A Marriage of Utility

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We sometimes use the notion of "utility" to reason about preferences, as in the stable marriage problem.

Rather than having all women **rank** all men and vice versa, we have each woman **rate** each man (and vice versa). To rate man m_j, woman w_i assigns him an integer from 0 to 100—which we designate $w_i(m_j)$. w_i prefers m_j to m_k if and only if she rates m_j higher than m_k; that is, $w_i(m_j) > w_i(m_k)$. Furthermore, we'll insist that a woman's ratings for any two men are distinct (i.e., for any woman w_i there are no two different men m_j and m_k such that $w_i(m_j) = w_i(m_k)$) and similarly for men rating women.

Finally, we assume that a difference of 1 unit of utility means the same thing for everyone at all points in their scales (so we can compare people's ratings to each other, meaningfully add and subtract ratings, and so forth).

1 Failure of Distinctness

Explain why we cannot possibly ensure our "distinctness" criterion holds as the instance's size scales up. (In subsequent parts, assume this problem has been fixed.)

2 Converting to SMP

Given a list L of one woman's ratings of all the men—where $L[1]$ is her rating for m_1, $L[2]$ is her rating for m_2, $\ldots$, and finally $L[n]$ is her rating for m_n—give an algorithm to convert that into a preference list. Again, assume all ratings are distinct.

3 Comparing Utilities and Preferences

In this part, you will show that utility **may** be a better measure than stability for the quality of an assignment. Be sure to read **all** the questions here before answering any.

1. Give a **small** instance of this utility-rating problem (that will satisfy all subsequent parts).
2. Give the corresponding preference lists for men and women.
3. Give a stable matching for this instance.
4. Give an **unstable** matching for the same instance that is **much better** in terms of utilities than the stable one. **Explain** why the unstable matching is so much better in terms of utilities!

4 Maximum Matching

For a weighted bipartite graph, it's possible to efficiently find a maximum matching: a matching with maximal total edge weight.[1]

[1] You probably do not know what some of these terms mean. If not, find out!

1. Give a reduction from the utility-based marriage problem to maximum matching on a weighted, bipartite graph.
2. Give at least one measure of the "goodness" of a solution for which your reduction produces an optimal result. Briefly explain why your reduction produces an optimal result.
3. Give at least one measure of the "goodness" of a solution for which your reduction does **not** produce an optimal result. Use a small example to illustrate how the reduction fails.