THE UNIVERSITY OF BRITISH COLUMBIA
CPSC 320 2016WT2: WEEKLY QUIZZES

Full Name: ______________________ Exam ID: |__|__|__|

Signature: ______________________ UBC Student #: |__|__|__|__|__|__|__|__|

Important notes about this examination

1. You have 25 minutes to complete this quiz.
2. **Answer all questions in PEN and write CLEARLY and LEGIBLY.**
3. You are allowed to bring up to (the equivalent of) a 3-inch 3-ring binder of notes and 3 textbooks, and nothing else. Justify all you answers.
4. Use the back of the pages for your notes, or if you need extra space for the answer to any question.
5. Good luck!

Please do not write in this space:

Quiz Number:

Tutorial Section:

1 236413 048929

3-SAT Variations

March 28, 2017

Recall the 3-SAT problem. Given a collection of 3-literal clauses, we want to decide whether there is a satisfiable assignment (of $true/false$ values to variables) that satisfies each clause. This problem is NP-complete even if we assume that clauses cannot contain the same variable multiple times. For instance, clauses $(x_1 \vee x_1 \vee x_1)$ or $(x_1 \vee \overline{x_1} \vee x_2)$ are not allowed.

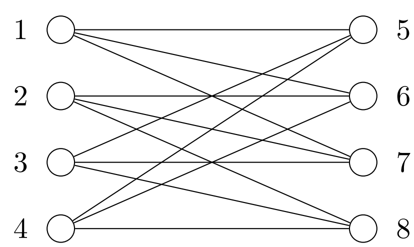
The number of occurrence of a variable x_i in the instance of 3-SAT is the number of times literals x_i or $\overline{x_i}$ appears in the instance. For example, in instance

$$(x_1 \vee \overline{x_2} \vee x_6) \wedge (\overline{x_6} \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (\overline{x_2} \vee x_3 \vee x_5)$$

there is 1 occurrence of x_1, 3 occurrences of x_2, 3 occurrences of x_3, 1 occurrence of x_4, 2 occurrences of x_5 and 2 occurrences of x_6.

A *bipartite graph* is a undirected graph $G = (V, E)$ in which vertices can be partitioned into two sets V_1, V_2 (so $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \{\}$) such that every edge $(u, v) \in E$ has one end point in V_1 and the other end point in V_2. A bipartite graph is often denoted as $G = (V_1, V_2, E)$.

A graph is called *regular*, if every vertex has the same degree. For example, the following graph is a regular bipartite graph:



Note that the sizes of the two partitions (the left and right set of vertices) in a regular bipartite graph are the same.

BIPARTITE MATCHING problem: Given a bipartite graph $G = (V_1, V_2, E)$. Find a maximal matching $E' \subset E$. (Recall: no two edges in a matching share a vertex.)

The BIPARTITE MATCHING problem can be solved in time $O(|V||E|)$ by Ford-Fulkerson algorithm.

In addition, we have the following theorem:

Theorem 1 (Hall's Theorem). *A regular bipartite graph has a matching with exactly $|V_1| = |V_2|$ edges (so it involves all vertices).*

1 Darn 2-clauses!

Consider a special version of 3-SAT that requires that each variable has exactly 3 occurrences, but allows each clause to have either 2 or 3 literals. Let's call this problem 2,3-SAT3. Here is an example of an instance of 2,3-SAT3 problem:

$$(x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_4) \wedge (x_1 \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3}) \wedge (x_3 \vee x_4)$$

Note that each variable has exactly 3 occurrences.

We want to show that 2,3-SAT3 is NP-hard[1]. Follow the following steps to reduce 3-SAT to 2,3-SAT3. For simplicity, you may assume that in each instance of 3-SAT, every variable has at least 2 occurrences.

(a) Given variables $y_1, y_2, \ldots, y_k$, $k \geq 2$, design a set of 2-literal clauses such that there are exactly two assignments for these variables that satisfy the set of your designed clauses: (i) $y_1 = y_2 = \cdots = y_k = false$, and (ii) $y_1 = y_2 = \cdots = y_k = true$. Each variable should have exactly 2 occurrences in these clauses.

(b) Now consider an instance S of 3-SAT in which variable x_i has $k \geq 2$ occurrences. Using part (a) construct a new instance S' by replacing all occurrences of x_i with new variables $y_1, \ldots, y_k$. Add necessary clauses to make the original instance S and the new instance S' equivalent (one is satisfiable if and only if the other is satisfiable). Argue why each of these new variables has exactly 3 occurrences in S'.

[1]This is somewhat surprising, since both 2-SAT and 3-SAT3 are in P.

(c) Describe a reduction from 3-SAT to 2,3-SAT3 as an algorithm.

You don't need to prove correctness of your reduction!