CPSC 320 Notes: The Futility of Laying Pipe, Part 3

November 23, 2016

First, an apparently unrelated side-track: **How many edges must there be in a tree with n nodes? So...if you use at most k edges to form a connected graph, how many nodes can you connect?**

Now, we'll prove that your reduction is correct. Remember: your reduction is correct exactly when the answer to an instance of 3-SAT is YES if and only if the answer to the corresponding instance of SP is YES. So, your proof should proceed in two steps:

- Assume the answer to the 3-SAT instance is YES. Prove that the answer to the SP instance your reduction creates is also YES.

- Assume the answer to the SP instance (which your reduction creates from the 3-SAT instance) is YES. Prove that the answer to the corresponding 3-SAT instance is also YES.

1   If 3-SAT answer is YES, SP answer is YES

We'll start by assuming that the answer to the 3-SAT instance is YES and proving that the answer to the SP instance your reduction creates is also YES.

1. We usually want to go further than the assumption that the original instance's answer is YES and say that a solution to the orginal instance (i.e., a working certificate) exists.

   Consider what a solution for 3-SAT looks like and use it to finish the statement: "Since the answer to the 3-SAT instance is YES, there must be..."

2. To prove the underlying instance's answer is YES, we similarly try to show the existence of a solution (working certificate) to that instance and conclude that therefore the answer to the instance is YES.

   What does a solution for SP look like?

3. By assumption (because you have a certificate), you now know the truth value of each variable in the 3-SAT instance. What parts of the solution to (i.e., certificate for) the SP instance follow from these truth assignments?

4. By assumption (because your certificate works), every clause in the 3-SAT problem has at least one true literal. What parts of the solution to the SP instance follow from this fact?

5. Finish the proof to show that the SP certificate actually works, i.e., is a solution to the instance.

2   If SP answer is YES, 3-SAT answer is YES

Now go the other way. Assume that the answer to the SP instance (created by your reduction from the 3-SAT instance) is YES and prove that the answer to the 3-SAT instance is also YES.

Hints: Many pieces of this proof will be very similar to the previous one, but you'll also need to show that any solution (working certificate) for the SP instance has its edges where you intended them to be when you designed your reduction. It may help to think about the number of nodes you can possibly connect, given a maximum of k edges.

You've now shown that SP is in NP and is NP-hard (because 3-SAT—which is known to be NP-hard—is polynomial-time-reducible to SP). Therefore, SP is NP-complete!