or be horizontal) for the same reasons as those given for the APR_x curve.

One of the major advantages of our diagrammatic presentation is that, despite the

[5] The mathematical requirement for a declining average revenue product curve (APR_x) is

$$\sum_{i=1}^{x} (q_{i/x}/x) \leq \sum_{i=1}^{x-1} [q_{i/x-1}/(x - 1)].$$

This would not hold in the general case of a multiproduct firm where the products may be complements.

[6] The fact that, with the uniform price, the consumers' marginal rates of substitution of products 1 to x must be equal to unity under competitive conditions on the consumers' side of the market does not preclude this possibility.

existence of interdependent demands which cause output costs to change with the addition of an extra product, the curves showing the cost of an extra product do not shift.[7]

The Average Variable Product Cost Curve (labelled $AVPC_x$) shows the product cost f plus the average of output costs per product when x products are produced.

$$AVPC_x = f + \sum_{i=1}^{x} (C(q_{i/x})/x). \tag{8}$$

Under some conditions[8] the $AVPC_x$ curve must decline or remain horizontal. If it does keep declining, it must be asymptotic to a horizontal line lying at the height of f units above the horizontal axis.

The Average Total Cost Curve (labelled ATC_x) is

$$ATC_x = TC_x/x. \tag{9}$$

ATC_x declines not only because of the reasons which make $AVPC_x$ decline but also because of the distribution of overhead costs onto more products.

Finally, the Marginal Product Cost Curve (MPC_x) shows the change in total costs with the introduction of the xth product.

$$\begin{aligned} MPC_x &= TC_x - TC_{x-1} \\ &= f + C(q_{x/x}) + \sum_{i=1}^{x-1} \{C(q_{i/x}) \\ &\qquad - C(q_{i/x-1})\} \\ &= f + L(x), \end{aligned} \tag{10}$$

where $L(x)$ is the net output cost, that is, the cost due to the output of the xth product plus the reduction in output costs of the previous $x - 1$ products. Given gross substitutability, this change will always be a negative amount, and it is conceivable that it may be greater in absolute value than the other two elements of the MPC_x together; in this case, MPC becomes negative. This is possible only if average output cost is increasing with the level of output so that reduced output of each product reduces average output cost of that product.

On the assumption that the demands are independent, MPC_x is simply the fixed product cost (f) plus the output costs of the xth product; i.e.,

$$MPC_x = f + C(q_x). \tag{11}$$

Now that all the cost concepts have been defined it is appropriate to discuss a conceptual difficulty concerning the nature of marginal costs when output is changed by the addition of differentiated products. The difficulty arises if one considers the total units of output from all products as a measure of the firm's activity and insists on treating the marginal cost concept in the conventional sense. Even in the case of independent demands this concept of marginal cost could lead to faulty decision-making since it does not take into account the fixed product cost f. In contrast, the MPC includes f which would conventionally be considered an average cost component. In the case of interdependent demands, the conventional concept used in this way is not even clearly defined. A change in output will be accompanied by a change in the way output is distributed among the number of products, quite likely causing the total cost of any

[7] This follows directly from the mathematical statement (3) in the text but is perhaps not easy to grasp intuitively. If the 11th product is introduced, the output costs of the previous 10 products will decline given that 11 products are being produced. The output cost of 10 products given that 10 products are being produced is of course not affected. This is the cost that is plotted on the diagram.

[8] The $AVPC_x$ curve will decline if the output of the xth product when x products are produced is less than or equal to the average output per product when $x - 1$ products are produced; i.e.,

$$q_{x/x} \leq \sum_{i=1}^{x-1} [q_{i/x-1}/(x - 1).]$$

The $AVPC_x$ curve will decline in these circumstances even if the average output cost of each product $C(q_i)/q_i$ increases as each output $q_{i/x}$ is reduced with expanded number of products. Notice that the condition for the APR_x curve to decline (see footnote 5) is slightly weaker than this, so that even with a downward sloping revenue curve the $AVPC_x$ curve may increase.

level of "output" to shift.[9] For example, when the introduction of an additional product reduces the sales of the previous products by an amount equal to the sales of the additional product,[10] the total costs of the firm will most probably change while the total number of units of output will remain the same. In another case, when the introduction of an additional product reduces the sales of the previous products by an amount greater than the sales of the additional product,[11] the total number of units of output decreases; if, then, the total cost increases because of the fixed product cost f, it may seem that the conventionally defined marginal cost is negative, although in the sense given above it is again undefined.

