Save the Last Dance for Someone Other Than Me

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You're arranging a formal dance evening. n people have signed up to lead and n people to follow. There will be a series of k dances (with $k \leq n$), each of which will match each leader with one follower. However, the same leader and follower will never be paired for more than one dance. (So, every person will dance with k different people across the k dances.) Your job is to create the series of k matchings for the k dances.

As with SMP, you have complete preference lists over the followers for each leader and complete preference lists over the leaders for each follower. (Where leaders and followers in this problem play similar roles to men and women—or vice versa—in SMP.)

1 Almost the Two-Step

Someone suggests that for $k = 2$, we can run Gale-Shapley for the first matching and then simply have each partner in a pair from the first round move their first round partner to the end of their preference lists and run Gale-Shapley again for the second round. (I.e., pairs from the first round mutually declare each other their least favorite options for the second round.)

Finish this proof that this strategy never re-pairs **more than a single pair** in the second round.

Proof: Imagine that a couple **is** re-paired in the second round (which is possible) and that—without loss of generality, since the proof's structure remains the same if we make the opposite assumption—we run Gale-Shapley with leaders proposing. The couple that are re-paired listed each other as their least-preferred partners. The leader in that couple must have proposed to **everyone else** on their list prior to proposing to their eventual partner (because G-S has them propose in order, one after another, down their preference list).

At the moment when the leader's second-to-last offer is rejected, therefore, everyone but the last person on their preference list must already be engaged because...