Little-o and Little-ω

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Big O, Θ, and Ω are **roughly** equivalent to asymptotic $\leq$, $=$, and $\geq$ comparisons on functions. That naturally leaves analogues of $<$ and $>$ to define.

1 Formal Definitions via Logic

A function $f(n)$ is little-o of another function $g(n)$—i.e., $f(n) \in o(g(n))$—exactly when: for all positive real numbers c, there is a positive integer n_0 such that for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$.

That's a lot like the big-O definition, except that c is not a constant. For **every** possible scaling factor (including very small ones like $\frac{1}{10000}$), once n is large enough, $g(n)$ is **still** bigger than $f(n)$.

Little-ω is exactly the converse definition. For our purposes, $f(n) \in \omega(g(n))$ exactly when $g(n) \in o(f(n))$.

2 Formal Definitions via Limits

A **very** handy tool is to compare the ratios of two functions: $\frac{f(n)}{g(n)}$. This can tell you quite a bit about how they compare asymptotically.

In particular:

1. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, then $g(n) \in o(f(n))$ and $f(n) \in \omega(g(n))$.
2. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f(n) \in o(g(n))$ and $g(n) \in \omega(f(n))$. (Notice that this just means $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$.)
3. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ for some constant real number $0 < c$, then $f(n) \in \Theta(g(n))$ (and so $g(n) \in \Theta(f(n))$).

It turns out we can prove that the limit definitions are equivalent to the logical definitions above (since limits also have quantifier-based definitions!). With a bit of calculus (remind yourself of "L'Hôpital's Rule"), using the limits technique is often **much** easier than using the logical definitions.

Try these out to compare: $n + 3$, $3n$, $n^2 - 1$, and 2^n.

3 Little-o is Not Really Big-O But Not Θ

Consider the function $n|\sin n|$. Because $|\sin n|$ oscilllates between 0 and 1, $n|\sin n|$ oscillates between 0 and n. If we compare that to n asymptotically, we find that $n|\sin n| \in O(n)$ (with the constant scaling factor $c = 1$, in fact!) but $n|\sin n| \notin \Theta(n)$ and $n|\sin n| \notin o(n)$. (In the case of the limit, the ratio of these two functions is just $|\sin n|$ which oscillates between 0 and 1 and so does not approach either value or anything in between!) So our analogy to $<$, $\leq$, $=$, $\geq$, and $>$ is useful but not exact.