IV. PROFIT MAXIMIZATION AND THE NUMBER OF PRODUCTS

Let total profit from the enterprise when x products are produced be

$$\pi_x = TR_x - TC_x. \tag{12}$$

For a local maximum of profit with respect to the number of products it is sufficient that with x products profits increase, but that with $x + 1$ products profits diminish,[12] i.e.,

$$\pi_x - \pi_{x-1} > 0 \quad \text{and} \quad \pi_x - \pi_{x+1} > 0. \tag{13}$$

An equivalent expression is

$$MPR_x - MPC_x > 0 \quad \text{and} \quad MPR_{x+1} - MPC_{x+1} < 0. \tag{14}$$

Using (6) and (10), (14) becomes

$$pn(x) > f + L(x) \quad \text{and} \quad pn(x+1) < f + L(x+1). \tag{15}$$

The first expression from (15) states that the net new sales (whether positive or negative) must exceed the fixed product cost and the net output cost of the xth product. The second expression from (15) states that for optimality of the xth product, the $x+1$ product must reduce total enterprise profit. Notice that overhead costs F do not enter the decision to add an extra product.

In Figure 1b, the profit-maximizing number of products is oa, where $MPR = MPC$. The maximum profit is shown as the area $bcgh$. As mentioned above, although strictly the cost and revenue curves are discontinuous with each change in the number of products, for convenience they are drawn as continuous. Thus the condition for a maximum of profit appears as an equality $MPR_x = MPC_x$.

In the more general case where costs and prices differ among products and where goods are not necessarily substitutes, we have no difficulty in determining mathematically the most profitable number of products. This is a two-stage procedure. We can first assume that a previous decision has been made concerning the level of these prices, or a series of trial prices can be used. Given these, the total profit from x products is as follows:

$$\pi_x = \sum_{i=1}^{x} p_i q_{i/x} - [F + x f_i(x) + \sum_{i=1}^{x} C_i(q_{i/x})]. \tag{16}$$

The products which are included are the most profitable in combination when x goods are to be produced. The condition for the most profitable number of products is then given in (13) above.

When is it profitable for the firm to operate at all? It will pay the firm to operate if at the optimum product number all costs, including overhead, are covered. In our case of

[9] The different distribution of output within each product will of course not affect costs if $C(q_i)$ is a linear function of q_i and C is the same for all products. Because of the fixed product cost f a new product will always change total costs.

[10] i.e., when $MPR_x = 0$.

[11] i.e., when $MPR_x < 0$.

[12] If $\pi_x = \pi_{x-1}$, then this profit level will be a local maximum if $\pi_{x-1} - \pi_{x-2} > 0$ and $\pi_x - \pi_{x+1} > 0$. This maximum condition assumes a fixed price for all products. The price can be fixed for reasons of marketing or because of collusion. Alternatively, for any given number of products a price can be found which will maximize profits by simple differentiation of π_x with respect to p.

gross substitutes, if for the single most profitable product MPR does not cover MPC then it would definitely not pay to produce other products except at a higher price.[13] Thus, in Figure 1b the marginal product revenue curve must start above the marginal product cost curve. When they cut, the second-order conditions for a maximum of profit will hold. If for the single most profitable product MPR is less than MPC, then we have a sufficient but not a necessary condition for the firm to shut down. It may be that overhead costs are so high that total costs are covered at any scale, while $MPR > MPC$ for the single most profitable product.

The possibility of MPC and/or MPR being negative was pointed out in the previous section. Despite their special nature it may be instructive to consider these cases as they pertain to the decision of the firm on the optimum number of products. Figure 2 illustrates the two conceivable cases. Suppose, first, that both MPC and MPR can become negative. As mentioned above, for MPC to become negative average output cost must be increasing in that range. If they cut as in the Figure,[14] the most profitable number of products is again indicated by their point of intersection. At point a in the Figure, the introduction of the new product has added some net new sales, as is obvious from the positive MPR_i. It has also taken away sales of relatively expensively produced units of output from the previous a-1 products, with the result that production costs of the ath product are equal to the cost reduction effected in the a-1 products. At point b, the introduction of the new product simply detracts from the sales of the previous products, and hence the MPR is zero. Costs are reduced in total, however, indicating that a better distribution of units within products has been achieved from the point of view of cost. "Expansion" (in terms of the number of products, that is) from b to the optimal point c entails a net reduction in the total units of output sold from all products together (the MPR is negative), but the redistribution of the remaining output within more products produces cost economies more than the losses in revenue. Although logically possible, the situation at b or c may not

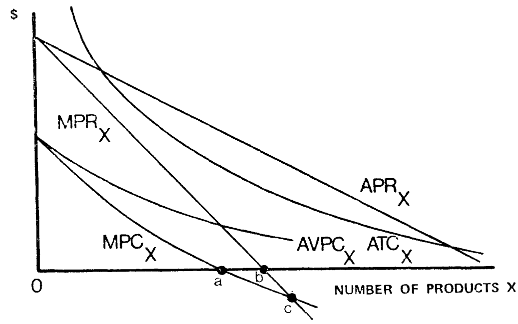

FIGURE 2.

