

CPSC 320 Notes: What's in a Reduction?

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To reduce a problem A to another problem B , we give one algorithm that takes a (legal) instance a of A and converts it into a legal instance b of B and a second algorithm that takes the corresponding solution s_b to b and transforms it into a solution s_a to a . We've used reductions to solve new problems based on problems we could already solve. For example, reducing hospital/intern matching to stable marriage.

But... there's another way to use reductions. A more **sinister** way.¹

1 Boolean Satisfiability

Boolean satisfiability (SAT) is—as far as Computer Scientists know—a hard problem. In SAT, you're given a propositional logic expression like: $(x_1 \vee \overline{x_2} \vee x_3 \vee x_4) \wedge (x_5) \wedge (\overline{x_1}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (\overline{x_2} \vee x_3)$ and must determine whether any assignment of truth values to variables (the x_i 's) makes the expression true.

Here's a formal definition of an instance of SAT. A *literal* is an integer i or $-i$ (for $i > 0$), meaning x_i or its negation $\overline{x_i}$, respectively. A *clause* is a number $k \geq 0$ followed by k literals, meaning the result of "ORing together" k variables or their negations. An instance of SAT is a number c followed by c clauses. (For convenience, we'll insist on using the variables x_1, x_2, \dots, x_n for some n , without skipping any.) A solution to SAT is simply YES (there is an assignment that makes the expression true) or NO (there isn't).

1. Finish the input to SAT for the example above:

```
5           // 5 clauses
4 1 -2 3 4  // a clause with 4 literals: x1, not x2, x3, or x4
1 5        // a clause with 1 literal: x5
...
```

2. Is the example SAT instance above satisfiable? If not, explain why not. If so, prove it by giving an assignment that makes the statement true.
3. Build two small examples of SAT and check whether they're satisfiable.

¹Well, OK. Just **another** way.

4. If I gave an assignment of truth values to the variables and said it satisfies the expression, how long would it take (in terms of the length of the input) to test whether my statement is true?
5. A brute force algorithm could make a list of the variables x_1, \dots, x_n in the problem, try every assignment of truth values to these variables, and return **YES** if any satisfies the expression or **NO** otherwise. Asymptotically, how many truth assignments might this algorithm try (in terms of n)?

2 3-SAT and SAT

The 3-SAT problem is just like SAT, except **every** clause must be **exactly** of length 3. Let's build a reduction from SAT to 3-SAT. (So, we're solving SAT in terms of 3-SAT.)

1. If you really wanted a clause like (x_5) in 3-SAT, how could you represent it? (Hint: one variable can appear multiple times in a clause. Challenge: how can you do it if one variable is **not** allowed to appear multiple times?)
2. If you really wanted a clause like $(x_1 \vee \overline{x_2} \vee x_3 \vee x_4)$ in 3-SAT, how could you represent it? (Hint: break it into two clauses.)
3. If you really wanted a clause of 100 literals in 3-SAT, how could you represent it?
4. Give a reduction from SAT to 3-SAT.

4 What does NP-completeness tell us?

Most Computer Scientists think “ $P \neq NP$ ”. If that’s true, then there is no correct, deterministic algorithm for any NP-complete problem that runs in polynomial time.² Specifically: if a problem is NP-complete, it’s hopeless to write an algorithm that scales to arbitrarily large problem sizes and definitely, precisely solves every possible instance of those sizes correctly for exactly that problem.

1. I once visited the largest, seated, outdoor, bronze Buddha in the world (at the time). It was pretty impressive, but presumably there was a larger **standing** (reclining?), outdoor, bronze Buddha; a larger seated, **indoor**, bronze Buddha; and a larger, seated, outdoor Buddha **in some other material**.

List as many ways as you can think of to “get around” an NP-complete problem.

2. Go solve a big NP-complete problem in your browser, on your phone, and laugh in the face of NP-completeness: <http://www.msoos.org/2013/09/minisat-in-your-browser/>.

Note, however: There really are an enormous number of NP-complete problems that are **HARD** and **important** to solve, including many interesting instances of SAT. There are also many interesting problems that are either definitely or probably (if $P \neq NP$ or similar conditions) **harder** than NP, for example the problem of “AI Planning”.

3. A **huge** number of problems are solved using SAT solvers because “SAT is an easy target for reductions”.

Explain that quote.

5 Challenge

1. Why wouldn’t our “trick” for reducing SAT to 3-SAT work in “2-SAT”?
2. Give a polynomial-time algorithm to solve 2-SAT.
3. Find a good bound on the length of the 3-SAT instance created by our 3-SAT to SAT reduction in terms of the length of the initial SAT instance. (We take “length” to be $1 + \sum_{i=1}^c (1 + k_i)$.)

Fun Communications of the ACM reference with discussion of industrial and research applications of SAT: <http://goo.gl/KQoKFd>

Solving NP-complete problems is not just for industry and research, it’s for art as well (Travelling Salesperson Problem): <http://www.cgl.uwaterloo.ca/~cšk/projects/tsp/>

²Technical note: that doesn’t mean the algorithm has to run in exponential time. There are options in between, like $2^{\sqrt{n}}$.