CPSC 320 Notes, Reductions and Analysis on Stable Marriage

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Note for all three problems defined below, we want a "stable" solution. We've defined an "instability" for the first and will leave it to you for the next two.

Definitions: An *instance* of a problem is a particular input drawn from the space of possible inputs the problem allows. For example, the 4-element array $[5, 1, 4, 3]$ is an instance of the problem of sorting arrays of integers.

A (simple) *reduction* from problem P_1 to problem P_2 is a pair of algorithms that work together. The first algorithm takes any instance of P_1 and transforms it into a corresponding instance of P_2. The second algorithm takes a solution to an instance of P_2 generated by the first algorithm and transforms it into a solution to the corresponding instance of P_1. (If needed, the second algorithm has access to bookkeeping information from the first.)

An instance of the *stable marriage problem (SMP)* is: a number n, n permutations of the numbers $1, \ldots, n$ (the women's lists of the men, listed in order from most to least preferred), and another n permutations (the men's preferences for the women). For example (adding some labels and using some handy visual structure, as we will often do):

```
n=3

w1: m2 m1 m3        m1: w3 w1 w2
w2: m1 m2 m3        m2: w3 w2 w1
w3: m2 m1 m3        m3: w2 w1 w3
```

An instance of the /"unequal" stable marriage problem (USMP)/ is a number W and a number $M < W$, W permutations of the numbers $1, \ldots, M$ (the women's preferences for the men), and M permutations of the numbers $1, \ldots, W$ (the men's preferences for the women). For this problem, we assume all women prefer being married to being unmarried. For example:

```
W=3 M=2

w1: m1 m2      m1: w2 w1 w3
w2: m2 m1      m2: w1 w2 w3
w3: m1 m2
```

Similar to USMP instances, an instance of the *stable resident/hospital problem (R/H)* is a number R, a number $H \leq R$, R permutations of the numbers $1, \ldots, H$, and H permutations of the numbers $1, \ldots, R$. In addition, there are H positive integers summing to R that represent the hospitals' capacities (number of residents they'll match with). For example:

```
R=3 H=2

r1: h2 h1      h1: r1 r2 r3, capacity 5
r2: h2 h1      h2: r3 r1 r2, capacity 2
r3: h1 h2
```

1 Solve the SMP instance above

1. Give a stable solution to the instance:

2. Are there any other stable solutions?

2 By-Hand Reduction USMP to SMP

1. Convert the example instance of USMP above into an instance of SMP:

2. Now, provide a solution to this instance of SMP:

3. And, finally, convert it back to a solution to the original USMP instance:

3 By-Hand Reduction R/H to SMP

1. Convert the example instance of R/H above into an instance of SMP:

2. Now, provide a solution to this instance of the SMP:

3. And, finally, convert it back to a solution to the original R/H instance:

4 General Reduction USMP to SMP

Provide a general reduction from USMP to SMP.

1. Algorithm to convert an instance of USMP to an instance of SMP:

2. Algorithm to convert a solution to an instance of SMP (generated by your algorithm) to a solution to the corresponding instance of USMP:

5 General Reduction R/H to SMP

Provide a general reduction from R/H to SMP.

1. Algorithm to convert an instance of R/H to an instance of SMP:

2. Algorithm to convert a solution to an instance of SMP (generated by your algorithm) to a solution to the corresponding instance of R/H:

6 Analyze USMP

1. Formally define an "instability" in the resident/hospital scenario:

2. Using your reduction and the G-S algorithm where **women** propose first: use proposer-optimality to prove that no woman who ends up unmarried could be married in any stable match. (You'll need more space for this page. Sketch ideas here and then grab a sheet of paper!)

7 Analyze R/H

1. Formally define an "instability" in the resident/hospital scenario:

2. Prove that using your reduction and any stable-solution-producing algorithm for SMP yields a stable-solution-producing algorithm for the R/H problem.

8 Challenge Problems

1. Using the USMP-SMP reduction and the G-S algorithm where **men** propose first: prove that no woman who ends up unmarried could be married in any stable match.

2. Create a variant of R/H in which hospitals can have too much or too little capacity and reduce it to SMP.

3. Establish or refute the quality of your reduction (i.e., prove or disprove that stable solutions to the underlying problem produce stable solutions to the general R/H problem).

4. Create a reduction from the "truncated" stable marriage to SMP. In truncated stable marriage, anyone can leave off as many people from the other category as they wish, all of whom are assumed to be worse than everyone they listed.

5. Analyse the properties of your reduction.