[13] $MPR_1 > MPC_1$ is the same as $APR_1 > AVPC_1$ and in conventional terms amounts to price less than average total cost excluding F or

$$P < (f/q_{1/1}) + C(q_{1/1})/q_{1/1}.$$

The proof that if $MPR_1 < MPC_1$ then further products at the same price will not produce overall profitability follows most easily by first considering the case of constant output costs. Let k equal the constant average output cost $C(q_{i/x})/q_{i/x}$. The total profit from x goods is then

$$\pi_x = (p - k) \sum_{i=1}^{x} q_{i/x} - fx - F.$$

If the profit from the single most profitable product excluding overhead costs F is negative, i.e., $MPR_1 - MPC_1 > \pi_1 + F < 0$, then the assumption of gross substitutes ensures that $q_{i/x} < q_{i/x-1}$ so that from π_x above production will remain unprofitable however many goods are produced jointly. If average output costs are not constant but decreasing an extra product will cause costs to decrease less than in proportion to the decrease in average output per product, giving a worse picture. The case of increasing average output costs is more difficult. Mathematically it is true that even if $MPR_1 < MPC_1$ the reduction in output per product caused by increased products may so reduce costs that a positive profit is made. However, this makes economic nonsense. Under these circumstances the entrepreneur would raise his price achieving a higher return with the reduction in output. This is particularly true if the negative profit for the single most profitable good is due to marginal and average output costs rising so fast that to meet demand fully the firm must operate past the point where price equals average total cost. Profit could be made positive simply by raising the price.

[14] The only other conceivable way for them to cut without making the firm shut down is as in Figure 1.

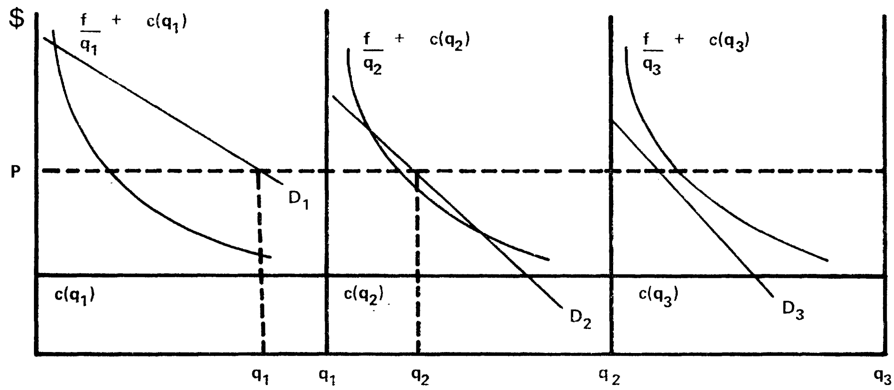

FIGURE 3.

continue, as it is then likely to be more profitable for the firm to raise the fixed price.

If the possibility of a negative *MPR* is denied, the curve coincides with the horizontal axis from point *b* onwards. As long as the *MPC* remains negative, it pays the firm to "expand," spreading the constant output onto more products. With *U*-shaped output cost curves, the *MPC* curve must eventually turn up and cut the horizontal axis at a point, which is the optimum position in this case.[15] Needless to say, further products with $MPR = 0$ may not exist either.

V. COMPARISONS WITH CONVENTIONAL TREATMENT

It is instructive to compare Figure 1b to a diagram with total units of output on the horizontal axis (Figure 3). As pointed out earlier, conventional treatment breaks down totally in the cases described by Figure 2. But even without those cases, the conventional output diagram suffers from several disadvantages.

Assume first the simple case where demands are independent. Here it is clearly apparent that average costs per unit of output are involved in the decision to add an extra product. Our marginal product cost concept takes care of this fact, and in this simple case the more conventional output diagram can be used to illustrate this point.

More formally from (14), (7) and (11), the requirement for profit maximization is:

$$pq_x > f + C_x(q_x) \quad \text{and} \quad pq_{x+1} < f + C_{x+1}(q_{x+1}). \tag{17}$$

If $c_i(q_i)$ is the average output cost of the *i*th product, then (17) can be written:

$$p > (f/q_x) + c_x(q_x) \quad \text{and} \quad p < (f/q_{x+1}) + c_{x+1}(q_{x+1}). \tag{18}$$

That is, from (18), with independent demands the condition for profit maximization is that, for each product, price must be greater than the sum of the fixed product cost and output cost both taken per unit of output of that product, yet these costs are not covered if a further product is produced.

Assume, for purposes of illustration only, that the average output cost $C(q_i)/q_i$ is constant. Figure 3 illustrates the decision-making of the firm concerning the number of products: with the price fixed at p, the optimum level of output is $q_1 + q_2$, with two products being produced.

