

CPSC 320 Notes: What's in a Reduction?

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To reduce a problem A to another problem B , we typically proceed as follows: give one algorithm that takes a (legal) instance a of A and converts it into a legal instance b of B and a second algorithm that takes the corresponding solution s_b to b and transforms it into a solution s_a to a . (The second algorithm can use whatever bookkeeping information it needs from the first.)

We've used reductions to solve new problems based on problems we could already solve. For example, reducing hospital/intern matching to stable marriage.

But... there's another way to use reductions. A more **sinister** way.¹

1 Boolean Satisfiability

Boolean satisfiability (SAT) is—as far as Computer Scientists know—a hard problem. In the version of SAT we discuss here, you're given a propositional logic expression like: $(x_1 \vee \overline{x_2} \vee x_3 \vee x_4) \wedge (x_5) \wedge (\overline{x_1}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (\overline{x_2} \vee x_3)$ and must determine whether any assignment of truth values to variables (the x_i 's) makes the expression true.

Here's a formal definition of an instance of SAT. A *literal* is an integer i or $-i$ (for $i > 0$), meaning x_i or its negation $\overline{x_i}$, respectively. A *clause* is a number $k \geq 0$ followed by k literals, meaning the result of “ORing together” k variables or their negations. An instance of SAT is a number c followed by c clauses. (For convenience, we'll insist on using the variables x_1, x_2, \dots, x_n for some n , without skipping any.) A solution to SAT is simply YES (there is an assignment that makes the expression true) or NO (there isn't).

1. Finish the input to SAT for the instance above:

```
5                // 5 clauses
4 1 -2 3 4       // a clause with 4 literals: x1, not x2, x3, or x4
1 5              // a clause with 1 literal: x5
...
```

2. Is the example SAT instance above satisfiable? If not, explain why not. If so, prove it by giving an assignment that makes the statement true.

¹Well, OK. Just **another** way.

3. Give the trivial instance(s) of SAT.²

4. Build two small but non-trivial instances of SAT and check whether they're satisfiable.

5. If I gave an assignment of truth values to the variables and said it satisfies the expression, how long would it take (in terms of the length of the input, i.e., the total number of literals in all the clauses) to test whether my statement is true?

6. A brute force algorithm could make a list of the variables x_1, \dots, x_n in the problem, try every assignment of truth values to these variables, and return YES if any satisfies the expression or NO otherwise. Asymptotically, how many truth assignments might this algorithm try (in terms of n)?

²The conjunction (“and”) of zero conjuncts is **true**. The disjunction (“or”) of zero disjuncts is **false**. Why? These are the identity elements for “and” and “or”.

2 3-SAT and SAT

The 3-SAT problem is just like SAT, except **every** clause must be **exactly** of length 3. Let's build a reduction from SAT to 3-SAT. (So, we're solving SAT in terms of 3-SAT.)

1. If you really wanted a clause like (x_5) in 3-SAT, how could you represent it? *Hint:* one variable can appear multiple times in a clause. *Challenge:* How can you do it if one variable is **not** allowed to appear multiple times?
2. If you really wanted a clause like $(x_1 \vee \overline{x_2} \vee x_3 \vee x_4)$ in 3-SAT, how could you represent it? Note that the 3-SAT instance you create must be satisfiable if and only if the original SAT instance was. *Hint:* Manufacture a new variable and then use it to help break this one clause into two.
3. Extend your 4-literal clause plan above to a 5-literal clause like $(x_1 \vee \overline{x_2} \vee x_3 \vee x_4 \vee \overline{x_5})$.
4. Extend your 5-literal clause plan to a 100-literal clause. How could you represent such a clause in 3-SAT?
5. Give a reduction from SAT to 3-SAT.

3 What does a reduction tell us?

Considering a reduction to be two algorithms that “connect” one problem to another:

1. **IMAGINARY SCENARIO #1:** Say our reduction’s two algorithms take $O(f(n))$ time and we have a solution to the underlying problem that also takes $O(f(n))$ time. What do we know about the original problem?
2. **IMAGINARY SCENARIO #2:** Say our reduction’s two algorithms take $O(g(n))$ time and we know that there is **no solution** to the original problem that runs in $O(g(n))$ time. What do we know about the underlying problem? Why?
3. **NOT-SO-IMAGINARY SCENARIO #3:** Say that we know (which we do) that if SAT can be solved in polynomial time, then **any** problem in the large set called “NP” can also be solved in polynomial time. What does our reduction from SAT to 3-SAT tell us? Why?

4 What does NP-completeness tell us?

Most Computer Scientists think “ $P \neq NP$ ”. If that’s true, then there is no correct, deterministic algorithm for any NP-complete problem that runs in polynomial time.³ Specifically: if a problem is NP-complete, it’s hopeless to write an algorithm that scales to arbitrarily large problem sizes and definitely, precisely solves every possible instance of those sizes correctly for exactly that problem.

1. Imagine you visit the largest, seated, outdoor, bronze Buddha in the world. It’s pretty impressive... but presumably there’s a larger **standing** (reclining?), outdoor, bronze Buddha; a larger seated, **indoor**, bronze Buddha; and a larger, seated, outdoor Buddha **in some other material**.

List as many ways as you can think of to “get around” an NP-complete problem.

2. Go solve a big NP-complete problem in your browser, on your phone, and laugh in the face of NP-completeness: <http://www.msoos.org/2013/09/minisat-in-your-browser/>.

Note, however: There really are an enormous number of NP-complete problems that are **HARD** and **important** to solve, including many interesting instances of SAT. There are also many interesting problems that are either definitely or probably (if $P \neq NP$ or similar conditions) **harder** than NP, for example the problem of “AI Planning”.

3. A **huge** number of problems are solved using SAT solvers because “SAT is an easy target for reductions”.

Explain that quote.

³Technical note: that doesn’t mean the algorithm has to run in exponential time. There are options in between, like $2^{\sqrt{n}}$.

5 Challenge

1. Why wouldn't our "trick" for reducing SAT to 3-SAT work in "2-SAT"?
2. Give a polynomial-time algorithm to solve 2-SAT.
3. Find a good bound on the length of the 3-SAT instance created by our SAT to 3-SAT reduction in terms of the length of the initial SAT instance. (We take "length" to be $1 + \sum_{i=1}^c (1 + k_i)$.)

Fun Communications of the ACM reference with discussion of industrial and research applications of SAT: <http://goo.gl/KQoKFd>.

Solving NP-complete problems is not just for industry and research, it's for art as well (Travelling Salesperson Problem): <http://www.cgl.uwaterloo.ca/csk/projects/tsp/>.