A major defect of Figure 3 is that it does not show whether the enterprise as a whole operates at a profit, since the overhead costs do not appear. Total cost, which includes F, must be covered if the enterprise is to operate at all. Also notice that if the fixed product cost f did not exist we could draw continuous

[15] Naturally the firm can only expand if new products exist for which $MPR_x = 0$. The marginal product cost curve must eventually rise again if the output of each product becomes so low that the firm suffers from diseconomies of low production of each product.

average cost curves for output Σq_i itself, as is done by Clemens [3].

In fact, Clemens uses a similar diagram to our Figure 3. His differs in that, with the assumption of a single plant, the marginal cost of a unit of output is the same regardless of the product mix. He can thus work with a horizontally-added demand curve to get the standard result for a price discrimination model that marginal revenue must be the same in each market and equal to the common marginal cost. Clemens' case, with his simple assumption of independent demand can be dealt with in our model by dropping the assumption of a fixed price. Then, in our model, for maximum profit prices should be chosen so that marginal revenue equals marginal output cost for each product. With respect to product number, Clemens can state that a firm will not be in equilibrium if there are any other markets in which price is still greater than conventionally defined marginal cost. With the introduction of the fixed product cost f, the rule for the profitable addition of an extra market is no longer the same. Price can be greater than marginal cost with respect to output of the ith market but less than average output cost $(f/q_i) + (C(q_i)/q_i)$ in the ith market. Although the market satisfies Clemen's rule, it may not be profitable if there are any fixed product costs.

We can now return to the case of interdependent demands. As we claimed above, the diagram with output along the horizontal axis is no longer useful. Every addition of an extra product will shift down the demand curves of previous products. Along with the change in total revenue from previous products there is a change in their total costs. Although the cost curves do not shift, outputs are altered, thus changing output cost; therefore, the marginal product revenue and the marginal product cost of the addition of an extra product are obscured. Price may be greater than average total cost for product x alone, yet product x may reduce the total profit of the firm. We can no longer say in a situation as depicted in Figure 3 that just two products should be produced. Total profit may be higher with just one product. Our product diagram 1b does not suffer from this deficiency.

We have examined some of the consequences of the fixed product cost, f, and of interdependent demand in the theory of the multiproduct firm. We shall now briefly consider the fixed overhead cost F and its effect on the degrees of freedom of the multiproduct firm versus those of the single-product firm. While it is still planning its form of production, all costs (apart from planning costs) will initially be variable for the single-product firm. This is not true for the multiproduct firm since in this partially shortrun situation it has fixed overhead costs F arising from its production of other products. If the resources found in F contribute to the new operation, the multiproduct firm will be willing to enter a market with less profitable demand conditions than are necessary for a new firm to begin production. For any given demand conditions, the multiproduct firm will be at a profit advantage.

VI. EMPIRICAL COUNTERPARTS

This section is devoted to mentioning some problems in the analysis of which this paper is directly applicable.

It should be noted that very common dimensions of differentiation are time and space. For example, the service rendered by a flight of an airline differs according to the time of day, so that the services from each flight should be considered as separate products. Within each flight, the number of seats sold can vary, and this is the level of output. Other examples of time and space differentiation are ferry services, fast food chains, hotel chains, gas station chains, and food chain stores, provided that they are managed centrally. Most of these cases are characterised by overhead costs F, uniform fixed costs of continued operation f, and output costs $C(q_{i/x})$ which are in most cases not only the same for each product but also constant per unit of output. Also, the price per unit of output is in most of these cases uniform for all products and independent of their total number. Thus, the assumptions of Section II of this paper, although simplifying, are seen to apply to a wide range of actual problems, notwithstanding the fact that most of them are not essential to the mathematical argument.

The differentiation of products can also be partly contrived by the producer. Again this paper is directly applicable. Examples are cases in which the product is simply packaged differently and sold under another name brand, and at a price quite close to the "prevailing" price for similar products. Output costs can then safely be assumed the same, while f may be considered to consist of promotion costs plus costs of changeover of equipment.

VII. CONCLUSIONS

The theory of the multiproduct firm is developed in this paper under the assumptions of uniform price per unit of output and the same output cost conditions for all products. Though these assumptions seem overly specialized, a multitude of actual situations is seen to correspond quite closely to them. The existence of a fixed product cost and/or of interdependent demands necessitates new concepts for the treatment of the firm's problems, and renders treatment by conventional tools at best inadequate[16] and at worst impossible.[17]

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[16] The uniformity of the fixed product cost f and of the output cost $C(q_i)$ for all products are not actually essential for the argument although they make the graphical presentation much easier.

[17] In Figure 3 above, there is no indication as to whether the firm should operate at all. See above in text.